U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 3, 2015 ISSN 1223-7027

ON PSEUDO-CHEBYSHEV SUBSPACES IN QUOTIENT GENERALIZED 2-NORMED SPACES

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In this paper, we study the concept of best simultaneous approximation in quotient generalized 2-normed linear spaces. We will determined under what conditions pseudo-Chebyshev subspaces are transmitted to and from quotient spaces. Also we shall give a characterization of simultaneous pseudo-Chebyshev subspaces on these spaces.

Keywords: Generalized 2-normed space, 2-bounded, 2-best simultaneous approximation, simultaneous 2-pseudo-Chebyshev.

MSC2010: 41A 65, 52A 21, 54B 15.

1. Introduction and preliminaries

Approximation theory has many important applications in various areas of functional analysis, computer science, numerical solutions of differential and integral equations. A generalization of normed spaces is 2-normed spaces plays a very important role in functional analysis. The concept of linear 2-normed spaces was initiated by Gähler in 1965 ([8]) and has been developed extensively in different subjects by others. Later, in 1999-2004, Z. Lewandwoska published a series of papers on 2-normed sets and generalized 2-normed spaces, investigating some properties of these spaces. ([10]-[14]). The concept of generalized 2-normed space is a generalization of the concepts of a normed space and of a 2-normed space. In fact, generalized 2-normed spaces are part of locally convex spaces. Recently, some results on best approximation theory in generalized 2-normed spaces have been obtained by Sh. Rezapour, M. Acikgoz and others (for example [1]-[5] and [16]-[22]). The theory of best simultaneous approximation has been studied by many authors (for example [6],[7],[9]). In [9], M. Iranmanesh and H. Mohebi get some results on best simultaneous approximation in quotient normed spaces. In this paper, we shall introduce the notions of 2-best simultaneous approximation in quotient generalized 2-normed spaces and we shall give some results in this field.

Definition 1.1. [8] *Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following conditions:*
(G1)$\|x, y\| = 0$ *if and only if x and y are linearly dependent vectors.*
(G2)$\|x, y\| = \|y, x\|$ *for all* $x, y \in X$.

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(G3)$\|\alpha x, y\| = |\alpha|\|x, y\|$ *for every real number* α.
(G4)$\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all* $x, y, z \in X$.
Then $\|., .\|$ *is called a 2-norm on* X *and the pair* $(X, \|., .\|)$ *is called a linear 2-normed space.*

There are no remarkable relations between normed spaces and 2-normed spaces. We could not construct any 2-norm on X by a normed space $(X, \|.\|)$, and this could be a motive for definition of generalized 2-normed spaces.

Definition 1.2. [10],[11] *Let* X *and* Y *be linear spaces,* D *be a non-empty subset of* $X \times Y$ *such that for every* $x \in X$ *and* $y \in Y$ *, the sets*

$$D_x = \{y \in Y : (x, y) \in D\} \ ; \ D_y = \{x \in X : (x, y) \in D\}$$

are linear subspaces of the spaces Y *and* X*, respectively. A function* $\|., .\| : D \to [0, \infty)$ *is called a generalized 2-norm on* D *if it satisfies the following conditions:*
$(N1)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ *for all* $(x, y) \in D$ *and every scalar* α.
$(N2)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for all* $(x, y), (x, z) \in D$.
$(N3)\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all* $(x, z), (y, z) \in D$.
Then $(D, \|., .\|)$ *is called a 2-normed set. In particular, if* $D = X \times Y$, $(X \times Y, \|., .\|)$ *is called a generalized 2-normed space. Moreover, if* $X = Y$*, then generalized 2-normed space is denoted by* $(X, \|., .\|)$.

