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FUNDAMENTAL Γ-SEMIGROUPS THROUGH H_v-Γ-SEMIGROUPS

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In this paper, we consider the notions of H_v-Γ-semigroup and regular relation. Firstly we prove that any semigroup endowed with an equivalence relation can induce an H_v-Γ-semigroup. Secondly, by regular relations, isomorphism theorems on H_v-Γ-semigroups are proved and discussed. Finally, as a strongly regular relation, we point out the fundamental relation on H_v-Γ-semigroups and create a functor between the category of H_v-Γ-semigroups and the category of fundamental Γ-semigroups.

Keywords: H_v-Γ-semigroup, regular relation, fundamental relation, isomorphism theorem, fundamental Γ-semigroup

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1. Introduction and preliminaries

In 1986, Sen and Saha [1] defined the notion of a Γ-semigroup as a generalization of a semigroup. Many classical properties of semigroups have been extended to Γ-semigroups that have been investigated by a lot of mathematicians, for instance, Chattopadhyay [2, 3], Hila [4, 5], Saha [6], Sen et. al. [7]- [10], Seth [11] and many others.

Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then S is called a Γ-*semigroup* [1, 6] if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ written as $(a, \gamma, b) \mapsto a\gamma b$ satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. An unique element $e \in S$ is called an *identity element* if $e\gamma x = x = x\gamma e$, for all $x \in S$ and $\gamma \in \Gamma$. Let S be an arbitrary semigroup and Γ any non-empty set. Define a map $S \times \Gamma \times S \longrightarrow S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus, any semigroup can be considered as a Γ-semigroup.

Example 1.1. (1) Let $S = [0, 1]$ and $\Gamma = \{\frac{1}{n} \mid n \text{ is a positive integer}\}$. Then S is a Γ-semigroup under the usual multiplication.

(2) Let $S = \{-i, 0, i\}$ a subset of the complex numbers $\mathbb{C}$ and $\Gamma = S$. We notice that S is not a semigroup under complex numbers multiplication, while it is a Γ-semigroup under the same operation.

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(3) Let S be the set of all $m \times n$ matrices, with $m \neq n$ and Γ be the set of all $n \times m$ matrices over the same field. Then for $A, B \in S$, the product AB can not be defined i.e., S is not a semigroup under the usual matrix multiplication. But, for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence S is a Γ-semigroup.

These examples illustrate the motivation of the study of Γ-semigroups like an independent class of algebraic structures.

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty [12]. In a classical algebraic structure, the composition of two elements is an element (so the operation is a single valued function), while in an algebraic hyperstructure, the composition of two elements is a set, that is the hyperoperation, called also hyperproduct, is a multivalued function. The principal notions of algebraic hyperstructure theory and many examples can be found in [13]-[16]. Many authors studied different aspects of semihypergroups or semihyperrings, for instance, see [17, 18, 19], their connections with Γ-semihypergroups [20, 21]. Hedayati, Davvaz and Shum studied on some aspects of Γ-semirings and Γ-hyperrings in [22, 23]. On the other hand, H_v-structures have been first introduced by Vougiouklis in Fourth AHA Congress (1990) [24] as a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. The reader will find in [25, 16] some basic definitions and theorems regarding the H_v-structures. Since then the study of H_v-structure theory has been pursued in many directions by Vougiouklis, Davvaz, Spartalis and others, for example see [26]-[29].
On a hyperstructure one may define two types of fundamental relations. The first one, is connected with the regular relations. For example, if S is a hyperstructure, i.e. a multivalued structure (in particular a semihypergroup, a hypergroup, a hyperring, a hypermodule or a multialgebra/hyperalgebra), then the quotient by the fundamental relation ξ^* is a single valued structure of the same type (a semigroup, a group, a ring, a module, or an algebra respectively) [30, 22, 31]. Besides, Jantosciak [32] defined other three equivalences on a hypergroup, called fundamental relations too, in order to obtain the reduced hypergroups. The study of the hypergroups can be therefore divided into two parts: the study of the reduced hypergroups and that of the hypergroups with the same reduced form (see [33, 34]).
Let us recall this basic definition. Let S be a non-empty set. Then, the map $\circ : S \times S \longrightarrow \wp^*(S)$ is called a *hyperoperation*, where $\wp^*(S)$ is the family of non-empty subsets of S. Also, $(S, \circ)$ is called an *H_v-semigroup* [16, 27] if, for every $x, y, z \in S$, we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$. In this definition, if A and B are two non-empty subsets of S and $x \in S$, then we define

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

The rest of the paper is organized as follows. In Section 2, after defining the H_v-Γ-semigroups and the regular relations on them, we discuss the isomorphism theorems. Section 3 is dedicated to the study of the fundamental relation in a H_v-Γ-semigroup and the fundamental Γ-semigroup. Moreover we established a covariant functor between the category of H_v-Γ-semigroups and the category of fundamental Γ-semigroups. We conclude with final remarks and few open problems.

2. Isomorphism theorems on H_v-Γ-semigroups based on regular relations

The regular relations are a particular case of the ideal congruence relations introduced by Picket in the context of multialgebras [35]. Later on they have been studied for the hypergroups, hyperrings, hypermodules and the connected hyperstructures in order to obtain the corresponding factor hyperstructures.
Our intent here is to discuss on the three isomorphism theorems for the H_v-Γ-semigroups by means of regular relations. We expect that these theorems can be stated and proved as for the other structures/hyperstructures, but we will see that this doesn't happen for the second theorem. We recall that all these theorems have been proved for Γ-semigroups [36], Γ- semihypergroups [21], Γ-hyperrings [22], Γ-hypermodules [37]. On the other hand, isomorphism theorems for universal hyperalgebras (multialgebras) were proved by Ebrahimi et al. in [38].

Definition 2.1. Let S and Γ be non-empty sets. Then S is called an H_v-Γ-*semigroup* if there exists a mapping $\cdot : S \times \Gamma \times S \longrightarrow \wp^*(S)$ such that $(x\gamma y)\beta z \approx x\gamma(y\beta z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$, where by $A \approx B$ we mean $A \cap B \neq \varnothing$. An unique element $e \in S$ is called an *identity element* if $e\gamma x = x = x\gamma e$, for all $x \in S$ and $\gamma \in \Gamma$.

