ON BI-Γ-HYPERIDEALS OF Γ-SEMIHYPERGROUPS

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In this paper, we introduce the strongly prime, prime, semiprime, strongly irreducible and irreducible bi-Γ-hyperideals of Γ-semihypergroups. The space of strongly prime bi-Γ-hyperideals is topologized. We also characterize those Γ-semihypergroups for which each bi-Γ-hyperideal is strongly prime.

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1. Introduction

The algebraic hyperstructure notion was introduced in 1934 by a French mathematician F. Marty [16], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

In 1986, Sen and Saha [23] defined the notion of a Γ-semigroup as a generalization of a semigroup. One can see that Γ-semigroups are generalizations of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [4, 5], Chinram and Jirojkul [6], Chinram and Siammai [7], Hila [12, 13], Saha [16], Sen and et. al. [18, 19, 20, 21, 22, 25] and Seth [23]. Recently, Davvaz, Hila and et. al. [2, 11, 14, 17] introduced the notion of Γ – semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ – semigroup. They

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presented many interesting examples and obtained a several characterizations of $\Gamma-$semihypergroups. In [2] the notion of a $\Gamma-$hyperideal of a $\Gamma-$semihypergroup is introduced. A $\Gamma-$hyperideal of a $\Gamma-$semihypergroup is a generalization of an ideal of a semigroup, a generalization of a hyperideal of a semihypergroup and a generalization of a $\Gamma-$ideal of a $\Gamma-$semigroup. The notion of a bi-ideal was first introduced by Good and Hughes [10], as early as 1952, and it has been widely studied. Recently, Davvaz et al. generalized this notion introducing the notion of bi-$\Gamma-$hyperideal in $\Gamma-$semihypergroups [2]. This paper deals with some classes of bi-$\Gamma-$hyperideals of $\Gamma-$semihypergroups.

In this paper, we introduce the strongly prime, strongly semiprime, strongly irreducible and irreducible bi-$\Gamma-$hyperideals of $\Gamma-$semihypergroups. The space of strongly prime bi-$\Gamma-$hyperideals is topologized. We also characterize those $\Gamma-$semihypergroups for which each bi-$\Gamma-$hyperideal is strongly prime. We also prove, a $\Gamma-$semihypergroup S is completely regular if and only if every bi-$\Gamma-$hyperideal of S is semiprime.

2 Preliminaries

In this section, we recall certain definitions and results needed for our purpose.

Definition 2.1 *[18, 22] Let $S=\{a,b,c,...\}$ and $\Gamma=\{\alpha,\beta,\gamma,...\}$ be two non-empty sets. Then S is called a $\Gamma-$semigroup if there exists a mapping $S\times\Gamma\times S\to S$ written as $(a,\gamma,b)\to a\gamma b$ satisfying the following identity $(a\gamma b)\beta c=a\gamma(b\beta c)$ for all $a;b,c\in S$ and for all $\alpha,\beta\in\Gamma$. Let K be a non-empty subset of S. Then K is called a sub $\Gamma-$semigroup of S if $a\gamma b\in K$ for all $a;b\in S$ and $\gamma\in\Gamma$.*

Definition 2.2 *A map $\circ: S\times S\to P^*(S)$ is called hyperoperation or join operation on the set S, where S is a nonempty set and $P^*(S)=P(S)\setminus\{\varnothing\}$ denotes the set of all nonempty subsets of S. A hypergroupoid is a set S with together a (binary) hyperoperation. A hypergroupoid $(S,\circ)$, which is associative, that is $x\circ(y\circ z)=(x\circ y)\circ z$, $\forall x,y,z\in S$, is called a semihypergroup.*

Let A and B be two non-empty subset of S. Then we define

$$A\circ B=\bigcup_{a\in A,b\in B} a\circ b, a\circ A=\{a\}\circ A \text{and} a\circ B=\{a\}\circ B.$$

Definition 2.3 *[17]Let S and* Γ *be two non-empty sets. Then* S *is called a* Γ*-semihypergroup if every* $\gamma \in \Gamma$ *is a hyperoperation on* S*, i.e.,* $x\gamma y \subseteq S$ *for every* $x; y \in S$*, and for every* $\alpha, \beta \in \Gamma$ *and* $x; y; z \in S$ *we have* $x\alpha(y\beta z) = (x\alpha y)\beta z$.