Definition 1.3. [14] *Let* X *be a real linear space. Denote by* $\mathcal{X}$ *a non empty subset of* $X \times X$ *with the property* $\mathcal{X} = \mathcal{X}^{-1}$*(Symmetric) and such that the set* $\mathcal{X}^y = \{x \in \mathcal{X}; (x; y) \in \mathcal{X}\}$ *is a linear subspace of* X*, for all* $y \in X$*. A function* $\|., .\| : \mathcal{X} \to [0, \infty)$ *satisfying the following conditions:*
$(S1)\|x, y\| = \|y, x\|$*for all* $(x; y) \in \mathcal{X}$,
$(S2)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ *for any real number* α *and all* $(x, y) \in \mathcal{X}$,
$(S3)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for all* $x, y, z \in X$ *such that* $(x, y), (x, z) \in \mathcal{X}$,
will be called a generalized symmetric 2-norm on $\mathcal{X}$*. The set* $\mathcal{X}$ *is called a symmetric 2-normed set. In particular, if* $\mathcal{X} = X \times X$*, the function* $\|., .\|$ *will be called a generalized symmetric 2-norm on* X *and the pair* $(X; \|., .\|)$ *a generalized symmetric 2-normed space.*

The following examples are some generalized 2-normed spaces and symmetric generalized 2-normed spaces.

Example 1.1. [15] *1) Let* X *be a real linear space having two norms* $\|.\|_1$ *and* $\|.\|_2$*. Then* $(X, \|., .\|)$ *is a generalized 2-normed space with the 2-norm defined by*

$$\|x, y\| = \|x\|_1.\|y\|_2 \ ; x, y \in X.$$

Specially if $\|.\|_1 = \|.\|_2$*, our generalized 2-normed space will be a generalized symmetric 2-normed space.*
2) Let X *be a real inner product space. Then* X *is a symmetric generalized 2-normed space under the 2-norm*

$$\|x, y\| = |\langle x, y\rangle| \ ; \ \forall \ x, y \in X.$$

3) Let X *be the linear space of all sequence of real numbers. Put*

$$\|x, y\| = \sum_{1}^{\infty} |x_n||y_n|,$$

where $x = \{x_n\}, y = \{y_n\} \in X$. *Then* $D = \{(x,y) \in X \times X \ : \ \|x,y\| < \infty\}$ *is a symmetric 2-normed set and the function* $\|.,.\| : D \to [0,\infty)$ *is a generalized symmetric 2-normed on* D.
4) Let A *be a Banach algebra and* $\|a,b\| = \|ab\|$ *for all* $a, b \in A$. *Then,* $(A, \|.,.\|)$ *is a generalized 2-normed space.*

$S_1 \times S_2$ is called a 2-bounded subset of $X \times Y$ if there exists $r > 0$ such that $\|s_1, s_2\| < r$ for all $(s_1, s_2) \in S_1 \times S_2$.

Lemma 1.1. *Let* $(X, \|.\|)$ *be a normed space, and let* X *be equipped with the following generalized 2-norm*

$$\|x,y\| = \|x\|.\|y\|; \ \forall x, y \in X.$$

If S *is a bounded set in* X*, then* $S \times S$ *is a 2-bounded subset of* $X \times X$.

Proof. Let S be a bounded set in X. Then there exists $r > 0$ such that $\|x\| < r$, for each $x \in S$. Then we have

$$\|x,y\| = \|x\|.\|y\| < r.r = r^2,$$

for each $x, y \in S$. Therefore $S \times S$ is a 2-bounded subset of $X \times X$. □

Definition 1.4. *Let* $X \times Y$ *be a generalized 2-normed linear space,* $W_1 \times W_2$ *a subset of* $X \times Y$ *and* $S_1 \times S_2$ *a 2-bounded subset of* $X \times Y$. *We define*

$$d(S_1 \times S_2, W_1 \times W_2) = \inf_{(w_1,w_2) \in W_1 \times W_2} \ \sup_{(s_1,s_2) \in S_1 \times S_2} \|s_1 - w_1, s_2 - w_2\|,$$

if there exists some $(w_1, w_2) \in W_1 \times W_2$ *such that* $\sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1, s_2 - w_2\| < \infty$. $S_1 \times S_2$ *is called 2-simultaneous proximinal if for every* $(s_1, s_2) \in S_1 \times S_2$ *there exists an element* $(w_{01}, w_{02}) \in W_1 \times W_2$ *such that*

$$d(S_1 \times S_2, W_1 \times W_2) = \sup_{(s_1,s_2) \in S_1 \times S_2} \|s_1 - w_{01}, s_2 - w_{02}\|.$$