In the next example, we will see that each semigroup endowed with an equivalence relation can induce an H_v-Γ-semigroup.

Example 2.1. (1) Let $(S, \cdot)$ be a semigroup, σ an equivalence relation in S and $\sigma(x)$ the equivalence class of $x \in S$. If $\emptyset \neq \Gamma \subseteq S$ and R is an equivalence relation in Γ, then S/σ is an H_v-Γ/R-semigroup, where $S/\sigma = \{\sigma(x) \mid x \in S\}$ and $\Gamma/R = \{R(\gamma) \mid \gamma \in \Gamma\}$. Define $\odot : S/\sigma \times \Gamma/R \times S/\sigma \longrightarrow \wp^*(S/\sigma)$ by $\sigma(x) \odot R(\gamma) \odot \sigma(y) = \{\sigma(z) \mid z \in \sigma(x)R(\gamma)\sigma(y)\}$. It is easy to verify that $\odot$ is well-defined. Also, $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$, which implies that $(\sigma(x) \odot R(\gamma) \odot \sigma(y)) \odot R(\beta) \odot \sigma(z) \cap \sigma(x) \odot R(\gamma) \odot (\sigma(y) \odot R(\beta) \odot \sigma(z)) \neq \emptyset$. Therefore, S/σ is an H_v-Γ/R-semigroup.

(2) Let $\emptyset \neq \Gamma \subseteq \mathbb{Z}_m$. Define $\oplus : \mathbb{Z}_m \times \Gamma \times \mathbb{Z}_m \longrightarrow \wp^*(\mathbb{Z}_m)$ by $x \oplus \gamma \oplus y = \{x + y, \gamma\}$ for all $x, y \in \mathbb{Z}_m$ and $\gamma \in \Gamma$. Then, for all $x, y, z \in \mathbb{Z}_m$ and $\gamma, \beta \in \Gamma$ we have $(x \oplus \gamma \oplus y) \oplus \beta \oplus z = \{x+y+z, \beta, \gamma+z\}$ and $x \oplus \gamma \oplus (y \oplus \beta \oplus z) = \{x+y+z, \gamma, x+\beta\}$. Thus, $(x \oplus \gamma \oplus y) \oplus \beta \oplus z \cap x \oplus \gamma \oplus (y \oplus \beta \oplus z) \neq \emptyset$. Therefore, $\mathbb{Z}_m$ is an H_v-Γ-semigroup.

(3) Let $\emptyset \neq \Gamma \subseteq \mathbb{Z}^n$. Define $\oplus : \mathbb{Z}^n \times \Gamma \times \mathbb{Z}^n \longrightarrow \wp^*(\mathbb{Z}^n)$ by

$$(m_1, \ldots, m_n) \oplus (\gamma_1, \ldots, \gamma_n) \oplus (0, \ldots, 0) = \{(m_1 + \gamma_1, \ldots, m_n + \gamma_n), (0, \ldots, 0)\},$$

$$(m_1, \ldots, m_n) \oplus (\gamma_1, \ldots, \gamma_n) \oplus (m'_1, \ldots, m'_n) = (m_1 + \gamma_1 + m'_1, \ldots, m_n + \gamma_n + m'_n).$$

It is easy to verify that $\mathbb{Z}^n$ is an H_v-Γ-semigroup.

In the following theorem, by a Γ-semigroup S and every non-empty subset of S, we construct an H_v-Γ-semigroup.

Theorem 2.1. *Let S be a Γ-semigroup and I a non-empty subset of S. Then, S is an H_v-Γ-semigroup with the mapping $\odot_I : S \times \Gamma \times S \longrightarrow \wp^*(S)$ defined by $x \odot_I \gamma \odot_I y = x\Gamma I\gamma y$ for all $x, y \in S$ and $\gamma \in \Gamma$.*

Proof. It is easy to verify that $\odot_I$ is well-defined. Then, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ we have

$$\begin{aligned}
&(x \odot_I \alpha \odot_I y) \odot_I \beta \odot_I z\\
&= \{t \in S \mid t \in x\Gamma I\alpha y\} \odot_I \beta \odot_I z\\
&= \{w \in S \mid w \in t\Gamma I\beta z,\ t \in x\Gamma I\alpha y\}\\
&= \{w \in S \mid w \in (x\Gamma I\alpha y)\Gamma I\beta z\} \approx \{w' \in S \mid w' \in x\Gamma I\alpha(y\Gamma I\beta z)\}\\
&= x \odot_I \alpha \odot_I \{t' \in S' \mid t' \in y\Gamma I\beta z\} = x \odot_I \alpha \odot_I (y \odot_I \beta \odot_I z).
\end{aligned}$$

Therefore, S is an H_v-Γ-semigroup. □

In Theorem 2.1, if we define $\odot_I$ by $x \odot_I \gamma \odot_I y = x\gamma I\Gamma y$, for all $x, y \in S$ and $\gamma \in \Gamma$, then it is easy to prove that S is an H_v-Γ-semigroup, too.

Let S be an H_v-Γ-semigroup and θ an equivalence relation in S. Then, we extend the relation θ to the non-empty subsets A and B of S as follows: $A\overline{\theta}B$ if and only if $\forall\ a \in A\ \exists\ b \in B$, such that $a\theta b$ and $\forall\ b \in B\ \exists\ a \in A$, such that $b\theta a$, where by $a\theta b$, we mean $(a, b) \in \theta$. An equivalence relation θ on S is said to be *regular* if, for all $x, y, z \in S$ and $\alpha \in \Gamma$, $x\theta y$ implies that $(x\alpha z)\overline{\theta}(y\alpha z)$ and $(z\alpha x)\overline{\theta}(z\alpha y)$. By S/θ we mean the set of all equivalence classes of the elements of S with respect to the relation θ, that is $S/\theta = \{\theta(x) \mid x \in S\}$. In what follows, S is an H_v-Γ-semigroup unless otherwise specified. In the next lemma, we have a well-known property of regular relations.

