OSCILLATORY BEHAVIOR OF A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH DAMPING

Da-Xue Chen[1]

This paper deals with the oscillatory behavior of the fractional differential equation with damping

$$(D_-^{1+\alpha}y)(t) - p(t)(D_-^{\alpha}y)(t) + q(t)f\left(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv\right) = 0 \quad \text{for} \quad t > 0,$$

where $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y. We obtain some sufficient conditions for the oscillatory behavior of the equation by employing a generalized Riccati transformation technique and certain parameter functions. Examples are given to show the significance of our results. To the best of our knowledge, nothing is known regarding the oscillatory behavior of the equation, so this paper initiates the study.

Keywords: oscillation, fractional differential equation, damping
MSC2000: 34A 08, 34C 10.

1. Introduction

In this paper, we discuss the oscillatory behavior of the fractional differential equation with damping

$$(D_-^{1+\alpha}y)(t) - p(t)(D_-^{\alpha}y)(t) + q(t)f\left(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv\right) = 0 \quad \text{for} \quad t > 0, \quad (1.1)$$

where $\alpha \in (0,1)$ is a constant, $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order α of y defined by $(D_-^{\alpha}y)(t) := -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv$ for $t \in \mathbb{R}_+ := (0,\infty)$, here Γ is the gamma function defined by $\Gamma(t) := \int_0^{\infty} v^{t-1}e^{-v}dv$ for $t \in \mathbb{R}_+$, and the following conditions are assumed to hold:

(S) $p \geq 0$ and $q > 0$ are continuous functions on $[t_0,\infty)$ for a certain $t_0 > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(u)/u \geq K$ for a certain constant $K > 0$ and for all $u \neq 0$, and $\int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t} p(v)dv\right)dt = \infty$.

By a solution of (1.1) we mean a nontrivial function $y \in C(\mathbb{R}_+,\mathbb{R})$ such that $\int_t^{\infty}(v-t)^{-\alpha}y(v)dv \in C^1(\mathbb{R}_+,\mathbb{R})$, $D_-^{\alpha}y \in C^1(\mathbb{R}_+,\mathbb{R})$ and satisfying (1.1) on $\mathbb{R}_+$. Our attention is restricted to those solutions of (1.1) which exist on $\mathbb{R}_+$ and satisfy $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \geq 0$. A solution y of (1.1) is said to be oscillatory if

Foundation item: This work was supported by the National Natural Science Foundation of P. R. China (Grant No. 11271311) and the Natural Science Foundation of Hunan of P. R. China (Grant No. 11JJ3010).

[1]Professor, College of Science, Hunan Institute of Engineering, 88 East Fuxing Road, Xiangtan 411104, P. R. China, e-mail: cdx2003@163.com

it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Fractional differential equations are generalizations of classical differential equations of integer order and have gained considerable popularity and importance during the past three decades or so, due mainly to their demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Nowadays the number of scientific and engineering problems involving fractional calculus is already very large and still growing. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. Fractional differentials and integrals provide more accurate models of systems under consideration. Some of the areas of present applications of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics and signal processing, economics and so on; for example, see [1–6] and the references cited therein.

Recently, there have been some books on the subject of fractional calculus and fractional differential equations, such as the books [7–11]. Many papers have investigated some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions, and we refer to [12–19].

However, to the best of our knowledge very little is known regarding the oscillatory behavior of fractional differential equations up to now. Especially, nothing is known regarding the oscillatory behavior of (1.1) up to now. To develop the qualitative properties of fractional differential equations, it is of great interest to study the oscillatory behavior of (1.1). In this paper, we establish several oscillation criteria for (1.1) by applying a generalized Riccati transformation technique and certain parameter functions. Our results are essential new. We also provide several examples to illustrate the results.

2. **Preliminaries and lemmas**

In this section, we present the definitions of fractional integrals and fractional derivatives, which are used throughout this paper. More details can be found in [7–11]. We also give several lemmas, which are useful in the proof of our results.

There are several kinds of definitions of fractional integrals and fractional derivatives, such as the Riemann-Liouville definition, the Caputo definition, the Liouville definition, the Grünwald-Letnikov definition, the Erdélyi-Kober definition and the Hadamard definition. We adopt the Liouville right-sided definition on the half-axis $\mathbb{R}_+$ for the purpose of this paper.