In this case $(w_{01}, w_{02}) \in W_1 \times W_2$ *is called a 2-best simultaneous approximation to* $S_1 \times S_2$ *from* $W_1 \times W_2$. *The set of all 2-best simultaneous approximation to* $S_1 \times S_2$ *from* $W_1 \times W_2$ *will be denoted by* $\boldsymbol{S}_{W_1 \times W_2}(S_1 \times S_2)$. *If* $S_1 \times S_2 = \{(x,y)\}$ *where* $(x,y) \in X \times Y$ *then* $\boldsymbol{S}_{W_1 \times W_2}(S_1 \times S_2)$ *is the set of all 2-best approximation of* (x,y) *in* $W_1 \times W_2$ *that denoted by* $P_{W_1 \times W_2}(x,y)$ *and also* $W_1 \times W_2$ *is called a 2-proximinal subspace of* $X \times Y$.

We recall that for an arbitrary nonempty convex set A in X the linear manifold spanned by A which is denoted by $\ell(A)$ is defined as follows

$$\ell(A) := \{\alpha x + (1-\alpha)y \ : \ x, y \in A : \alpha \text{ is a scalar}\}.$$

For every fixed $y \in A$ the set $\ell(A - y)$ is a linear subspace of X satisfying

$$\ell(A - y) = \ell(A) - y := \{x - y \ : \ x \in \ell(A)\}.$$

It is clear that for an arbitrary nonempty convex set A in X

$$\ell(\pi(A)) = \pi(\ell(A)),$$

where $\pi : X \times Y \longrightarrow \frac{X}{M_1} \times \frac{Y}{M_2}$ which is defined by $\pi(x,y) = (x + M_1, y + M_2)$, is the canonical map. The dimension of A is defined by

$$\dim A := \dim \ell(A).$$

Then for every $y \in A$ we have

$$\dim A := \dim \ell(A) = \dim[\ell(A) - y] = \dim \ell(A - y) = \dim(A - y).$$

Definition 1.5. *Let $X \times Y$ be a generalized 2-normed linear space, $W_1 \times W_2$ a subspace of $X \times Y$ and $S_1 \times S_2$ a 2-bounded set in $X \times Y$. Then, $W_1 \times W_2$ is called 2-simultaneous pseudo-Chebyshev subspace if $\boldsymbol{S}_{W_1 \times W_2}(S_1 \times S_2)$ is finite dimensional subset of $W_1 \times W_2$ for all 2-bounded subset $S_1 \times S_2$ in $X \times Y$.*

Theorem 1.1 ([1])**.** *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed linear space, and M_1 and M_2 be subspaces of X and Y respectively. Define*

$$\|.,.\| : \frac{X}{M_1} \times \frac{Y}{M_2} \longrightarrow [0, +\infty)$$
$$\|x + M_1, y + M_2\| = \inf_{(m_1,m_2) \in M_1 \times M_2} \|x + m_1, y + m_2\|$$

for every $x \in X$ and $y \in Y$. Then $\|.,.\|$ is a generalized 2-norm on $\frac{X}{M_1} \times \frac{Y}{M_2}$.

In [1], the authors have been shown that $\|.,.\|$ is a generalized 2-norm that it is not necessary a 2-norm.

2. Main Results

Lemma 2.1. *Let $X \times Y$ be a generalized 2-normed linear space and $M_1 \times M_2$ a 2-proximinal subset of $X \times Y$.Then for each nonempty 2-bounded subset $S_1 \times S_2$ in $X \times Y$ we have*

$$d(S_1 \times S_2, M_1 \times M_2) = \sup_{(s_1,s_2) \in S_1 \times S_2} \inf_{(m_1,m_2) \in M_1 \times M_2} \|s_1 - m_1, s_2 - m_2\|.$$

Proof. Since $M_1 \times M_2$ is 2-proximinal, it follows that for each $(s_1, s_2) \in S_1 \times S_2$, there exists $(m_{01}, m_{02}) \in M_1 \times M_2$ such that

$$\|s_1 - m_{01}, s_2 - m_{02}\| = \inf_{(m_1,m_2) \in M_1 \times M_2} \|s_1 - m_1, s_2 - m_2\|.$$