Lemma 2.1. *Let θ be a regular relation on S. Then, we have $\{\theta(z) \mid z\in\theta(x)\alpha\theta(y)\}=\{\theta(z) \mid z \in x\alpha y\}$, for all $x, y \in S$ and $\alpha \in \Gamma$.*

Proof. See [39]. □

Now, we will see that each H_v-Γ-semigroup with a regular relation can induce a new H_v-Γ-semigroup.

Theorem 2.2. *Let θ be a regular relation on S. Then S/θ is an H_v-Γ-semigroup with the mapping $\odot : S/\theta \times \Gamma \times S/\theta \longrightarrow \wp^*(S/\theta)$ defined by $\theta(x) \odot \alpha \odot \theta(y) = \{\theta(z) \mid z \in \theta(x)\alpha\theta(y)\}$ for all $\theta(x), \theta(y) \in S/\theta$ and $\alpha \in \Gamma$.*

Proof. It follows from Lemma 2.1 (for more details see [39]). □

Let S_1 and S_2 be two H_v-Γ-semigroups. A mapping $\varphi : S_1 \longrightarrow S_2$ is called a *homomorphism* if $\varphi(x\alpha y) = \varphi(x)\alpha\varphi(y)$, for all $x, y \in S_1$ and $\alpha \in \Gamma$. A homomorphism φ is called an *isomorphism* if φ is 1-1 and onto. Two H_v-Γ-semigroups S_1 and S_2 are *isomorphic* if there exists an isomorphism $\varphi : S_1 \longrightarrow S_2$ between them; it is denoted by $S_1 \cong S_2$. Let $\varphi : S_1 \longrightarrow S_2$ be a homomorphism of H_v-Γ-semigroups. We define a relation $\mathcal{K}$ on S_1 as follows: $\mathcal{K} = \varphi^{-1} \circ \varphi = \{(x, y) \in S_1 \times S_1 \mid \varphi(x) = \varphi(y)\}$.

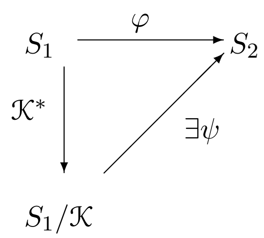
In the next theorems, we consider the regular relation induced by homomorphisms and investigate the corresponding results and properties associated with this regular relation.

Lemma 2.2. *The relation $\mathcal{K}$ is a regular relation on S_1.*

Proof. Straightforward. □

Since $\mathcal{K}$ is a regular relation in S_1, then by Theorem 2.2, $S_1/\mathcal{K}$ is an H_v-Γ-semigroup. Now, we have the following theorem.

Theorem 2.3. *Let S_1 and S_2 be two H_v-Γ-semigroups and $\varphi : S_1 \longrightarrow S_2$ a homomorphism. Then, there is a monomorphism $\psi : S_1/\mathcal{K} \longrightarrow S_2$ such that $Im\varphi = Im\psi$ and the diagram*



commutes, i.e. $\psi \circ \mathcal{K}^* = \varphi$, where the mapping $\mathcal{K}^* : S_1 \longrightarrow S_1/\mathcal{K}$ is defined by $\mathcal{K}^*(x) = \mathcal{K}(x)$, for all $x \in S_1$.

Proof. Define $\psi : S_1/\mathcal{K} \longrightarrow S_2$ by $\psi(\mathcal{K}(x)) = \varphi(x)$ for all $x \in S_1$. We have

$$\mathcal{K}(x) = \mathcal{K}(y) \Longleftrightarrow (x, y) \in \mathcal{K} \Longleftrightarrow \varphi(x) = \varphi(y) \Longleftrightarrow \psi(\mathcal{K}(x)) = \psi(\mathcal{K}(y)).$$

Then, ψ is well-defined and 1-1. Also, ψ is a homomorphism since, for all $x, y \in S_1$ and $\alpha \in \Gamma$, we have

$$\psi(\mathcal{K}(x) \odot \alpha \odot \mathcal{K}(y)) = \{\psi(\mathcal{K}(z)) \mid z \in x\alpha y\} = \{\varphi(z) \mid z \in x\alpha y\}$$
$$= \varphi(x\alpha y) = \varphi(x)\alpha\varphi(y) = \psi(\mathcal{K}(x))\alpha\psi(\mathcal{K}(y)).$$

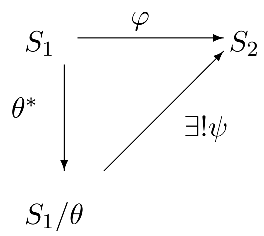
It is easy to prove that $Im\varphi = Im\psi$. Also, the diagram is commutative, because for all $x \in S_1$ we have $(\psi \circ \mathcal{K}^*)(x) = \psi(\mathcal{K}^*(x)) = \psi(\mathcal{K}(x)) = \varphi(x)$. This completes the proof. □

Now, by the help of the regular relation $\mathcal{K}$, we state the first isomorphism theorem.

Theorem 2.4. *(First Isomorphism Theorem) Let S_1 and S_2 be H_v-Γ-semigroups and $\varphi : S_1 \longrightarrow S_2$ a homomorphism. Then $S_1/\mathcal{K} \cong Im\varphi$.*

Proof. It follows immediately from Theorem 2.3. □

Theorem 2.5. *Let S_1 and S_2 be H_v-Γ-semigroups and $\varphi : S_1 \longrightarrow S_2$ a homomorphism. If θ is a regular relation on S_1 such that $\theta \subseteq \mathcal{K}$, then there is an unique monomorphism $\psi : S_1/\theta \longrightarrow S_2$ such that $Im\varphi = Im\psi$ and the diagram*



commutes, i.e. $\psi \circ \theta^* = \varphi$, *where the mapping* $\theta^* : S_1 \longrightarrow S_1/\theta$ *is defined by* $\theta^*(x) = \theta(x)$, *for all* $x \in S_1$.