If every $\gamma \in \Gamma$ is an operation, then S is a Γ – semigroup. If (S, γ) is a hypergroup for every $\gamma \in \Gamma$, then S is called a Γ – hypergroup.

Let A and B be two non-empty subset of S. Then we define

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Let $(S, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then S is Γ-semihypergroup. So every semihypergroup is Γ-semihypergroup.

Let S be a Γ-semihypergroup and $\gamma \in \Gamma$. A non-empty subset A of S is called a sub Γ-semihypergroup of S if $x\gamma y \subseteq A$ for every $x, y \in A$. A Γ-semihypergroup S is called *commutative* if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$.

Example 2.4 *Let* $S = [0,1]$ *and* $\Gamma = \mathbb{N}$*. For every* $x, y \in S$ *and* $\gamma \in \Gamma$*, we define* $\gamma : S \times S \to \wp^*(S)$ *by* $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$*. Then* γ *is hyperoperation. For every* $x, y, z \in S$ *and* $\alpha, \beta \in \Gamma$*, we have* $(x\alpha y)\beta z = \left[0, \frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z)$*. This means that* S *is a* Γ *– semihypergroup.*

Example 2.5 *Let S be a non-empty set and let* Γ *be a non-empty subset of* S*. If we define* $x \circ y = \{x, \gamma, y\}$*, for every* $x; y \in S$ *and* $\gamma \in \Gamma$*, then* S *is a* Γ *– semihypergroup.*

Example 2.6 *Let* $(S, \circ)$ *be a semihypergroup and* Γ *be a non-empty subset of* S*. We define* $x\gamma y = x \circ y$ *for every* $x; y \in S$ *and* $\gamma \in \Gamma$*. Then* S *is a* Γ *– semihypergroup.*

Definition 2.7 *[2] A non-empty subset* A *of a* Γ*-semihypergroup* S *is a right (left)* Γ*-hyperideal of* S *if* $A\Gamma S \subseteq A$ *(*$S\Gamma A \subseteq A$*), and is a* Γ*-hyperideal of* S *if it is both a right and a left* Γ*-hyperideal.*

Definition 2.8 *A non-empty subset* B *of a* Γ*-semihypergroup* S *is called bi-*Γ*-hyperideal of* S *if the following two conditions hold*

(1) $B\Gamma B \subseteq B$

(2) $B\Gamma S\Gamma B \subseteq B$

A bi-$\Gamma-$hyperideal B of $\Gamma-$semihypergroups S is proper if $B \neq S$.

Example 2.9 *Let* $S = (0,1)$, $\Gamma = \{\gamma_n | n \in \mathbb{N}\}$ *and for every* $n \in \mathbb{N}$ *we define the hyperoperation* γ_n *on* S *as follows*

$$x\gamma_n y = \left\{\frac{xy}{2^k} | 0 \leq k \leq n\right\}, \forall x, y \in S.$$

Then, $x\gamma_n y \subset S$ and for every $m; n \in \mathbb{N}$ and $x; y; z \in S$

$$(x\gamma_n y)\gamma_m z = \left\{\frac{xyz}{2^k} | 0 \leq k \leq n+m\right\} = x\gamma_n (y\gamma_m z)$$

So S is a $\Gamma-$semihypergroup.Now let $S = \left(0, \frac{1}{2^i}\right)$, where $i \in \mathbb{N}$. It easily to see that S_i is a bi-Γ-hyperideal of S and $\bigcap_{i\in I} S_i = \Phi$

Example 2.10 *Let S be the* Γ*-semihypergroup in Example 2.9, and let for every* $i \in \mathbb{N}$, $B_i = \{i, i+1, ...\}$. *Therefore* B_i *is a bi-*Γ*-hyperideal of* S *and* $\bigcap_{i\in I} S_i = \Phi$.