Definition 2.1. (*Kilbas et al. [10]*) The Liouville right-sided fractional integral of order $\beta > 0$ of a function $g : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(I_-^\beta g)(t) := \frac{1}{\Gamma(\beta)} \int_t^\infty (v-t)^{\beta-1} g(v)dv \quad \text{for} \quad t > 0, \tag{2.1}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2.2. (*Kilbas et al. [10]*) The Liouville right-sided fractional derivative of order $\beta > 0$ of a function $g : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$\begin{aligned}(D_-^{\beta} g)(t) &:= (-1)^{\lceil\beta\rceil} \frac{d^{\lceil\beta\rceil}}{dt^{\lceil\beta\rceil}} (I_-^{\lceil\beta\rceil-\beta} g)(t) \\ &= (-1)^{\lceil\beta\rceil} \frac{1}{\Gamma(\lceil\beta\rceil - \beta)} \frac{d^{\lceil\beta\rceil}}{dt^{\lceil\beta\rceil}} \int_t^{\infty} (v-t)^{\lceil\beta\rceil-\beta-1} g(v)dv \quad \text{for} \quad t > 0, \end{aligned} \tag{2.2}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil\beta\rceil := \min\{z \in \mathbb{Z} : z \geq \beta\}$ is the ceiling function.

Lemma 2.1. *If y is a solution of (1.1), then $(D_-^{1+\alpha} y)(t) = -(D_-^{\alpha} y)'(t)$ for $\alpha \in (0,1)$ and $t > 0$.*

Proof. From (2.2), for $\alpha \in (0,1)$ and $t > 0$ we have

$$\begin{aligned}&(D_-^{1+\alpha} y)(t) \\ &= (-1)^{\lceil 1+\alpha\rceil} \frac{1}{\Gamma(\lceil 1+\alpha\rceil - (1+\alpha))} \frac{d^{\lceil 1+\alpha\rceil}}{dt^{\lceil 1+\alpha\rceil}} \int_t^{\infty} (v-t)^{\lceil 1+\alpha\rceil-(1+\alpha)-1} y(v)dv \\ &= -(-1)^{\lceil\alpha\rceil} \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil+1}}{dt^{\lceil\alpha\rceil+1}} \int_t^{\infty} (v-t)^{\lceil\alpha\rceil-\alpha-1} y(v)dv \\ &= -\frac{d}{dt}\left[(-1)^{\lceil\alpha\rceil} \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_t^{\infty} (v-t)^{\lceil\alpha\rceil-\alpha-1} y(v)dv\right] \\ &= -(D_-^{\alpha} y)'(t).\end{aligned}$$

The proof is complete. □

Lemma 2.2. *Let y be a solution of (1.1) and*

$$G(t) := \int_t^{\infty} (v-t)^{-\alpha} y(v)dv \quad for \quad \alpha \in (0,1) \quad and \quad t > 0, \tag{2.3}$$

then

$$G'(t) = -\Gamma(1-\alpha)(D_-^{\alpha} y)(t) \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{2.4}$$

Proof. From (2.3) and (2.2), for $\alpha \in (0,1)$ and $t > 0$ we obtain

$$\begin{aligned}G'(t) &= \Gamma(1-\alpha) \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^{\infty} (v-t)^{-\alpha} y(v)dv \\ &= -\Gamma(1-\alpha)\left[(-1)^{\lceil\alpha\rceil} \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_t^{\infty} (v-t)^{\lceil\alpha\rceil-\alpha-1} y(v)dv\right] \\ &= -\Gamma(1-\alpha)(D_-^{\alpha} y)(t).\end{aligned}$$

The proof is complete. □

Lemma 2.3. *Let $a \geq 0, b, X \in \mathbb{R}$, then $b\sqrt{a}X - aX^2 \leq \frac{b^2}{4}$.*

Proof. $b\sqrt{a}X - aX^2 = \frac{b^2}{4} - \left(\sqrt{a}X - \frac{b}{2}\right)^2 \leq \frac{b^2}{4}$. The proof is complete. □

