MULTITIME DIFFERENTIABLE STOCHASTIC PROCESSES, DIFFUSION PDES, TZITZEICA HYPERSURFACES

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În această lucrare extindem teoria integrabilităţii complete la sisteme diferenţiale stochastice multitemporale, utilizând integrale curbilinii independente de drum. Rezultatele principale includ procesele stochastice multitemporale cu dependenţă volumetrică, derivata unui proces stochastic în raport cu un proces Wiener multitemporal şi descrierea lor prin EDP de difuzie, polinoame Hermite şi hipersuprafeţe Tzitzeica. Orice proces stochastic multitemporal diferenţiabil admite o dezvoltare in serie de polinoame Hermite. Geometric, mulţimile de nivel constant ale proceselor stochastice multitemporale cu dependenţă volumetrică sunt reuniuni de hipersuprafeţe Tzitzeica. Rezultatele principale pot fi utilizate pentru ameliorarea tehnicilor de spirometrie.

In this paper we address the problem of extending the complete integrability theory to multitime stochastic differential systems, using path independent curvilinear integrals. The main results include the multitime stochastic processes with volumetric dependence, the derivative of a stochastic process with respect to a multitime Wiener process and their description via the multitime diffusion PDEs, Hermite polynomials and Tzitzeica hypersurfaces. Any differentiable multitime stochastic process admits an expansion in series of Hermite polynomials. Geometrically, the constant level sets of multitime stochastic processes with volumetric dependence are union of Tzitzeica hypersurfaces. The main results can be used to improve the spirometry techniques.

Keywords: Multitime stochastic process, volumetric functions, multitime diffusion PDEs, Hermite polynomials, Tzitzeica hypersurfaces, spirometry.
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1. **Introduction**

The paper [2] described the possibility of introducing stochastic curvilinear integrals along all sufficiently smooth curves in $\mathbb{R}^m_+$. The most simple situation is that of increasing curves. Our papers [9], [13] extended this point of view to stochastic curvilinear integrals and to completely integrable stochastic differential systems in $\mathbb{R}^m_+$ (non-negative orthant of R^m defined via the *product order*). These research trends and the original results are based on *Itô-Udrişte stochastic calculus*

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rules [13]

$$dW_t^a\, dW_t^b = \delta^{ab}\, c_\alpha(t)dt^\alpha, dW_t^a\, dt^\alpha = dt^\alpha\, dW_t^a = 0, dt^\alpha\, dt^\beta = 0,$$

for any $a, b = \overline{1,d}; \alpha, \beta = \overline{1,m}$, where $t = (t^1, ..., t^m) \in \mathbb{R}_+^m$ means the *multitime*, δ^{ab} is the *Kronecker symbol*, $v = t^1 \cdots t^m$ is the *volume* of the hyperparallelepiped $\Omega_{0t} \subset R_+^m$, $c_\alpha(t) = \frac{\partial v}{\partial t^\alpha}$ and the tensorial product $\delta^{ab}\, c_\alpha(t)$ represents the *correlation coefficients*.

Section 2 studies differentiable multitime stochastic processes, highlighting their volumetric character. Section 3 describes the differentiable multitime stochastic processes as solutions of multitime backward diffusion PDEs or as sums of series of Hermite polynomials. Section 4 studies unions of Tzitzeica hypersurfaces and their connection to differentiable multitime stochastic processes. Section 5 underlines possible applications in Thermodynamics, Biology, Chemistry, Medicine etc.

2. Multitime differentiable stochastic processes

Let $(W_t)_t$, $t = (t^1, ..., t^m) \in \mathbb{R}_+^m$ be a multitime Wiener process [9] and let $f\,(t, x)\,, t = (t^1, ..., t^m) \in \mathbb{R}_+^m$, $x \in \mathbb{R}$ be a real-valued function, with $f(t, 0) = 0$, which has continuous partial derivatives of the first order with respect to t^α, $\alpha = \overline{1,m}$ and of the second order in x. Such a function defines a stochastic process

$$y_t = f\,(t, W_t)\,,\ t \in \mathbb{R}_+^m$$

By Itô-Udrişte Lemma [13], the foregoing process is involved in the associated stochastic equation

$$dy_t = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,(t, W_t)\, c_\alpha\,(t) + \frac{\partial f}{\partial t^\alpha}\,(t, W_t)\right) dt^\alpha + \frac{\partial f}{\partial x}\,(t, W_t)\, dW_t. \tag{1}$$