Proof. Define $\psi : S_1/\theta \longrightarrow S_2$ by $\psi(\theta(x)) = \varphi(x)$, for all $x \in S_1$. Suppose that $x, y \in S_1$ such that $\theta(x) = \theta(y)$. That is $(x, y) \in \theta$, which implies that $(x, y) \in \mathcal{K}$. Hence, $\varphi(x) = \varphi(y)$. Thus, ψ is well-defined. Now, suppose that $x, y \in S_1$ and $\alpha \in \Gamma$. Then,

$$\psi(\theta(x) \odot \alpha \odot \theta(y)) = \psi(\{\theta(z) \mid z \in x\alpha y\}) = \{\psi(\theta(z)) \mid z \in x\alpha y\}$$
$$= \{\varphi(z) \mid z \in x\alpha y\} = \varphi(x\alpha y) = \varphi(x)\alpha\varphi(y) = \psi(\theta(x))\alpha\psi(\theta(y)).$$

Therefore, ψ is a homomorphism. It is easy to see that $Im\varphi = Im\psi$. Suppose that $x \in S_1$. Then, $(\psi \circ \theta^*)(x) = \psi(\theta^*(x)) = \psi(\theta(x)) = \varphi(x)$. It implies that $\psi \circ \theta^* = \varphi$. Finally, let $\psi^* : S_1/\theta \longrightarrow S_2$ be any homomorphism satisfying $\psi^* \circ \theta^* = \varphi$. Then, for all $x \in S_1$, we have $\psi^*(\theta(x)) = \psi^*(\theta^*(x)) = \psi^* \circ \theta^*(x) = \varphi(x) = \psi(\theta(x))$. Therefore, $\psi^* = \psi$ and the proof is completed. □

Let θ and μ be two relations in the H_v-Γ-semigroup S with $\theta \subseteq \mu$. Define the relation μ/θ on S/θ by $\mu/\theta = \{(\theta(x), \theta(y)) \in S/\theta \times S/\theta \mid (x, y) \in \mu\}$. Suppose that $\theta(x) = \theta(y)$. Then, $(x, y) \in \theta \subseteq \mu$ which implies that $(\theta(x), \theta(y)) \in \mu/\theta$ and so $\mu/\theta(\theta(x)) = \mu/\theta(\theta(y))$. Therefore, μ/θ is well-defined.

Lemma 2.3. *If* θ *and* μ *are regular relations on* S, *then* μ/θ *is a regular relation on* S/θ.

Proof. Suppose that $x \in S$. Then $(x, x) \in \mu$ and thus $(\theta(x), \theta(x)) \in \mu/\theta$. Hence μ/θ is reflexive. Also, let $x, y \in S$ such that $(\theta(x), \theta(y)) \in \mu/\theta$. Then, $(x, y) \in \mu$. Since μ is symmetric, $(y, x) \in \mu$, which implies that $(\theta(y), \theta(x)) \in \mu/\theta$. Hence, μ/θ is symmetric. Also, let $x, y, z \in S$ such that $(\theta(x), \theta(y)) \in \mu/\theta$ and $(\theta(y), \theta(z)) \in \mu/\theta$. Then $(x, y) \in \mu$ and $(y, z) \in \mu$. Since μ is transitive, $(x, z) \in \mu$, which implies that $(\theta(x), \theta(z)) \in \mu/\theta$. Hence, μ/θ is transitive. Therefore, μ/θ is an equivalence relation on S/θ. Now, we prove that μ/θ is regular. Suppose that $x, y, z \in S$ and $\alpha \in \Gamma$. We have

$$\begin{aligned}
\theta(x)(\mu/\theta)\theta(y) &\Longrightarrow (x, y) \in \mu \Longrightarrow x\mu y \Longrightarrow (x\alpha z)\bar{\mu}(y\alpha z) \\
&\Longrightarrow \{\theta(u) \mid u \in x\alpha z\}\overline{\mu/\theta}\{\theta(v) \mid v \in y\alpha z\} \\
&\Longrightarrow (\theta(x) \odot \alpha \odot \theta(z))\overline{\mu/\theta}(\theta(y) \odot \alpha \odot \theta(z)).
\end{aligned}$$

Similarly, we can show that $(\theta(z) \odot \alpha \odot \theta(x))\overline{\mu/\theta}(\theta(z) \odot \alpha \odot \theta(y))$. Therefore, μ/θ is a regular relation on S/θ. □

If μ and θ are regular relations in S with $\theta \subseteq \mu$, then we know that $(S/\theta)/(\mu/\theta)$ is an H_v-Γ-semigroup. In the next theorem, by the help of regular relations, we prove the third isomorphism theorem.

Theorem 2.6. *(Third Isomorphism Theorem) Let* θ *and* μ *be regular relations on* S *with* $\theta \subseteq \mu$. *Then,* $(S/\theta)/(\mu/\theta) \cong S/\mu$.

Proof. Define $\varphi : (S/\theta)/(\mu/\theta) \longrightarrow S/\mu$ by $\varphi(\mu/\theta(\theta(x))) = \mu(x)$, for all $x \in S$. Clearly, φ is onto. Also, for all $x, y \in S$, we have

$$\mu/\theta(\theta(x))=\mu/\theta(\theta(y)) \Longleftrightarrow (\theta(x),\theta(y))\in\mu/\theta \Longleftrightarrow (x,y)\in\mu \Longleftrightarrow \mu(x)=\mu(y).$$

Therefore, φ is well-defined and 1-1. Clearly, φ is onto. Now, we prove that φ is a homomorphism. Suppose that $x, y \in S$ and $\alpha \in \Gamma$, then

$$\begin{aligned}\varphi(\mu/\theta(\theta(x)) \odot \alpha \odot \mu/\theta(\theta(y)))&=\varphi(\{\mu/\theta(\theta(z)) \mid \theta(z) \in \theta(x) \odot \alpha \odot \theta(y)\})\\ =\varphi(\{\mu/\theta(\theta(z)) \mid z\in x\alpha y\})&=\Big\{\varphi(\mu/\theta(\theta(z))) \mid z\in x\alpha y\Big\}=\{\mu(z) \mid z \in x\alpha y\}\\ &= \mu(x) \odot \alpha \odot \mu(y) = \varphi(\mu/\theta(\theta(x))) \odot \alpha \odot \varphi(\mu/\theta(\theta(y))).\end{aligned}$$