Hence we have

$$\begin{aligned} d(S_1 \times S_2, M_1 \times M_2) &= \inf_{(m_1,m_2) \in M_1 \times M_2} \sup_{(s_1,s_2) \in S_1 \times S_2} \|s_1 - m_1, s_2 - m_2\| \\ &\leq \sup_{(s_1,s_2) \in S_1 \times S_2} \|s_1 - m_{01}, s_2 - m_{02}\| \\ &= \sup_{(s_1,s_2) \in S_1 \times S_2} \inf_{(m_1,m_2) \in M_1 \times M_2} \|s_1 - m_1, s_2 - m_2\| \\ &\leq \inf_{(m_1,m_2) \in M_1 \times M_2} \sup_{(s_1,s_2) \in S_1 \times S_2} \|s_1 - m_1, s_2 - m_2\| \\ &= d(S_1 \times S_2, M_1 \times M_2). \end{aligned}$$

Which completes the proof. ☐

Lemma 2.2. *Let $W_1 \times W_2$ be a 2-simultaneous proximinal subspace of a generalized 2-normed space $X \times Y$, $M_1 \times M_2$ a 2-proximinal subspace of $X \times Y$ and $M_1 \times M_2 \subseteq W_1 \times W_2$. Then for each nonempty 2-bounded set $S_1 \times S_2$ with $M_1 \times M_2 \subseteq S_1 \times S_2 \subseteq X \times Y$ we have*

$$d(S_1 \times S_2, W_1 \times W_2) = d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big).$$

Proof. It is easy to see that $d(S_1 \times S_2, W_1 \times W_2) \geq d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big)$. Fix $(w_1, w_2) \in W_1 \times W_2$. Then, $\sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1 + M_1, s_2 - w_2 + M_2\| \geq \|s_1 - w_1 + M_1, s_2 - w_2 + M_2\|$ for all $(s_1, s_2) \in S_1 \times S_2$. Since $M_1 \times M_2$ is 2-proximinal, there exists $(m_{01}, m_{02}) \in M_1 \times M_2$ such that

$$\begin{aligned}\|s_1 - w_1 + M_1, s_2 - w_2 + M_2\| &= \|s_1 - w_1 + m_{01}, s_2 - w_2 + m_{02}\| \\ &\geq \inf_{(w_1^{'},w_2^{'})\in W_1\times W_2} \|s_1 - w_1^{'}, s_2 - w_2^{'}\|.\end{aligned}$$

Thus, $\sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1 + M_1, s_2 - w_2 + M_2\| \geq \inf_{(w_1^{'},w_2^{'})\in W_1\times W_2} \|s_1 - w_1^{'}, s_2 - w_2^{'}\|$ for all $(s_1, s_2) \in S_1 \times S_2$. Hence by lemma 2.1,

$$\begin{aligned}\sup_{(s_1,s_2)\in S_1\times S_2} &\|s_1 - w_1 + M_1, s_2 - w_2 + M_2\| \\ &\geq \sup_{(s_1,s_2)\in S_1\times S_2} \inf_{(w_1^{'},w_2^{'})\in W_1\times W_2} \|s_1 - w_1^{'}, s_2 - w_2^{'}\| \\ &= \inf_{(w_1^{'},w_2^{'})\in W_1\times W_2} \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1^{'}, s_2 - w_2^{'}\| \\ &= d(S_1 \times S_2, W_1 \times W_2),\end{aligned}$$

for all $(w_1, w_2) \in W_1 \times W_2$. Therefore,

$$\begin{aligned}d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big) &= \inf_{(w_1^{'},w_2^{'})\in W_1\times W_2} \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1 + M_1, s_2 - w_2 + M_2\| \\ &\geq d(S_1 \times S_2, W_1 \times W_2)\end{aligned}$$

□

Lemma 2.3. *Let $W_1 \times W_2$ be a 2-simultaneous proximinal subspace of a generalized 2-normed space $X \times Y$, $M_1 \times M_2$ a 2-proximinal subspace of $X \times Y$, $S_1 \times S_2$ a 2-bounded set in $X \times Y$, $M_1 \times M_2 \subseteq W_1 \times W_2$. Then,*

$$\pi\Big(\boldsymbol{S}_{W_1\times W_2}(S_1 \times S_2)\Big) \subseteq \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big).$$