2.1. Multitime backward diffusion PDE

In order that the stochastic process y_t be a *martingale*, the *drift coefficients* in formula (1) must vanish, i.e., f is a solution of the *backward diffusion-like system*

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,(t, W_t)\, c_\alpha\,(t) + \frac{\partial f}{\partial t^\alpha}\,(t, W_t) = 0,\ \text{for } \alpha = \overline{1,m}. \tag{2}$$

Theorem 2.1. *The solution f of the diffusion system (2) depends on the point $(t^1, ..., t^m)$ only through the product of components $t^1 \cdots t^m$, i.e., it is a function of the volume $v = t^1 \cdots t^m$ of the hyperparallelepiped $\Omega_{0t} \subset \mathbb{R}_+^m$.*

Proof From (2) it follows

$$c_\beta(t)\,\frac{\partial f}{\partial t^\alpha}\,(t, W_t) = c_\alpha(t)\,\frac{\partial f}{\partial t^\beta}\,(t, W_t)\,,\ \alpha \neq \beta.$$

The general solution of this PDEs system is $f(t) = \varphi(t^1 \cdots t^m, W_t)$, $t \in \mathbb{R}_+^m$.

Let us take into account the shape of volumetric features, recalling that a volumetric function is invariant under the subgroup of central equi-affine (i.e., volume-preserving with no translation) transformations, where the determinant of the representing matrix is 1.

If $v = t^1 \cdots t^m$ is the volume of $\Omega_{0t} \subset \mathbb{R}^m_+$ and $g(v,x) \overset{def}{=} \varphi\left(t^1 \cdots t^m, x\right)$ has continuous partial derivatives of the first order in v and of the second order in x, then from (2) it follows that g satisfies the *backward multitime heat PDE*

$$\frac{\partial g}{\partial v} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0. \tag{3}$$

Consequently, the relation (1) reduces to

$$dg(v, W_t) = \frac{\partial g}{\partial x}(v, W_t)\, dW_t.$$

Thus, if $\mathbb{E}\left[\left(\frac{\partial g}{\partial x}(v, W_t)\right)^2\right]$ is bounded with respect to t, in bounded subsets of $\mathbb{R}^m_+$, then the stochastic process $g(v, W_t)$ is *differentiable* with the *stochastic derivative* $\frac{\partial g}{\partial x}(v, W_t)$.

2.2. Path independent stochastic curvilinear integral

The foregoing theory suggests introducing the notion of multitime differentiable stochastic processes. For this purpose we need a multitime Wiener process $(W_t)_t,\ t = (t^1, ..., t^m) \in \mathbb{R}^m_+$.

Definition 2.1. *Let $\gamma_{0t} \subset \mathbb{R}^m_+$ be an increasing curve joining the points $0, t \in \mathbb{R}^m_+$. A multitime stochastic process $\Phi_t = \Phi(t, W_t),\ t \in \mathbb{R}^m_+$ is called differentiable with respect to W_t, on $\mathbb{R}^m_+$, if there exists a multitime adapted measurable process $\phi_t = \phi(t, W_t),\ t \in \mathbb{R}^m_+$ such that $\mathbb{E}\left[\phi_t^2\right]$ is bounded for t in compact sets of $\mathbb{R}^m_+$ and*

$$\Phi_t = \Phi_0 + \int_{\gamma_{0t}} \phi_s\, dW_s, \tag{4}$$

where the stochastic curvilinear integral is path independent.

In terms of stochastic differentials, the multitime stochastic process Φ_t is differentiable if the stochastic system

$$d\Phi_t = \phi_t\, dW_t.$$

is completely integrable, i.e., $\Phi(t)$ is a function of $v = t^1 \cdots t^m$ and hence $\phi(t)$ is a function of v [13].

The multitime process ϕ_t is called the *derivative* of the multitime process Φ_t with respect to W_t (see also [2], [3]).

Remark A differentiable multitime stochastic process has properties similar to those of a holomorphic function: a differentiable process has a differentiable derivative, the curvilinear integral primitive of a differentiable process is differentiable, and each differentiable process admits a power series expansion.