Example 2.11 *Let* $S = [0,1]$ *and* $\Gamma = \mathbb{N}$. *Then with hyperoperation* $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$ *is a* Γ*-semihypergroup. Let* $T = [0,t]$ *be sub set of* $S = [0,1]$, *where* $t \in (0,1]$. *Then,* T *is a left (right, bi)-*$\Gamma-$*hyperideal of* S.

3. Prime Bi-Γ-hyperideals

In this section, we study some properties of prime bi-Γ-hyperideals in a Γ-semihypergroup.

Definition 3.1 *A bi-*Γ*-hyperideal* B *of a* Γ*-semihypergroup* S *is called a prime (strongly prime) bi-*Γ*-hyperideal if* $B_1\Gamma B_2 \subseteq B (B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B)$ *implies* $B_1 \subseteq B$ *or* $B_2 \subseteq B$ *for any bi-*Γ*-hyperideasl* B_1 *and* B_2 *of* S. *A bi-*Γ*-hyperideal* B *of a* Γ*-semihypergroup* S *is called a semiprime bi-*Γ*-hyperideal if*

$B_1^2 \subseteq B$ *implies* $B_1 \subseteq B$ *for any bi-*Γ*- hyperideal* B_1 *of* S.

Every strongly prime bi-$\Gamma-$hyperideal of a $\Gamma-$semihypergroup S is a prime bi-$\Gamma-$hyperideal and every prime bi-$\Gamma-$hyperideal is a semiprime bi-$\Gamma-$hyperideal. A prime bi-$\Gamma-$hyperideal is not necessarily strongly prime and a semiprime bi-$\Gamma-$hyperideal is not necessary prime.

Definition 3.2 *A bi-*Γ*-hyperideal* B *of a* Γ*-semihypergroup* S *is called an irreducible (strongly irreducible) bi-*Γ*-hyperideal if* $B_1 \cap B_2 = B(B_1 \cap B_2 \subseteq B)$ *implies* $B_1 = B$ *or* $B_2 = B$ *(*$B1 \subseteq B$ *or* $B_2 \subseteq B$*).*

Every strongly irreducible bi-Γ-hyperideal of a Γ- semihypergroup is an irreducible bi-Γ-hyperideall but the converse is not true.

Lemma 3.3 *The intersection of any family of prime bi-*Γ*-hyperideal of a* Γ*-semihypergroup is a semiprime bi-*Γ*-hyperideal.*

Proof. Straightforward.

Theorem 3.4 *Every strongly irreducible, semiprime bi-*Γ*-hyperideal of a* Γ*- semihypergroup* S *is a strongly prime bi-*Γ*-hyperideal.*

Proof. Let B be a strongly irreducible semiprime bi-Γ-hyperideal of S. Let B_1, B_2 be any bi-Γ-hyperideal of S such that $B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B$. Since $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2\Gamma B_1$, $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B$. Since B is a semiprime bi-Γ-hyperideal, $B_1 \cap B_2 \subseteq B$. Because B is a strongly irreducible bi-Γ-hyperideal of S, so either $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime bi-Γ-hyperideal of S.

Definition 3.5 *An element* a *of the* Γ*-semihypergroup* S *is called regular if there exists* $x \in S$ *and* $\alpha, \beta \in \Gamma$ *such that* $a \in a\alpha x\beta a$. *If every element of* Γ*-semihypergroup* S *is regular, then* S *is called a regular* Γ*-semihypergroup. The following are equivalent definitions*

(1) For every $A \subseteq S$, $A \subseteq A\Gamma S\Gamma A$.

(2) For every element $a \in S$, $a \in a\Gamma M\Gamma a$

Theorem 3.6 *Let* B *be a bi-*Γ*-hyperideal of a* Γ*-semihypergroup* S *and* $a \in S$ *such that* $a \notin B$. *Then there exists an irreducible bi-*Γ*-hyperideal* I *of* S *such that* $B \subseteq I$ *and* $a \notin I$.