3. Main results

Theorem 3.1. *Suppose that (S) holds and that there exists a positive function $r \in C^1[t_0, \infty)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[K r(s) q(s) V(s) - \mu r'_+(s) \right] ds = \infty \tag{3.1}$$

for any constant $\mu > 0$, where $r'_+(s) := \max\{r'(s), 0\}$ and

$$V(s) := \exp\left(\int_{t_0}^{s} p(v) dv \right) \quad for \quad s \geq t_0. \tag{3.2}$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0, \infty)$ such that

$$y(t) > 0 \quad \text{and} \quad G(t) > 0 \quad \text{for} \quad t \in [t_1, \infty), \tag{3.3}$$

where G is defined as in (2.3). Hence, from Lemma 2.1, (3.2) and (1.1) it follows that

$$\begin{aligned} \left[(D_-^{\alpha} y)(t) V(t)\right]' &= -(D_-^{1+\alpha} y)(t) V(t) + (D_-^{\alpha} y)(t) p(t) V(t) \\ &= q(t) f(G(t)) V(t) > 0 \quad \text{for} \quad t \in [t_1, \infty). \end{aligned} \tag{3.4}$$

Thus $(D_-^{\alpha} y)(t)V(t)$ is strictly increasing on $[t_1, \infty)$ and is eventually of one sign. Since $V(t) > 0$ for $t \in [t_0, \infty)$, we see that $(D_-^{\alpha} y)(t)$ is eventually of one sign. We now claim

$$(D_-^{\alpha} y)(t) < 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{3.5}$$

If not, then $(D_-^{\alpha} y)(t)$ is eventually positive and there exists $t_2 \in [t_1, \infty)$ such that $(D_-^{\alpha} y)(t_2) > 0$. Since $(D_-^{\alpha} y)(t)V(t)$ is strictly increasing on $[t_1, \infty)$, it is clear that $(D_-^{\alpha} y)(t)V(t) \geq (D_-^{\alpha} y)(t_2)V(t_2) := c_1 > 0$ for $t \in [t_2, \infty)$. Therefore, from (2.4) we have

$$-\frac{G'(t)}{\Gamma(1-\alpha)} = (D_-^{\alpha} y)(t) \geq c_1 V^{-1}(t) = c_1 \exp\Big(- \int_{t_0}^{t} p(v) dv \Big) \quad \text{for} \quad t \in [t_2, \infty).$$

Integrating both sides of the last inequality from t_2 to t, we get

$$\int_{t_2}^{t} \exp\Big(- \int_{t_0}^{s} p(v) dv \Big) ds \leq -\frac{G(t) - G(t_2)}{c_1 \Gamma(1-\alpha)} < \frac{G(t_2)}{c_1 \Gamma(1-\alpha)} \quad \text{for} \quad t \in [t_2, \infty).$$

Letting $t \to \infty$, we see $\int_{t_2}^{\infty} \exp\big(- \int_{t_0}^{s} p(v) dv \big) ds \leq \frac{G(t_2)}{c_1 \Gamma(1-\alpha)} < \infty$. This contradicts the assumption $\int_{t_0}^{\infty} \exp\big(- \int_{t_0}^{t} p(v) dv \big) dt = \infty$ in (S). Hence, (3.5) holds. Define the function w by the generalized Riccati substitution

$$w(t) = r(t)\frac{-(D_-^\alpha y)(t)V(t)}{G(t)} \quad \text{for} \quad t \in [t_1, \infty).$$

It is easy to see that $w(t) > 0$ for $t \in [t_1, \infty)$. Since $0 < -(D_-^\alpha y)(t)V(t) \le -(D_-^\alpha y)(t_1)V(t_1)$ and $G(t) \ge G(t_1) > 0$ for $t \in [t_1, \infty)$, from (3.4), (2.4) and (S) we have

$$\begin{aligned}
w'(t) &= r'(t)\frac{-(D_-^\alpha y)(t)V(t)}{G(t)} + r(t)\left[\frac{-(D_-^\alpha y)(t)V(t)}{G(t)}\right]' \\
&\le r'_+(t)\frac{-(D_-^\alpha y)(t)V(t)}{G(t)} + r(t)\left[\frac{[-(D_-^\alpha y)(t)V(t)]'}{G(t)} + \frac{(D_-^\alpha y)(t)V(t)G'(t)}{G^2(t)}\right] \\
&\le r'_+(t)\frac{-(D_-^\alpha y)(t_1)V(t_1)}{G(t_1)} \\
&\quad + r(t)\left[\frac{-q(t)f(G(t))V(t)}{G(t)} + \frac{(D_-^\alpha y)(t)V(t)[-\Gamma(1-\alpha)(D_-^\alpha y)(t)]}{G^2(t)}\right] \\
&< r'_+(t)\frac{-(D_-^\alpha y)(t_1)V(t_1)}{G(t_1)} + r(t)\frac{-q(t)f(G(t))V(t)}{G(t)} \\
&\le \mu r'_+(t) - Kr(t)q(t)V(t) \quad \text{for} \quad t \in [t_1, \infty),
\end{aligned}$$