Hence, φ is a homomorphism. Therefore, $(S/\theta)/(\mu/\theta) \cong S/\mu$. □

Remark 2.1. It is quite easy to notice that the second isomorphism theorem can not be proved by the help of regular relation: if we consider $\theta\mu/\mu$ or $\mu/\theta \cap \mu$, for θ and μ regular relation, we don't obtain H_v-Γ-semigroups. So we are forced to work with other entities, like the hyperideals [39]. But the problem is not so easy as it seems to be. If I and J are hyperideals on a H_v-Γ-semigroup, then we have to prove the second isomorphism theorem by the form $I\Gamma J/I \cong J/I \cap J$. In this case we meet some problems. The quotient $I\Gamma J/I$ is not well-defined because we don't know if $I \subset I\Gamma J$. Besides, how can we construct a well-defined map $\varphi : I\Gamma J/I \longrightarrow J/I \cap J$? Till now we have no answer to these questions, it reamins an open problem to investigate in future.

3. Fundamental relation in H_v-Γ-semigroups

In this section, we introduce the notion of fundamental relation in H_v-Γ-semigroups as a strongly regular relation. Also, by the help of the fundamental relation in H_v-Γ-semigroups we construct fundamental Γ-semigroups.
It is worth to mention that an H_v-Γ-semigroup can be viewed as a particular multialgebra $(S, (\circ_\gamma)_{\gamma\in\Gamma})$, where $\circ_\gamma : S \times S \longrightarrow \mathcal{P}^*(S)$ are binary hyperoperations, defined here by $x \circ_\gamma y = x\gamma y$, and which satisfies the weak associativity. The fundamental relation of a multialgebra and the relative quotient multialgebra are studied in [31, 40]. Here we focus on the same arguments in the particular case of H_v-Γ-semigroups, giving a detailed presentation of these properties, using the terminology and the tools of Γ-semigroups.

Definition 3.1. Let S be an H_v-Γ-semigroup. We define the relation ξ^* as the smallest equivalence relation such that the quotient S/ξ^* is a Γ-semigroup. Then, the relation ξ^* is called the *fundamental relation* in H_v-Γ-semigroup S, and S/ξ^* is called the *fundamental* Γ*-semigroup*. Let us denote the set $\mathcal{U}_{(S,\Gamma)} = \mathcal{U}$ as follows:

$$\mathcal{U} = \{a_1\gamma_1 a_2\gamma_2 \cdots a_n\gamma_n a_{n+1} \mid a_i \in S,\ \gamma_i \in \Gamma,\ \forall i \in \{1, \ldots, n\},\ n \in \mathbb{N}\}.$$

In fact, $\mathcal{U}$ is the set of all finite products of elements of S and Γ. It is easy to see that $S \subseteq \mathcal{U}$, that is $\mathcal{U}$ contains all singletons of the elements of S. Now, we define the relation ξ on S as follows:

$$x\xi y \Longleftrightarrow \exists\ u \in \mathcal{U},\ \{x, y\} \subseteq u.$$

Let $\overline{\xi}$ be the transitive closure of ξ. For all $a, b \in S$ and $\gamma \in \Gamma$, we define $\overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b) = \{\overline{\xi}(c) \mid c \in \overline{\xi}(a)\gamma\overline{\xi}(b)\}$. Now, we obtain some interesting results concerning $\overline{\xi}$.

Lemma 3.1. *The set* $\overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b)$ *is a singleton, i.e.,* $|\overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b)| = 1$.

Proof. Let $\overline{\xi}(c) \in \overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b)$. Then, $c \in \overline{\xi}(a)\gamma\overline{\xi}(b)$, so there exists $a' \in \overline{\xi}(a)$ and $b' \in \overline{\xi}(b)$ such that $c \in a'\gamma b'$. It is enough we prove that $\overline{\xi}(z) = \overline{\xi}(z')$, for all $z \in a\gamma b$ and $z' \in a'\gamma b'$. We know that $a'\overline{\xi}a$ if and only if there exist $x_1, \ldots, x_{m+1} \in S$ with $x_1 = a'$ and $x_{m+1} = a$ and there exist $u_1, \ldots, u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, for $i = 1, 2, \ldots, m$. Also, $b'\overline{\xi}b$ if and only if there exist $y_1, \ldots, y_{n+1} \in S$ with $y_1 = b'$ and $y_{n+1} = b$ and there exist $v_1, \ldots, v_n \in \mathcal{U}$ such that $\{y_j, y_{j+1}\} \subseteq v_j$, for $j = 1, 2, \ldots, n$. Therefore, we obtain

$$\begin{cases} \{x_i, x_{i+1}\}\gamma y_1 \subseteq u_i\gamma v_1, & i = 1, 2, \ldots, m-1, \\ x_{m+1}\gamma\{y_j, y_{j+1}\} \subseteq u_m\gamma v_j, & j = 1, 2, \ldots, n \end{cases} \qquad (*)$$

Therefore, $u_i\gamma v_1 = t_i \in \mathcal{U}$, for $i = 1, 2, \ldots, m-1$ and $u_m\gamma v_j = t_{m+j-1} \in \mathcal{U}$, for $j = 1, 2, \ldots, n$. Now, choose the elements $z_1, z_2, \ldots, z_{m+n}$ such that $z_i \in x_i\gamma y_1$, for $i = 1, 2, \ldots, m$ and $z_{m+j} \in x_{m+1}\gamma y_{j+1}$, for $j = 1, 2, \ldots, n$. Using $(*)$, we have $\{z_k, z_{k+1}\} \subseteq t_k$, for $k = 1, \ldots, m+n-1$. Thus, every element $z_1 \in x_1\gamma y_1 = a'\gamma b'$ is equivalent to every element $z_{m+n} \in x_{m+1}\gamma y_{n+1} = a\gamma b$ with respect to the relation $\overline{\xi}$. Therefore, $|\overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b)| = 1$ and we can write $\overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b) = \overline{\xi}(c)$, for all $c \in \overline{\xi}(a)\gamma\overline{\xi}(b)$. This completes the proof. □

Now, by the help of an H_v-Γ-semigroup and the relation $\overline{\xi}$, we construct a Γ-semigroup.