Proof. If $(w_{01}, w_{02}) \in \mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$, we have

$$\begin{aligned}\|s_1 - w_{01} + M_1, s_2 - w_{02} + M_2\| &= \inf_{(m_1,m_2)\in M_1\times M_2} \|s_1 - w_{01} + m_1, s_2 - w_{02} + m_2\| \\ &\leq \|s_1 - w_{01}, s_2 - w_{02}\|.\end{aligned}$$

So by lemma 2.2 we obtain

$$\begin{aligned}\sup_{(s_1 s_2)\in S_1\times S_2} \|s_1 - w_{01} + M_1, s_2 - w_{02} + M_2\| &\leq \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_{01}, s_2 - w_{02}\| \\ &= d(S_1 \times S_2, W_1 \times W_2) = d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big).\end{aligned}$$

Therefore, $(w_{01} + M_1, w_{02} + M_2) \in \mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)$. □

Lemma 2.4. *Let $W_1 \times W_2$ be a 2-simultaneous proximinal subspace of a generalized 2-normed space $X \times Y$, $M_1 \times M_2$ a 2-proximinal subspace of $X \times Y$, $S_1 \times S_2$ a 2-bounded set in $X \times Y$, $M_1 \times M_2 \subseteq W_1 \times W_2$. If $(w_{01}+M_1, w_{02}+M_2) \in \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times$*

$\left.\frac{S_2}{M_2}\right)$ *and* $(m_{01}, m_{02}) \in \boldsymbol{S}_{M_1 \times M_2}(S_1 - w_{01}, S_2 - w_{02})$*, then* $(w_{01} + m_{01}, w_{02} + m_{02}) \in \boldsymbol{S}_{W_1 \times W_2}(S_1 \times S_2)$.

Proof. By lemma 2.1 and 2.2, we have

$$\begin{aligned}
&\sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_{01} - m_{01}, s_2 - w_{02} - m_{02}\| \\
&\qquad = \inf_{(m_1,m_2)\in M_1\times M_2} \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_{01} - m_1, s_2 - w_{02} - m_2\| \\
&\qquad = \sup_{(s_1,s_2)\in S_1\times S_2} \inf_{(m_1,m_2)\in M_1\times M_2} \|s_1 - w_{01} - m_1, s_2 - w_{02} - m_2\| \\
&\qquad = \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_{01} + M_1, s_2 - w_{02} + M_2\| \\
&\qquad \le d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big) \ = d(S_1 \times S_2, W_1 \times W_2)
\end{aligned}$$

So, $(w_{01} + m_{01}, w_{02} + m_{02}) \in \mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$. □

Corollary 2.1. *Let* $W_1 \times W_2$ *be a 2-simultaneous proximinal subspace of a generalized 2-normed space* $X \times Y$, $M_1 \times M_2$ *a 2-proximinal subspace of* $X \times Y$, $S_1 \times S_2$ *a 2-bounded set in* $X \times Y$ *and* $M_1 \times M_2 \subseteq W_1 \times W_2$. *Then,*

$$\pi\Big(\boldsymbol{S}_{W_1\times W_2}(S_1 \times S_2)\Big) = \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big).$$

Proof. By lemma 2.3, we have

$$\pi\Big(\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)\Big) \subseteq \mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big).$$

Now, suppose that $(w_{01} + M_1, w_{02} + M_2) \in \mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)$. Since $M_1 \times M_2$ is 2-simultaneous proximinal, there exists $(m_{01}, m_{02}) \in M_1 \times M_2$ such that $(m_{01}, m_{02}) \in \mathbf{S}_{M_1\times M_2}(S_1 - w_{01}, S_2 - w_{02})$. Now by lemma 2.4, $(w_{01} + m_{01}, w_{02} + m_{02}) \in \mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$. So $(w_{01} + M_1, w_{02} + M_2) \in \pi\Big(\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)\Big)$. □

Theorem 2.1. *Let* $M_1 \times M_2$ *and* $W_1 \times W_2$ *be subspaces of a generalized 2-normed linear space* $X \times Y$ *such that* $W_1 \times W_2$ *is 2-simultaneous proximinal and* $M_1 \times M_2$ *is finite dimensional and 2-proximinal subspace of* $W_1 \times W_2$. *Then the following are equivalent.*
(i) $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ *is 2-simultaneous pseudo-Chebyshev subspace of* $\frac{X}{M_1} \times \frac{Y}{M_2}$.
(ii) $(W_1 + M_1) \times (W_2 + M_2)$ *is 2-simultaneous pseudo-Chebyshev subspace of* $X \times Y$.