where $\mu := \frac{-(D_-^\alpha y)(t_1)V(t_1)}{G(t_1)} > 0$. Integrating both sides of the last inequality from t_1 to t, we obtain

$$\int_{t_1}^t \Big[Kr(s)q(s)V(s) - \mu r'_+(s)\Big]ds \le w(t_1) - w(t) < w(t_1) \quad \text{for} \quad t \in [t_1, \infty).$$

Letting $t \to \infty$, we get $\limsup_{t\to\infty} \int_{t_1}^t \big[Kr(s)q(s)V(s) - \mu r'_+(s)\big]ds \le w(t_1) < \infty$, which implies a contradiction to (3.1). The proof is complete. □

Theorem 3.2. *Assume that (S) holds and that there exists a positive function $r \in C^1[t_0, \infty)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[Kr(s)q(s) - \frac{M_+^2(s)}{4\Gamma(1-\alpha)r(s)}\right]ds = \infty, \tag{3.6}$$

where $M_+(s) := \max\{0, r'_+(s) - r(s)p(s)\}$ and r'_+ is defined as in Theorem 3.1. Then all solutions of (1.1) are oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is an eventually positive solution of (1.1). Proceeding as in the proof of Theorem 3.1, we see that (3.3) and (3.5) hold. Define the function w by the generalized Riccati substitution

$$w(t) = r(t)\frac{-(D_-^\alpha y)(t)}{G(t)} \quad \text{for} \quad t \in [t_1, \infty). \tag{3.7}$$

Then we have $w(t) > 0$ for $t \in [t_1, \infty)$. From Lemma 2.1, (1.1), (2.4), (3.7) and (S), it follows that

$$\begin{aligned} w'(t) &= r'(t)\frac{-(D^{\alpha}_{-}y)(t)}{G(t)} + r(t)\left[\frac{-(D^{\alpha}_{-}y)(t)}{G(t)}\right]' \\ &\le r'_{+}(t)\frac{-(D^{\alpha}_{-}y)(t)}{G(t)} + r(t)\left[\frac{(D^{1+\alpha}_{-}y)(t)}{G(t)} + \frac{(D^{\alpha}_{-}y)(t)G'(t)}{G^2(t)}\right] \\ &= r'_{+}(t)\frac{-(D^{\alpha}_{-}y)(t)}{G(t)} \\ &\quad + r(t)\left[\frac{p(t)(D^{\alpha}_{-}y)(t) - q(t)f(G(t))}{G(t)} + \frac{(D^{\alpha}_{-}y)(t)[-\Gamma(1-\alpha)(D^{\alpha}_{-}y)(t)]}{G^2(t)}\right] \\ &\le r'_{+}(t)\frac{w(t)}{r(t)} - p(t)w(t) - Kr(t)q(t) - \frac{\Gamma(1-\alpha)}{r(t)}w^2(t) \\ &= -Kr(t)q(t) + \frac{r'_{+}(t) - r(t)p(t)}{r(t)}w(t) - \frac{\Gamma(1-\alpha)}{r(t)}w^2(t) \\ &\le -Kr(t)q(t) + \frac{M_{+}(t)}{r(t)}w(t) - \frac{\Gamma(1-\alpha)}{r(t)}w^2(t) \quad \text{for} \quad t \in [t_1, \infty), \end{aligned} \tag{3.8}$$

where M_{+} is defined as in Theorem 3.2. Taking $b = M_{+}(t)/\sqrt{\Gamma(1-\alpha)r(t)}$ and $a = \Gamma(1-\alpha)/r(t)$, from (3.8) and Lemma 2.3 we conclude

$$w'(t) \le -Kr(t)q(t) + \frac{M^2_{+}(t)}{4\Gamma(1-\alpha)r(t)} \quad \text{for} \quad t \in [t_1, \infty).$$

Integrating both sides of the last inequality from t_1 to t, we get

$$\int_{t_1}^{t}\left[Kr(s)q(s) - \frac{M^2_{+}(s)}{4\Gamma(1-\alpha)r(s)}\right]ds \le w(t_1) - w(t) < w(t_1) \quad \text{for} \quad t \in [t_1, \infty).$$