Lemma 3.2. $S/\overline{\xi}$ *is a* Γ*-semigroup.*

Proof. We define $* : S/\overline{\xi} \times \Gamma \times S/\overline{\xi} \longrightarrow S/\overline{\xi}$ by $(\overline{\xi}(a), \gamma, \overline{\xi}(b)) \mapsto \overline{\xi}(a) \circ \gamma \circ \overline{\xi}(b)$. For any $\overline{\xi}(x), \overline{\xi}(y), \overline{\xi}(z) \in S/\overline{\xi}$ and $\alpha, \beta \in \Gamma$ we prove that

$$\overline{\xi}(x) \circ \alpha \circ (\overline{\xi}(y) \circ \beta \circ \overline{\xi}(z)) = (\overline{\xi}(x) \circ \alpha \circ \overline{\xi}(y)) \circ \beta \circ \overline{\xi}(z).$$

Suppose that $\overline{\xi}(a) \in \overline{\xi}(x) \circ \alpha \circ (\overline{\xi}(y) \circ \beta \circ \overline{\xi}(z))$. By Lemma 3.2, we have $\overline{\xi}(a) = \overline{\xi}(a_1)$, where $a_1 \in x\alpha(y\beta z)$. Then,

$$\begin{aligned} \overline{\xi}(a) \quad &= \quad \overline{\xi}(a_1) \in \overline{\xi}(x) \circ \alpha \circ (\overline{\xi}(y) \circ \beta \circ \overline{\xi}(z)) \\ &\Longleftrightarrow \quad a_1 \in x\alpha(y\beta z) \Longleftrightarrow a_1 \in (x\alpha y)\beta z \\ &\Longleftrightarrow \quad \overline{\xi}(a_1) \in (\overline{\xi}(x) \circ \alpha \circ \overline{\xi}(y)) \circ \beta \circ \overline{\xi}(z). \end{aligned}$$

Hence, $\overline{\xi}(a) \in (\overline{\xi}(x) \circ \alpha \circ \overline{\xi}(y)) \circ \beta \circ \overline{\xi}(z)$. This implies that $\overline{\xi}(x) \circ \alpha \circ (\overline{\xi}(y) \circ \beta \circ \overline{\xi}(z)) = (\overline{\xi}(x) \circ \alpha \circ \overline{\xi}(y)) \circ \beta \circ \overline{\xi}(z)$. Therefore, $S/\overline{\xi}$ is a Γ-semigroup. □

In the next lemma, we will see that $\overline{\xi}$ is the smallest equivalence relation with the property that $S/\overline{\xi}$ is a Γ-semigroup.

Lemma 3.3. $\overline{\xi}$ *is the smallest equivalence relation in* S *such that* $S/\overline{\xi}$ *is a* Γ*-semigroup. In other words,* $\overline{\xi} = \xi^*$.

Proof. Let μ be an equivalence relation on S such that S/μ is a Γ-semigroup. We denote the equivalence class of $a \in S$ as usually by $\mu(a)$. Then, we have $\mu(a) \circ \gamma \circ \mu(b) = \mu(d)$, for all $d \in \mu(a)\gamma\mu(b)$. Thus, for every $A \subseteq \mu(a)$ and $B \subseteq \mu(b)$ we can write $\mu(a) \circ \gamma \circ \mu(b) = \mu(a\gamma b) = \mu(A\gamma B)$. By induction we can extend these relations on finite products. Then, for all $u \in \mathcal{U}$ and $x \in u$ we have $\mu(x) = \mu(u)$. Hence, for all $t \in S$, $x \in \xi(t)$ implies that $x \in \mu(t)$. Also, μ is transitivity closed, so if $(x,t) \in \overline{\xi}$, then it implies that $(x,t) \in \mu$. Therefore, $\overline{\xi}$ is the smallest equivalence relation such that $S/\overline{\xi}$ is a Γ-semigroup. □

Theorem 3.1. *The fundamental relation ξ^* is the transitive closure of the relation ξ.*

Proof. It is concluded by Lemmas 3.1, 3.2 and 3.3. □

In the following, we investigate some properties of the equivalence classes corresponding to the fundamental relation in H_v-Γ-semigroups.

Theorem 3.2. *Let S be an H_v-Γ-semigroup and ξ^* the fundamental relation on S. If S has the identity element e and $\xi^*(x) = \xi^*(x')$, then there exist $B, B' \subseteq \xi^*(b)$ and $C, C' \subseteq \xi^*(c)$, for some $b, c \in S$, such that $x\gamma C \subseteq B$ and $x'\gamma' C' \subseteq B'$, for all $\gamma, \gamma' \in \Gamma$.*

Proof. It is enough we take $B = B' = \xi^*(x) = \xi^*(x')$ and $C = C' = \xi^*(e)$. Now, let $z \in x\gamma C \subseteq \xi^*(x)\gamma\xi^*(e)$, then $\xi^*(z) \in \xi^*(x) \circ \gamma \circ \xi^*(e)$. But, we know that $\xi^*(x) \in \xi^*(x) \circ \gamma \circ \xi^*(e)$ (since $\xi^*(e)$ is the identity element of S/ξ^*). On the other hand, $|\xi^*(x) \circ \gamma \circ \xi^*(e)| = 1$, so $\xi^*(z) = \xi^*(x)$. Hence, $z \in \xi^*(x) = B$, which implies that $x\gamma C \subseteq B$. Similarly, we can prove that $x'\gamma' C' \subseteq B'$. □

In the next theorem, we will give a characterization of the equivalence class of the identity element of S.