Proof. Let $\mathcal{A}$ be the collection of all bi-$\Gamma-$hyperideals of S which contain B and do not contain a. Then it is non-empty, because $B \in \mathcal{A}$ The collection $\mathcal{A}$ is a partially ordered set under inclusion. If $\mathcal{C}$ is any totally ordered subset of A then $\cup\mathcal{C}$ is a bi-Γ-hyperideal of S containing B. Hence by Zorn's Lemma, there exists a maximal element I in $\mathcal{A}$. We show that I is an irreducible bi-$\Gamma-$hyperideal. Let C and D be two bi-Γ-hyperideals of S such that $I = C \cap D$. If both C and D properly contain I, then $a \in C$ and $a \in D$. Hence $a \in C \cap D = I$. This contradicts the fact that $a \notin I$. Thus $I = C$ or $I = D$.

Lemma 3.7 *A Γ-semihypergroup S is completely regular if and only if $A \subseteq (A\Gamma A)\Gamma S\Gamma(A\Gamma A)$ for every $A \subseteq S$. Equivalently, a Γ-semihypergroup S is completely regular if and only if $a \in a\Gamma a\Gamma S\Gamma a\Gamma a$ for all $a \in S$.*

Theorem 3.8 *A $\Gamma-$semihypergroup S is completely regular if and only if every bi-$\Gamma-$hyperideal of S is semiprime.*

Proof. Suppose S is a completely regular $\Gamma-$semihypergroup S. Let B be a bi-Γ-hyperideal, $a \in S$ and $a\Gamma a \subseteq B$. Then for some $x, y, z \in S$ and $\alpha, \beta, \gamma, \rho, \tau, \eta, \mu \in \Gamma$, it holds

$$a \in a\alpha x\beta a = (a\gamma a\rho y)\alpha x\beta(z\tau a\eta a) = a\gamma(a\rho y\alpha x\beta z\tau a)\eta a \subseteq B\Gamma S\Gamma B \subseteq B$$
$$\Rightarrow a \in B$$

Thus B is semiprime.

Conversely, let $a \in S$. Then $a\Gamma a\Gamma S\Gamma a\Gamma a$ is a non-empty subset of S. Let $x, y \in (a\Gamma a)\Gamma S\Gamma(a\Gamma a)$ and $z \in S$. Then for some $s, t \in S$ and $\alpha, \beta, \gamma, \rho, \tau, \eta, \mu \in \Gamma$,

$$x\alpha z\beta y = (a\gamma a\rho u\tau a\gamma a)\alpha z\beta(a\gamma a\eta v\mu a\gamma a)$$
$$= a\gamma a\rho(u\tau a\gamma a\alpha z\beta a\gamma a\eta v)\mu a\gamma a \subseteq (a\Gamma a)\Gamma M\Gamma(a\Gamma a)$$

Thus $x\alpha z\beta y \subseteq a\Gamma a\Gamma S\Gamma a\Gamma a$. Then

$$((a\Gamma a)\Gamma S\Gamma(a\Gamma a))\Gamma((a\Gamma a)\Gamma S\Gamma(a\Gamma a)) \subseteq ((a\Gamma a)\Gamma S\Gamma(a\Gamma a))$$
$$\text{and}((a\Gamma a)\Gamma S\Gamma(a\Gamma a))\Gamma S\Gamma((a\Gamma a)\Gamma S\Gamma(a\Gamma a)) \subseteq (a\Gamma a)\Gamma S\Gamma(a\Gamma a)$$

Hence $(a\Gamma a)\Gamma S\Gamma(a\Gamma a)$ is a bi-Γ-hyperideal of S for all $a \in S$. Since

$$a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a = (a\Gamma a)\Gamma(a\Gamma a\Gamma a\Gamma a\Gamma a)\Gamma(a\Gamma a) \subseteq ((a\Gamma a)\Gamma S\Gamma(a\Gamma a))$$

and $(a\Gamma a)\Gamma S\Gamma(a\Gamma a)$ is semiprime, we get $a\Gamma a\Gamma a\Gamma a\Gamma a$, $a\Gamma a \subseteq a\Gamma a\Gamma S\Gamma a\Gamma a$ and so $a \in (a\Gamma a)\Gamma S\Gamma(a\Gamma a)$. Hence by Lemma 3.7, S is a completely regular $\Gamma-$semihypergroup.