Letting $t \to \infty$, we have $\limsup_{t\to\infty}\int_{t_1}^{t}\left[Kr(s)q(s) - \frac{M^2_{+}(s)}{4\Gamma(1-\alpha)r(s)}\right]ds \le w(t_1) < \infty$, which contradicts (3.6). The proof is complete. □

Theorem 3.3. *Suppose that (S) holds. Furthermore, assume that there exist a positive function* $r \in C^1[t_0, \infty)$ *and a function* $H \in C(\mathbb{D}, \mathbb{R})$*, where* $\mathbb{D} := \{(t, s) : t \ge s \ge t_0\}$*, such that*

$$H(t,t) = 0 \quad for \quad t \ge t_0 \quad and \quad H(t,s) > 0 \quad for \quad (t,s) \in \mathbb{D}_0,$$

where $\mathbb{D}_0 := \{(t, s) : t > s \ge t_0\}$*. Suppose also that* H *has a nonpositive continuous partial derivative* $H'_s(t,s) := \frac{\partial H(t,s)}{\partial s}$ *on* $\mathbb{D}_0$ *with respect to the second variable and that there exists a function* $h \in C(\mathbb{D}, \mathbb{R})$ *such that*

$$H'_s(t,s) + H(t,s)\frac{M_{+}(s)}{r(s)} = \frac{h(t,s)}{r(s)}\sqrt{H(t,s)} \quad for \quad (t.s) \in \mathbb{D} \tag{3.9}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[r(s)q(s)H(t,s) - \frac{h^2_{+}(t,s)}{4K\Gamma(1-\alpha)r(s)}\right]ds = \infty, \tag{3.10}$$

where M_+ is defined as in Theorem 3.2 and $h_+(t,s) := \max\{0, h(t,s)\}$. Then all solutions of (1.1) are oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is an eventually positive solution of (1.1). We proceed as in the proof of Theorem 3.2 to get that (3.8) holds. Multiplying (3.8) by $H(t,s)$ and integrating from t_1 to t, we get for $t \in [t_1, \infty)$,

$$\int_{t_1}^{t} Kr(s)q(s)H(t,s)ds \le -\int_{t_1}^{t} H(t,s)w'(s)ds + \int_{t_1}^{t} H(t,s)\frac{M_+(s)}{r(s)}w(s)ds - \int_{t_1}^{t} H(t,s)\frac{\Gamma(1-\alpha)}{r(s)}w^2(s)ds. \tag{3.11}$$

Using the integration by parts formula, we obtain for $t \in [t_1, \infty)$,

$$\begin{aligned} -\int_{t_1}^{t} H(t,s)w'(s)ds &= \Big[-H(t,s)w(s)\Big]_{s=t_1}^{s=t} + \int_{t_1}^{t} H'_s(t,s)w(s)ds \\ &= H(t,t_1)w(t_1) + \int_{t_1}^{t} H'_s(t,s)w(s)ds. \end{aligned} \tag{3.12}$$

Substituting (3.12) in (3.11), for $t \in [t_1, \infty)$ we have

$$\begin{aligned} &K\int_{t_1}^{t} r(s)q(s)H(t,s)ds \\ &\le H(t,t_1)w(t_1) \\ &\quad + \int_{t_1}^{t} \left\{ \left[H'_s(t,s) + H(t,s)\frac{M_+(s)}{r(s)}\right]w(s) - \Gamma(1-\alpha)\frac{H(t,s)}{r(s)}w^2(s)\right\}ds. \end{aligned}$$

In view of (3.9), from the last inequality we get

$$\begin{aligned} &K\int_{t_1}^{t} r(s)q(s)H(t,s)ds \\ &\le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left[\frac{h(t,s)}{r(s)}\sqrt{H(t,s)}w(s) - \Gamma(1-\alpha)\frac{H(t,s)}{r(s)}w^2(s)\right]ds \\ &\le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left[\frac{h_+(t,s)}{r(s)}\sqrt{H(t,s)}w(s) - \Gamma(1-\alpha)\frac{H(t,s)}{r(s)}w^2(s)\right]ds \end{aligned} \tag{3.13}$$

for $t \in [t_1, \infty)$, where h_+ is defined as in Theorem 3.3. Taking $b = \frac{h_+(t,s)}{\sqrt{\Gamma(1-\alpha)r(s)}}$ and $a = \Gamma(1-\alpha)\frac{H(t,s)}{r(s)}$, by using Lemma 2.3 and (3.13), we obtain for $t \in [t_1, \infty)$,

$$\int_{t_1}^{t} r(s)q(s)H(t,s)ds \le K^{-1}H(t,t_1)w(t_1) + K^{-1}\int_{t_1}^{t}\frac{h_+^2(t,s)}{4\Gamma(1-\alpha)r(s)}ds. \tag{3.14}$$