Theorem 3.3. *Let S be an H_v-Γ-semigroup and ξ^* the fundamental relation in S. If S has the identity element e, then $y \in \xi^*(e)$ if and only if there exists $B \subseteq \xi^*(b)$, for some $b \in B$, such that $y\gamma B \subseteq B$, for all $\gamma \in \Gamma$.*

Proof. Let $y \in \xi^*(e)$, $b \in S$, $\gamma \in \Gamma$ and $B = \xi^*(b)$. Suppose that $z \in y\gamma B$; we have $\xi^*(z) = \xi^*(y) \circ \gamma \circ \xi^*(b) = \xi^*(e) \circ \gamma \circ \xi^*(b) = \xi^*(b)$, so $z \in \xi^*(b) = B$, which implies that $y\gamma B \subseteq B$.
Conversely, suppose that there exists $B \subseteq \xi^*(b)$, for some $b \in S$, such that $y\gamma B \subseteq B$, for all $\gamma \in \Gamma$. Then $\xi^*(y) \circ \gamma \circ \xi^*(b) = \xi^*(b) = \xi^*(e) \circ \gamma \circ \xi^*(b)$. On the other hand, $\xi^*(e)$ is unique. Hence $\xi^*(y) = \xi^*(e)$, which implies that $y \in \xi^*(e)$. □

Proposition 3.1. *Let S be an H_v-Γ-semigroup.*
If $u = a_1\gamma_1 a_2\gamma_2 \ldots a_n\gamma_n a_{n+1} \in \mathcal{U}$, then $\xi^*(u) = \xi^*(a_1) \circ \gamma_1 \circ \xi^*(a_2) \circ \gamma_2 \circ \ldots \circ \xi^*(a_n) \circ \gamma_n \circ \xi^*(a_{n+1}) = \xi^*(z)$, for all $z \in u$.

Proof. We have

$$\begin{aligned} z \in u &= a_1\gamma_1 a_2\gamma_2 \ldots a_n\gamma_n a_{n+1} \\ &\Longrightarrow \xi^*(z) \in \xi^*(a_1) \circ \gamma_1 \circ \xi^*(a_2) \circ \gamma_2 \circ \ldots \circ \xi^*(a_n) \circ \gamma_n \circ \xi^*(a_{n+1}) \\ &\Longrightarrow \xi^*(z) = \xi^*(a_1) \circ \gamma_1 \circ \xi^*(a_2) \circ \gamma_2 \circ \ldots \circ \xi^*(a_n) \circ \gamma_n \circ \xi^*(a_{n+1}) \end{aligned}$$

Besides, clearly

$$\xi^*(u) = \xi^*(z) = \xi^*(a_1) \circ \gamma_1 \circ \xi^*(a_2) \circ \gamma_2 \circ \cdots \circ \xi^*(a_n) \circ \gamma_n \circ \xi^*(a_{n+1}),$$

which completes the proof. □

Lemma 3.4. *Let S be an H_v-Γ-semigroup and ξ^* the fundamental relation in S. Then, $\Pi_S : S \longrightarrow S/\xi^*$ defined by $\Pi_S(x) = \xi^*(x)$ is an epimorphism of H_v-Γ-semigroups.*

Proof. Clearly, Π_S is well-defined. We prove that $\Pi_S(x\gamma y) = \Pi_S(x)\circ\gamma\circ\Pi_S(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. Let $z \in x\gamma y \subseteq \Pi_S(x)\gamma\Pi_S(y)$. Then, $\Pi_S(z) \in \Pi_S(x) \circ \gamma \circ \Pi_S(y)$. By Lemma 3.1, we know that $|\Pi_S(x)\circ\gamma\circ\Pi_S(y)| = 1$, hence $\Pi_S(z) = \Pi_S(x)\circ\gamma\circ\Pi_S(y)$, consequently $\Pi_S(x\gamma y) = \Pi_S(x) \circ \gamma \circ \Pi_S(y)$. Therefore, Π_S is an epimorphism of H_v-Γ-semigroups from S to S/ξ^*. □

In the sequel, we prove that there exists a covariant functor between the category of H_v-Γ-semigroups and the category of fundamental Γ-semigroups. For this we need the following theorem.

Theorem 3.4. *Let S_1 and S_2 be H_v-Γ-semigroups, and ξ_1^* and ξ_2^* the fundamental relations in S_1 and S_2, respectively. If $f : S_1 \longrightarrow S_2$ is a homomorphism, then there exists an unique homomorphism $f^* : S_1/\xi_1^* \longrightarrow S_2/\xi_2^*$ such that the following diagram commutes:*

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ \Pi_{S_1} \downarrow & & \downarrow \Pi_{S_2} \\ S_1/\xi_1^* & \xrightarrow{f^*} & S_2/\xi_2^*, \end{array}$$

Moreover, if f is an isomorphism, then f^* is an isomorphism, too.

Proof. We define $f^* : S_1/\xi_1^* \longrightarrow S_2/\xi_2^*$ by $f^*(\xi_1^*(x)) = \xi_2^*(f(x))$ for all $\xi_1^*(x) \in S_1/\xi_1^*$. Clearly, $f^* \circ \Pi_{S_1} = \Pi_{S_2} \circ f$. Therefore, the diagram is commutative. We prove that f^* is a homomorphism. Let $\xi_1^*(x) = \xi_1^*(y)$, i.e. $x\xi_1^*y$. Then, there exist $a_1, \ldots, a_{m+1} \in S_1$ and $u_1, \ldots, u_m \in \mathcal{U}_{(S_1,\Gamma)}$ by $x = a_1$ and $y = a_{m+1}$ such that $\{a_i, a_{i+1}\} \subseteq u_i$, for all $1 \le i \le m$. Now, since f is a homomorphism we have

$$\begin{aligned} & f(u_i) \in \mathcal{U}_{(S_2,\Gamma)} \Longrightarrow \{f(a_i), f(a_{i+1})\} \subseteq f(u_i) \in \mathcal{U}_{(S_2,\Gamma)} \\ & \Longrightarrow (f(x), f(y)) \in \xi_2^* \Longrightarrow \xi_2^*(f(x)) = \xi_2^*(f(y)) \Longrightarrow f^*(\xi_1^*(x)) = f^*(\xi_1^*(y)). \end{aligned}$$