Proof. $(i) \Rightarrow (ii)$ let $S_1 \times S_2$ be an arbitrary 2-bounded subset in $X \times Y$ and (k_{01}, k_{02}) be an element of $\mathbf{S}_{(W_1+M_1,W_2+M_2)}(S_1 \times S_2)$. Then by using corollary 2.5 we have

$$\begin{aligned}
&\pi\Big(\ell(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2) - (k_{01}, k_{02}))\Big) \\
&\qquad\qquad = \ell\Big(\pi(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2) - (k_{01}, k_{02}))\Big) \\
&\qquad\qquad = \ell\Big(\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big) - (k_{01} + M_1, k_{02} + M_2)\Big).
\end{aligned}$$

Since $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is 2-simultaneous pseudo-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$, so

$$\dim \left[\ell\Big(\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big) - (k_{01} + M_1, k_{02} + M_2)\Big)\right] < \infty.$$

Hence,

$$\dim \left[\pi\Big(\ell(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2) - (k_{01}, k_{02}))\Big)\right] < \infty.$$

Since $M_1 \times M_2$ is finite dimensional, we have

$$\dim \left[\Big(\ell(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2) - (k_{01}, k_{02}))\Big)\right] < \infty.$$

Therefore, $(W_1 + M_1) \times (W_2 + M_2)$ is 2-simultaneous pseudo-Chebyshev subspace of $X \times Y$.
$(ii) \Rightarrow (i)$ Let $S_1 \times S_2$ be an arbitrary 2-bounded subset of $X \times Y$. Since $(W_1+M_1)\times(W_2+M_2)$ is 2-simultaneous pseudo-Chebyshev subspace of $X\times Y$, $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$ is finite dimensional. But since $\frac{W_1+M_1}{M_1} \times \frac{W_2+M_2}{M_2} = \frac{W_1}{M_1} \times \frac{W_2}{M_2}$, so we have

$$\begin{aligned}
\dim \left[\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)\right] &= \dim \left[\ell\Big(\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)\Big)\right] \\
&= \dim \left[\ell\Big(\mathbf{S}_{\frac{W_1+M_1}{M_1}\times\frac{W_2+M_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)\Big)\right] \\
&= \dim \left[\ell\Big(\pi\Big(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)\Big)\Big)\right] \\
&= \dim \left[\pi\Big(\ell\Big(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)\Big)\Big)\right] < \infty.
\end{aligned}$$

Thus, $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is 2-simultaneous pseudo-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$. □

Corollary 2.2. *Let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of generalized 2-normed linear space $X\times Y$ such that $M_1\times M_2$ is finite dimensional and 2-proximinal, $W_1\times W_2$ is 2-simultaneous proximinal and $M_1 \times M_2 \subseteq W_1 \times W_2$. Then the following are equivalent.*
(i) $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is 2-simultaneous pseudo-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$.
(ii) $W_1 \times W_2$ is 2-simultaneous pseudo-Chebyshev subspace of $X \times Y$.

3. **Conclusions**

In this paper, we investigated the concept of best simultaneous approximation in quotient generalized 2-normed linear spaces. We proved that under the 2-proximinality of the subspace $M_1\times M_2$ pseudo-Chebyshev subspaces are transmitted to and from quotient spaces. A characterization of simultaneous pseudo-Chebyshev subspaces is obtained. Also we introduced equivalent assertions between the 2-simultaneous pseudo-Chebyshevity of subspaces $W_1\times W_2$ and $(W_1+M_1)\times(W_2+M_2)$ and the quotient space $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$.

Acknowledgements. The authors express their gratitude to the referees for their helpful and effective suggestions which improved final version of this paper.

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