Since $H'_s(t,s) \le 0$ for $t > s \ge t_0$, we have $0 < H(t,t_1) \le H(t,t_0)$ for $t > t_1 \ge t_0$. Therefore, from (3.14) we get for $t \in [t_1, \infty)$,

$$\int_{t_1}^{t} \left[r(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4K\Gamma(1-\alpha)r(s)} \right] ds \le K^{-1}H(t,t_1)w(t_1) \le K^{-1}H(t,t_0)w(t_1). \tag{3.15}$$

Since $0 < H(t,s) \le H(t,t_0)$ for $t > s \ge t_0$, we have $0 < \frac{H(t,s)}{H(t,t_0)} \le 1$ for $t > s \ge t_0$. Hence, it follows from (3.15) that

$$\begin{aligned} &\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[r(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4K\Gamma(1-\alpha)r(s)} \right] ds \\ &= \frac{1}{H(t,t_0)} \int_{t_0}^{t_1} \left[r(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4K\Gamma(1-\alpha)r(s)} \right] ds \\ &\quad + \frac{1}{H(t,t_0)} \int_{t_1}^{t} \left[r(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4K\Gamma(1-\alpha)r(s)} \right] ds \\ &\le \frac{1}{H(t,t_0)} \int_{t_0}^{t_1} r(s)q(s)H(t,s)ds + \frac{1}{H(t,t_0)} K^{-1}H(t,t_0)w(t_1) \\ &\le \int_{t_0}^{t_1} r(s)q(s)ds + K^{-1}w(t_1) \quad \text{for} \quad t \in [t_1, \infty). \end{aligned}$$

Letting $t \to \infty$, we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[r(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4K\Gamma(1-\alpha)r(s)} \right] ds \\ &\le \int_{t_0}^{t_1} r(s)q(s)ds + K^{-1}w(t_1) < \infty, \end{aligned}$$

which yields a contradiction to (3.10). The proof is complete. □

Remark 3.1. From Theorems 3.1–3.3, we can obtain many different sufficient conditions for the oscillatory behavior of (1.1) with different choices of the functions r and H.

For example, let $r(s) = 1$, then from Theorem 3.2 we obtain the following result.

Corollary 3.1. *Assume that (S) and the following condition hold:*

$$\int_{t_0}^{\infty} q(t)dt = \infty. \tag{3.16}$$

Then all solutions of (1.1) are oscillatory.

Let $r(s) = 1$. Then Theorem 3.1 yields the following result.

Corollary 3.2. *Suppose that (S) and the following condition hold:*

$$\int_{t_0}^{\infty}\left[q(t)\exp\left(\int_{t_0}^{t}p(v)dv\right)\right]dt=\infty. \tag{3.17}$$

Then every solution of (1.1) is oscillatory.

Note that, since $q(t)\exp\left(\int_{t_0}^{t}p(v)dv\right)\geq q(t)$ for $t\geq t_0$, Corollary 3.1 can also be derived from Corollary 3.2. Obviously, Corollary 3.2 is better than Corollary 3.1.

Let $r(s)=s$. Then from Theorem 3.2 we conclude the following result.

Corollary 3.3. *Assume that (S) and the following condition hold:*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sq(s)-\frac{(\max\{0,1-sp(s)\})^2}{4K\Gamma(1-\alpha)s}\right]ds=\infty. \tag{3.18}$$

Then all solutions of (1.1) are oscillatory.

Let $r(s)=1$ and $H(t,s)=(t-s)^m$, where $m\geq 2$ is a constant. Then Theorem 3.3 implies the following result.

Corollary 3.4. *Suppose that (S) holds and that there exists a constant $m\geq 2$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}q(s)(t-s)^m ds=\infty.$$

Then every solution of (1.1) is oscillatory.