Theorem 3.9 *For a* $\Gamma-$*semihypergroup* S *,the following assertions are equivalent:*

(i) S *is both regular and intra-regular.*

(ii) $B\Gamma B = B$ *for every bi-*$\Gamma-$*hyperideal* B *of* S.

(iii) $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$ *for all bi-*$\Gamma-$*hyperideals* B_1 *and* B_2 *of* S.

(iv) *Each bi-*$\Gamma-$*hyperideal of S is semiprime.*

(v) *Each proper bi-*$\Gamma-$*hyperideal of* S *is the intersection of irreducible semiprime bi-*$\Gamma-$*hyperideals of* S *which contain it.*

Proof. $(i) \Leftrightarrow (ii)$ It follows by Theorem 3.8 and [24, Cor. 9.6 and Cor. 9.1]

$(ii) \Rightarrow (iii)$ Let B_1 and B_2 be any two bi-$\Gamma-$hyperideals of the $\Gamma-$semihypergroup S. Then by our hypothesis,

$$B_1 \cap B_2 = (B_1 \cap B_2)\Gamma(B_1 \cap B_2)$$

$$B_1 \cap B_2 \subseteq B_1\Gamma B_2$$

Similarly $\quad B_1 \cap B_2 \subseteq B_2\Gamma B_1$

Thus

$$B_1 \cap B_2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B_1 \qquad (1)$$

Now $B_1\Gamma B_2$ and $B_2\Gamma B_1$ are bi-$\Gamma-$hyperideals being the products of bi-$\Gamma-$hyperideals. Also, $B_1\Gamma B_2 \cap B_2\Gamma B_1$ is a bi-$\Gamma-$hyperideal. Then

$$B_1\Gamma B_2 \cap B_2\Gamma B_1 = (B_1\Gamma B_2 \cap B_2\Gamma B_1)\Gamma(B_1\Gamma B_2 \cap B_2\Gamma B_1)$$

$$\subseteq (B_1\Gamma B_2)\Gamma(B_2\Gamma B_1)$$

$$\subseteq B_1\Gamma S\Gamma B_1 \subseteq B_1$$

Similarly, $\quad B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B_2$

Thus,

$$B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B_1 \cap B_2 \qquad (2)$$

Hence, from (1) and (2) we have

$$B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$$

$(iii) \Rightarrow (iv)$ Let B_1 and B be bi-$\Gamma-$hyperideals of S such that $B_1\Gamma B_1 \subseteq B$. By hypothesis,

$$B_1 = B_1 \cap B_1 = B_1\Gamma B_1 \cap B_1\Gamma B_1 = B_1\Gamma B_1$$

Thus

$$B_1 \subseteq B$$

Hence every bi-Γ-hyperideal of S is semiprime.

$(iv) \Rightarrow (v)$ Let B be a proper bi-Γ-hyperideal of S. Then B is contained in the intersection of all irreducible bi-Γ- hyperideals of S which contain B. Theorem 3.6 guarantees the existence of such irreducible bi-Γ-hyperideals. If $a \notin B$ then there exists an irreducible bi-Γ-hyperideal of S which contains B but does not contain a. Hence B is the intersection of all bi-Γ-hyperideals of S which contain it. By our hypothesis, every bi-Γ-hyperideal is semiprime, and so each bi-Γ-hyperideal is the intersection of irreducible semiprime bi-Γ-hyperideals of S containing it.

$(v) \Rightarrow (ii)$ Let B be a bi-Γ-hyperideal of S. If $B\Gamma B = S$, then clearly B is idempotent, that is, $B\Gamma B = B$. If $B\Gamma B \neq S$, then $B\Gamma B$ is a proper bi-Γ-hyperideal of S containing $B\Gamma B$ and so by our hypothesis

$$B\Gamma B = \bigcap_{\alpha}\{B_\alpha : B_\alpha \text{isirreduciblesemiprimebi} - \Gamma - \text{hyperidealof}S\}$$

Since each B_α is a semiprime bi-Γ-hyperideal, $B \subseteq B_\alpha$ for all α and so $B \subseteq \bigcap_{\alpha} B_\alpha = B\Gamma B$. Hence each bi-$\Gamma$-hyperideal in S is idempotent.