3. Hermite polynomials and stochastic processes

There is a special class of solutions of the backward multitime heat equation which will be particularly interesting in stochastic problems. These are the *Hermite polynomials* (e.g., [1], [4]). Denote by $H_n(v,x)$ the n^{th} Hermite polynomial of two variables v and x, i.e.,

$$H_n(v,x) = \frac{(-v)^n}{n!}\, e^{\frac{x^2}{2v}}\, \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2v}}. \tag{5}$$

Examples:

$$H_2(v,x)=\frac{1}{2}(x^2-v);\ H_3(v,x)=\frac{1}{6}x^3-\frac{1}{2}xv;\ H_4(v,x)=\frac{1}{24}x^4-\frac{1}{4}x^2v+\frac{1}{8}v^2.$$

For $t=(t^1,...,t^m)\in\mathbb{R}^m_+$ and $v=t^1\cdots t^m$, the sequence $\{H_n(v,\cdot)\}_{n=0}^{\infty}$ is a complete orthogonal system with respect to the weight $(2\pi v)^{\frac{-1}{2}}\,e^{\frac{-x^2}{2v}}$. The orthogonality means

$$\mathbb{E}\left[H_m\left(v,W_t\right)H_n\left(v,W_t\right)\right]=\begin{cases}0, & \text{if } m\neq n,\\ \dfrac{v^n}{n!}, & \text{if } m=n.\end{cases} \tag{6}$$

Now, let us use the generating function expansion

$$e^{x\xi-\frac{1}{2}v\xi^2}=\sum_{n=0}^{\infty}H_n(v,x)\xi^n,\ x,\xi\in R,\ v\in R_+.$$

Taking the partial derivatives with respect to v and x, we replace the exponential by the corresponding series, and equating the coefficients of both series, we find

$$\frac{\partial}{\partial v}H_n=-\frac{1}{2}H_{n-2},\ \frac{\partial}{\partial x}H_n=H_{n-1}.$$

Consequently each Hermite polynomial $H_n(v,x)$ is a solution of the backward multitime heat PDE. Thus

Theorem 3.1. *Each Hermite polynomial $H_n(v,W_t)$ is a differentiable process and its derivative is $H_{n-1}(v,W_t)$.*

It follows that finite sums of Hermite polynomials processes are differentiable processes. We extend this statement to series of Hermite polynomials, following the ideas of Cairoli and Walsh [2].

Theorem 3.2. *Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that*

$$\sum_{n=0}^{\infty}a_n^2\,\frac{v^n}{n!}<\infty,\quad \textit{for all } v>0 \textit{ (the series is convergent).}$$

Then, the process Φ_t defined by

$$\Phi_t=\sum_{n=0}^{\infty}a_nH_n(v,W_t) \tag{7}$$

is differentiable with respect to W_t and its derivative ϕ_t is

$$\phi_t=\sum_{n=1}^{\infty}a_nH_{n-1}(v,W_t). \tag{8}$$

The convergence of the series is understood in L^2.

Proof. By the orthogonality relations (6), we find

$$\mathbb{E}\left[\left(\sum_{n=0}^{m}a_nH_n\left(v,W_t\right)\right)^2\right]=\sum_{n=0}^{m}a_n^2\,\frac{v^n}{n!}\,.$$

This mean value is bounded due to the convergence of the series in the right hand member. It follows that the series (7) converges in L^2 and the same is true for the series (8). Consider now the sequence of partial sums

$$\phi_t^{(m)} = \sum_{n=1}^{m} a_n H_{n-1}\left(v, W_t\right),$$

and let

$$\Phi_t^{(m)} = a_0 + \int_{\gamma_{0t}^{\alpha}} \phi_s^{(m)}\, dW_s = a_0 + \sum_{n=1}^{m} a_n H_n\left(v, W_t\right),$$

where

$$\gamma_{0t}^{\alpha} : t^1 = c^1, ..., t^{\alpha-1} = c^{\alpha-1}, t^{\alpha} = \tau^{\alpha} \in \left[0, t^{\alpha}\right], t^{\alpha+1} = c^{\alpha+1}, ..., t^m = c^m$$

is an increasing curve joining the points 0 and t, with $c^{\beta} = \text{const} > 0$, $\beta \neq \alpha$. It follows

$$\lim_{m\to\infty} \Phi_t^{(m)} = \Phi_t \text{ in } L^2.$$

To finish the proof, we need only to check that

$$\lim_{m\to\infty} \int_{\gamma_{0t}^{\alpha}} \phi_s^{(m)}\, dW_s = \int_{\gamma_{0t}^{\alpha}} \lim_{m\to\infty} \phi_s^{(m)}\, dW_s,\ \alpha = \overline{1, m}.$$