4. **Examples**

Example 4.1. Consider the fractional differential equation

$$(D_-^{1+\alpha}y)(t)-\frac{1}{t^2}(D_-^{\alpha}y)(t)+\frac{1}{t}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv=0 \quad \text{for} \quad t>0, \tag{4.1}$$

where $\alpha\in(0,1)$. In (4.1), $p(t)=\frac{1}{t^2}, q(t)=\frac{1}{t}$ and $f(u)=u$. Take $t_0>0$ and $K=1$. Since

$$\begin{aligned}\int_{t_0}^{\infty}\exp\left(-\int_{t_0}^{t}p(v)dv\right)dt&=\int_{t_0}^{\infty}\exp\left(-\int_{t_0}^{t}\frac{1}{v^2}dv\right)dt\\ =\int_{t_0}^{\infty}\exp\left(\frac{1}{t}-\frac{1}{t_0}\right)dt&\geq\int_{t_0}^{\infty}\exp\left(-\frac{1}{t_0}\right)dt=\infty,\end{aligned}$$

we see that (S) holds. Furthermore, we have $\int_{t_0}^{\infty}q(t)dt=\int_{t_0}^{\infty}\frac{1}{t}dt=\infty$, which implies that (3.16) holds. Therefore, by Corollary 3.1 all solutions of (4.1) are oscillatory.

Example 4.2. Consider the fractional differential equation

$$\begin{aligned}(D_-^{1+\alpha}y)(t) &- \frac{1}{t}(D_-^{\alpha}y)(t)\\ &+ \frac{1}{t^2}\left[2+\exp\left(\int_t^\infty (v-t)^{-\alpha}y(v)dv\right)\right]\int_t^\infty (v-t)^{-\alpha}y(v)dv = 0 \qquad (4.2)\end{aligned}$$

for $t>0$, where $\alpha\in(0,1)$. In (4.2), $p(t)=\frac{1}{t}, q(t)=\frac{1}{t^2}$ and $f(u)=(2+e^u)u$. Take $t_0>0$ and $K=2$. Since

$$\int_{t_0}^\infty \exp\left(-\int_{t_0}^t p(v)dv\right)dt = \int_{t_0}^\infty \exp\left(-\int_{t_0}^t \frac{1}{v}dv\right)dt = \int_{t_0}^\infty \frac{t_0}{t}dt = \infty,$$

we find that (S) holds. On the other hand, we have

$$\int_{t_0}^\infty \left[q(t)\exp\left(\int_{t_0}^t p(v)dv\right)\right]dt = \int_{t_0}^\infty \frac{1}{t^2}\frac{t}{t_0}dt = \int_{t_0}^\infty \frac{1}{t_0 t}dt = \infty,$$

which yields that (3.17) holds. Hence, by Corollary 3.2 every solution of (4.2) is oscillatory.

Remark 4.1. In Example 4.2, we get $\int_{t_0}^\infty q(t)dt = \int_{t_0}^\infty \frac{1}{t^2}dt = \frac{1}{t_0} < \infty$, which shows that (3.16) doesn't hold. Hence, Corollary 3.1 cannot be applied to (4.2).

Example 4.3. Consider the fractional differential equation

$$\begin{aligned}(D_-^{1+\alpha}y)(t) &- \frac{1}{t^3}(D_-^{\alpha}y)(t)\\ &+ \frac{1}{t^2}\left[\frac{1}{4}+\left(\int_t^\infty (v-t)^{-\alpha}y(v)dv\right)^2\right]\int_t^\infty (v-t)^{-\alpha}y(v)dv = 0 \qquad (4.3)\end{aligned}$$

for $t>0$, where $\alpha\in(0,1)$ satisfies $\Gamma(1-\alpha)>1$. In (4.3), $p(t)=\frac{1}{t^3}, q(t)=\frac{1}{t^2}$ and $f(u)=(\frac{1}{4}+u^2)u$. Take $t_0=1$ and $K=\frac{1}{4}$. Since

$$\begin{aligned}\int_{t_0}^\infty \exp\left(-\int_{t_0}^t p(v)dv\right)dt &= \int_1^\infty \exp\left(-\int_1^t \frac{1}{v^3}dv\right)dt\\ &= \int_1^\infty \exp\left(\frac{1}{2t^2}-\frac{1}{2}\right)dt \ge \int_1^\infty \exp\left(-\frac{1}{2}\right)dt = \infty,\end{aligned}$$