Therefore, f^* is well-defined. Now, we prove that

$$f^*(\xi_1^*(x) \circ \gamma \circ \xi_1^*(y)) \subseteq f^*(\xi_1^*(x)) \circ \gamma \circ f^*(\xi_1^*(y)).$$

Let $f^*(\xi_1^*(z)) \in f^*(\xi_1^*(x) \circ \gamma \circ \xi_1^*(y))$, for $z \in \xi_1^*(x)\gamma\xi_1^*(y)$. For all $t \in x\gamma y$, we have

$$\begin{aligned} & \xi_1^*(z) = \xi_1^*(t) \Longrightarrow f(t) \in f(x)\gamma f(y) \Longrightarrow \\ & f^*(\xi_1^*(t)) = \xi_2^*(f(t)) \in \xi_2^*(f(x)) \circ \gamma \circ \xi_2^*(f(y)) = f^*(\xi_1^*(x)) \circ \gamma \circ f^*(\xi_1^*(y)) \\ & \Longrightarrow f^*(\xi_1^*(z)) = f^*(\xi_1^*(t)) \in f^*(\xi_1^*(x)) \circ \gamma \circ f^*(\xi_1^*(y)). \end{aligned}$$

On the other hand, we know that $f^*(\xi_1^*(x) \circ \gamma \circ \xi_1^*(y))$ and $f^*(\xi_1^*(x)) \circ \gamma \circ f^*(\xi_1^*(y))$ are singletons. Thus,

$$f^*(\xi_1^*(x) \circ \gamma \circ \xi_1^*(y)) = f^*(\xi_1^*(x)) \circ \gamma \circ f^*(\xi_1^*(y)).$$

Therefore, f^* is a homomorphism.
Moreover, if f is an isomorphism, then we show that f^* is an isomorphism. It is enough we prove that f^* is 1-1 and onto. Let $f^*(\xi_1^*(x)) = f^*(\xi_1^*(y))$. Then, $\xi_2^*(f(x)) = \xi_2^*(f(y))$. Hence, there exist $t_1, \dots, t_{m+1} \in S_2$ and $w_1, \dots, w_m \in \mathcal{U}_{(S_2,\Gamma)}$ with $f(x) = t_1$ and $f(y) = t_{m+1}$ such that $\{t_i, t_{i+1}\} \subseteq w_i$, for all $1 \le i \le m$. Now, since f is onto, there exist $r_i \in S_1$ such that $f(r_i) = t_i$ for all $2 \le i \le m$, and hence there exists $u_i \in \mathcal{U}_{(S_1,\Gamma)}$ such that $f(u_i) = w_i$. Thus $\{f(r_i), f(r_{i+1})\} \subseteq f(u_i)$. Since f is 1-1, then $\{r_i, r_{i+1}\} \subseteq u_i$. It concludes that $x\xi_1^*y$, i.e., $\xi_1^*(x) = \xi_1^*(y)$. Therefore, f^* is 1-1. Also, clearly f^* is onto, which implies that f^* is an isomorphism. □

Theorem 3.5. *Let H_v-Γ-S be the category of H_v-Γ-semigroups and Γ-S be the category of fundamental Γ-semigroups. Then, there exists a covariant functor between H_v-Γ-S and Γ-S.*

Proof. We define $\mathcal{F} : H_v\text{-}\Gamma\text{-}S \longrightarrow \Gamma\text{-}S$ by $\mathcal{F}(S) = S/\xi^*$ and $\mathcal{F}(f) = f^*$, where S is an H_v-Γ-semigroup, ξ^* the fundamental relation in S and f is a homomorphism between H_v-Γ-semigroups. Let $\psi : S_1 \longrightarrow S_2$ and $\varphi : S_2 \longrightarrow S_3$ be homomorphisms of H_v-Γ-semigroups. We have $\varphi \circ \psi : S_1 \longrightarrow S_3$. We prove that $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$. We know that $(\varphi \circ \psi)^* : S_1/\xi_1^* \longrightarrow S_3/\xi_3^*$ and $\varphi^* \circ \psi^* : S_1/\xi_1^* \longrightarrow S_3/\xi_3^*$. By Theorem 3.4, we have

$$\begin{aligned}(\varphi \circ \psi)^*(\xi_1^*(x)) &= \xi_3^*(\varphi \circ \psi(x)) = \xi_3^*(\varphi(\psi(x))) \\ &= \varphi^*(\xi_2^*(\psi(x))) = \varphi^* \circ \psi^*(\xi_1^*(x)).\end{aligned}$$

Thus, $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$. Therefore, $\mathcal{F}(\varphi \circ \psi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi)$. Let $I_S : S \longrightarrow S$ be the identity homomorphism of the H_v-Γ-semigroup S. We have $\mathcal{F}(I_S) = I_S^* = I_{S/\xi^*}$, because I_S^* and I_{S/ξ^*} are identity homomorphisms of S/ξ^*. Therefore, $\mathcal{F}$ is a covariant functor. □

It is worth pointing out that the first functorial consideration of the fundamental algebra of a multialgebra belongs to Pelea [40]. Since then, this aspect has bbeen investigated for all the other hyperstructures.

4. **Conclusions and future work**

The study of the Γ-structures or Γ-hyperstructures ([22, 23, 37]) represents a new line of research in hyperstructure theory, motivated by the various examples of these mathematical objects. In this note we have investigated the class of H_v-Γ-semigroups, where the weak associativity is verified. A covariant functor between the category of the H_v-Γ-semigroups and that of fundamental Γ-semigroups it was defined. Moreover, we have proved the first and third isomorphism theorem using only regular relations. But what about the second isomorphism theorem? It can not be proved in the same way, but it should be necessary to introduce a new notion, that of hyperideal [39], or maybe another one. For the moment the problem remains an open one. Besides, another future problem could be to study other H_v-Γ-structures as H_v-Γ-rings or H_v-Γ-modules.

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