Again, by the orthogonality relation (6), we have

$$\mathbb{E}\left[\left(\phi_t - \phi_t^{(m)}\right)^2\right] = \sum_{n=m+1}^{\infty} a_n^2\, \frac{v^{n-1}}{(n-1)!},$$

and consequently

$$\mathbb{E}\left[\left(\int_{\gamma_{0t}^{\alpha}} \left(\phi_s - \phi_s^{(m)}\right) dW_s\right)^2\right] =$$

$$= \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{m} t^{\beta} \int_0^{t^{\alpha}} \sum_{n=m+1}^{\infty} a_n^2\, \frac{\left(t^1 ... t^{\alpha-1} \tau^{\alpha} t^{\alpha+1} ... t^m\right)^{n-1}}{(n-1)!}\, d\tau^{\alpha} = \sum_{n=m+1}^{\infty} a_n^2\, \frac{v^n}{n!}.$$

In other words,

$$\lim_{m\to\infty} \int_{\gamma_{0t}^{\alpha}} \phi_s^{(m)}\, dW_s = \int_{\gamma_{0t}^{\alpha}} \phi_s\, dW_s.$$

□

Theorem 3.3. *Suppose that $\varphi(v, x)$ has continuous partial derivatives of the first order in v and of the second order in x, and that $\{\varphi(v, W_t)\}$ is a differentiable process, where $v = t^1 \cdots t^m$. Then, for each $t \in \mathbb{R}_+^m$, we can write*

$$\varphi\left(v, W_t\right) = \sum_{n=0}^{\infty} a_n H_n\left(v, W_t\right),$$

where the convergence is understood in L^2 and, for $t = (t^1, ..., t^m) \in \mathbb{R}_+^m$,

$$a_n = \frac{n!}{v^n}\, \mathbb{E}\left[\varphi\left(v, W_t\right) H_n\left(v, W_t\right)\right]. \tag{9}$$

Proof. Define the process $\Phi_t = \sum_0^\infty a_n H_n(v, W_t)$. Due to the convergence

$$\sum_0^\infty a_n^2 \frac{v^n}{n!} < \infty,$$

the series defining Φ_t converges in L^2 and the process Φ_t is differentiable.

Since H_n is an orthogonal sequence, we get

$$\mathbb{E}\left[\Phi_t H_n(v, W_t)\right] = \mathbb{E}\left[a_n H_n^2(v, W_t)\right] = a_n \frac{v^n}{n!}.$$

But, by hypothesis, $\mathbb{E}\left[\varphi(v, W_t) H_n(v, W_t)\right] = \frac{v^n}{n!} a_n$. Thus,

$$\mathbb{E}\left[(\varphi(v, W_t) - \Phi_t) H_n(v, W_t)\right] = 0, \forall n \in \mathbb{N}.$$

Then, $\varphi(v, W_t) - \Phi_t \equiv 0$. □

Remark We fix the point $t = (t^1, ..., t^m) \in \mathbb{R}_+^m$ and the volume $v = t^1 \cdots t^m$. Using the fact that H_n is a complete orthogonal sequence, we can define

$$f(v, x) = \sum_0^\infty a_n H_n(v, x),$$

where

$$a_n = \frac{n!}{v^n} \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{+\infty} f(v, x) \exp\left(-\frac{x^2}{2v}\right) dx.$$

This is just another way to write the relation (9).

4. **Union of Tzitzeica hypersurfaces**

A hypersurface $M \subset \mathbb{R}_+^m$, $m \geq 3$, is called *Tzitzeica hypersurface*, provided there exists a constant $a \in \mathbb{R}$ such that we have $K = a\, d^{m+1}$, for all points $t = (t^1, ..., t^m) \in M$, where K is the Gauss curvature of the hypersurface and d is the distance from the origin of the space to the tangent hyperplane to the hypersurface at the current point t. Since the Gauss curvature K describes the shape of the hypersurface, a Tzitzeica hypersurface has a bending against the tangent hyperplane in fixed proportion to the normal component of the position vector t. The simplest Tzitzeica hypersurfaces are the constant level sets $M_c : t^1 \cdots t^m = c$ in $\mathbb{R}^m$ (2^{m-1} connected components) (see, also, [5], [7], [8] or [11]).