we see that (S) holds. Since

$$\lim_{s\to\infty}\left[\frac{1}{s}-\frac{(1-\frac{1}{s^2})^2}{\Gamma(1-\alpha)s}\right]\Big/\left(\frac{1}{s}\right) = 1-\frac{1}{\Gamma(1-\alpha)} > 0 \quad \text{and} \quad \int_1^\infty \frac{1}{s}ds = \infty,$$

we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^t \left[sq(s)-\frac{(\max\{0,1-sp(s)\})^2}{4K\Gamma(1-\alpha)s}\right]ds\\ &= \limsup_{t\to\infty}\int_1^t \left[\frac{1}{s}-\frac{(1-\frac{1}{s^2})^2}{\Gamma(1-\alpha)s}\right]ds = \infty,\end{aligned}$$

which implies that (3.18) holds. Therefore, by Corollary 3.3 all solutions of (4.3) are oscillatory when $\Gamma(1-\alpha) > 1$.

Remark 4.2. In Example 4.3, we have $\int_{t_0}^{\infty} q(t)dt = \int_1^{\infty} \frac{1}{t^2}dt = 1 < \infty$, which implies that (3.16) doesn't hold. Thus, Corollary 3.1 cannot be applied to (4.3). Furthermore, we get

$$\int_{t_0}^{\infty}\left[q(t)\exp\left(\int_{t_0}^{t}p(v)dv\right)\right]dt = \int_1^{\infty}\frac{1}{t^2}\exp\left(\frac{1}{2}-\frac{1}{2t^2}\right)dt$$
$$\leq \int_1^{\infty}\frac{1}{t^2}e^{\frac{1}{2}}dt = e^{\frac{1}{2}} < \infty,$$

which yields that (3.17) doesn't hold. Hence, Corollary 3.2 cannot also be applied to (4.3).

Acknowledgments

The author is very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

REFERENCES

[1] *R. L. Bagley and P. J. Torvik*, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheology, **27**(1983) 201–210.

[2] *F. Mainardi*, Fractional calculus: Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997, 291–348.

[3] *B. Mandelbrot*, Some noises with 1/f spectrum, a bridge between direct current and white noise, IEEE Trans. Inform. Theory, **13**(1967) 289–298.

[4] *Y. A. Rossikhin and M. V. Shitikova*, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, Appl. Mech. Rev., **50**(1997) 15–67.

[5] *R. T. Baillie*, Long memory processes and fractional integration in econometrics, J. Econometrics, **73**(1996) 5–59.

[6] *R. L. Magin*, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng., **32**(2004) 1–377.

[7] *K. B. Oldham and J. Spanier*, The Fractional Calculus, Academic Press, San Diego, 1974.

[8] *K. S. Miller and B. Ross*, An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, 1993.

[9] *I. Podlubny*, Fractional Differential Equations, Academic Press, San Diego, 1999.

[10] *A. A. Kilbas, H. M. Srivastava and J. J. Trujillo*, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V, Amsterdam, 2006.

[11] *S. Das*, Functional Fractional Calculus for System Identification and Controls, Springer, New York, 2008.

[12] *M. Muslim*, Existence and approximation of solutions to fractional differential equations, Math. Comput. Modelling, **49**(2009) 1164–1172.

[13] *Yong Zhou, Feng Jiao and Jing Li*, Existence and uniqueness for p-type fractional neutral differential equations, Nonlinear Anal., **71**(2009) 2724–2733.

[14] *L. Galeone and R. Garrappa*, Explicit methods for fractional differential equations and their stability properties, J. Comput. Appl. Math., **228**(2009) 548–560.

[15] *J. T. Edwards, N. J. Ford and A. C. Simpson*, The numerical solution of linear multi-term fractional differential equations: systems of equations, J. Comput. Appl. Math., **148**(2002) 401–418.

[16] *J. C. Trigeassou, N. Maamri and J. Sabatier*, A. Oustaloup, A Lyapunov approach to the stability of fractional differential equations, Signal Processing, **91**(2011) 437–445.

[17] *Weihua Deng*, Smoothness and stability of the solutions for nonlinear fractional differential equations, Nonlinear Anal., **72**(2010) 1768–1777.

[18] *F. Ghoreishi and S. Yazdani*, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, Comput. Math. Appl., **61**(2011) 30–43.

[19] *A. Saadatmandi and M. Dehghan*, A new operational matrix for solving fractional-order differential equations, Comput. Math. Appl., **59**(2010) 1326–1336.