Remark The Gauss curvature of a Cartesian implicit surface

$$M_c : F(t^1, t^2, t^3) = c$$

in $\mathbb{R}^3$ is the function

$$\begin{aligned} K &= \left[[F_3(F_3F_{11} - 2F_1F_{13}) + F_1^2F_{33}][F_3(F_3F_{22} - 2F_2F_{23}) + F_2^2F_{33}\right. \\ &- \left.(F_3(-F_1F_{23} + F_3F_{12} - F_yF_{13}) + F_1F_2F_{33})^2\right] [F_3^2(F_1^2 + F_2^2 + F_3^2)^2]^{-1}, \end{aligned}$$

where the indices mean partial derivatives. The surface M_c is curving like a paraboloid if $K(t) > 0$, hyperboloid if $K(t) < 0$, or a cylinder or plane if $K(t) = 0$, near a point $t = (t^1, t^2, t^3) \in M_c$.

Let us show that the constant level sets of the functions $\varphi(t^1 \cdots t^m, x)$, with respect to $t = (t^1, ..., t^m) \in \mathbb{R}_+^m$, are Tzitzeica hypersurfaces or unions of simple Tzitzeica hypersurfaces in $\mathbb{R}_+^m$ indexed by the points x.

Theorem 4.1. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2 nonconstant function and c is a noncritical value of φ, then the constant level set $N_c : \varphi(t^1 \cdots t^m) = c$ is a union of simple Tzitzeica hypersurfaces.*

Proof. Let A_c be the set of the solutions of the equation $\varphi(v) = c$ and $v = t^1 \cdots t^m$. Then the constant level set N_c is the union of the constant level sets $t^1 \cdots t^m = k,\ k \in A_c$. If c is not a critical value of φ, then the set N_c is a union of hypersurfaces.

Remark If c is a critical value of φ, then the set N_c is a union of constant level sets $t^1 \cdots t^m = k,\ k \in A_c$, but the level sets which correspond to $\varphi'(k) = 0$ are not hypersurfaces.

Theorem 4.2. *(1) Each section $x = c$ of the hypersurface $H_n(v, x) = 0$ in $\mathbb{R}^{m+1} = \{(t, x)\}$ is a union of Tzitzeica hypersurfaces in $\mathbb{R}^m_+$.*

(2) A cylinder $v = t^1 \cdots t^m = c$ in $\mathbb{R}^{m+1} = \{(t, x)\}$ intersects alternatively the constant level sets $H_n(v, x) = 0$ and $H_{n+1}(v, x) = 0$ in $\mathbb{R}^{m+1}$.

Proof Each Hermite polynomial H_n has n roots, real, distinct, and strictly inside the interval of orthogonality, as a polynomial in an orthogonal sequence. Also, the roots of each polynomial in an orthogonal sequence lie strictly between the roots of the next higher index polynomial in the sequence.

For each simple Tzitzeica hypersurface $v = t^1 \cdots t^m = c$, the zeroes of the Hermite polynomial $H_n(v, x)$ lie strictly between the simple zeroes of the Hermite polynomial $H_{n+1}(v, x)$.

The constant level set $\mathbb{E}\left[H_n\left(v, W_t\right) H_n\left(v, W_t\right)\right] = c > 0$ is a simple Tzitzeica hypersurface.

5. **Volumetric functions and their applications**

The applications of the foregoing theory are in domains based on volumetric functions indexed after additional variables, such as Thermodynamics (e.g., thermodynamic functions of volume and temperature), Chemistry (e.g., family of volume-dependent interatomic pair potentials), Biology, Medicine (e.g., spirometry) etc.

One important example is *Spirometry* [6] (meaning the measuring of breath) used for the Pulmonary Function Tests (PFTs), measuring Lung functions, i.e., functions of the amount (volume) and/or speed (flow) of air that can be inhaled and exhaled. Spirometry is an important tool used for generating pneumotachographs which are helpful in controlling certain ailing such as asthma, pulmonary fibrosis, cystic fibrosis, and COPD (chronic obstructive pulmonary disease).

The spirometry test is performed using a device called a *spirometer*, which comes in several different varieties. Most spirometers display the following graphs, called *spirograms*:

1) a *volume-time curve*, showing volume (liters) along the vertical axis and time (seconds) along the horizontal axis;

2) a *flow-volume loop*, which graphically depicts the rate of airflow on the vertical axis and the total volume inspired or expired on the horizontal axis (a graphic of the instantaneous rate of airflow during a forced expiration; it may be a maximum expiratory flow-volume curve or a partial expiratory flow-volume curve).

A canonical prediction PDE for spirometric parameters and maximal expiratory flows is the diffusion PDE.

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