Theorem 3.10 *Let S be a regular and intra-regular semigroup. Then the following assertions, for a bi-*Γ*-hyperideal* B *of* S*, are equivalent:*

(i) B *is strongly irreducible.*
(ii) B *is strongly prime.*

Proof. Straightforward.

Next we characterize those $\Gamma-$semihypergroups in which each bi-$\Gamma-$hyperideal is strongly prime and also those $\Gamma-$semihypergroups in which each bi-$\Gamma-$hyperideal is strongly irreducible.

Theorem 3.11 *Each bi-*$\Gamma-$*hyperideal of a* $\Gamma-$*semihypergroups* S *is strongly prime if and only if S is regular, intra-regular and the set of bi-*$\Gamma-$*hyperideals of* S *is totally ordered by inclusion.*

Proof. Suppose that each bi-Γ-hyperideal of S is strongly prime. Then each bi-Γ-hyperideal of S is semiprime. Thus by Theorem 3.9, S is both regular and intra-regular. We show that the set of bi-Γ-hyperideals of S is totally ordered. Let B_1 and B_2 be any two bi-Γ-hyperideal of S. Then by Theorem 3.9,

$B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$. As each bi-$\Gamma$-hyperideal is strongly prime, $B_1 \cap B_2$ is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$. If $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$.

Conversely, assume that S is regular, intra-regular and since the set of bi-Γ-hyperideals of S is totally ordered under inclusion. Then we want to show that each bi-Γ-hyperideal of S is strongly prime. Let B be an arbitrary bi-Γ-hyperideal of S and B_1, B_2 be bi-Γ-hyperideals of S such that

$$B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$$

Since S is both regular and intra-regular, by Theorem 3.9,

$$B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$$

Also $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$ implies $B_1 \cap B_2 \subseteq B$. Since the set of bi-Γ-hyperideals of S is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$.

Theorem 3.12 *If the set of bi-Γ-hyperideals of a semigroup S is totally ordered, then S is both regular and intra-regular if and only if each bi-Γ-hyperideal of S is prime.*

Proof. Suppose that S is both regular and intra-regular. Let B be any bi-Γ-hyperideal of S and B_1, B_2 be bi-Γ-hyperideals of S such that $B_1\Gamma B_2 \subseteq B$. Since the set of bi-$\Gamma$-hyperideals of S is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Suppose $B_1 \subseteq B_2$. Then $B_1\Gamma B_1 \subseteq B_1\Gamma B_2 \subseteq B$. By Theorem 3.9, B is semiprime so $B_1 \subseteq B$. Hence B is a semiprime bi-Γ- hyperideal of S.

Conversely, assume that every bi-Γ-hyperideal of S is prime. Since the set of bi-Γ-hyperideals of S is totally ordered so the concepts of prime and strongly prime coincide. Now, by Theorem 3.11, we see that S is both regular and intra-regular.

Theorem 3.13 *For a semigroup S the following assertions are equivalent:*

(i) *The set of bi-Γ-hyperideals of S is totally ordered under inclusion.*
(ii) *Each bi-Γ-hyperideal of S is strongly irreducible.*
(iii) *Each bi-$\Gamma-$hyperideal of S is irreducible.*

Proof. $(i)) \Rightarrow (ii)$ Let B be an arbitrary bi-Γ-hyperideal of S and B_1, B_2

two bi-Γ-hyperideals of S such that $B_1 \cap B_2 \subseteq B$. Since the set of bi-Γ-hyperideals of S is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is strongly irreducible bi-Γ- hyperideal.

$(ii) \Rightarrow (iii)$ Let B be an arbitrary bi-Γ-hyperideal of S and B_1, B_2 two bi-Γ-hyperideals of S such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. That is, B is irreducible bi-Γ-hyperideal.

$(iii) \Rightarrow (i)$ Let B_1 and B_2 be any two bi-Γ- hyperideals of S. Then $B_1 \cap B_2$ is a bi-Γ-hyperideal of S. Also $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Let S be a Γ-semihypergroup, B be the set of all bi-Γ-hyperideals of S and P be the set of all strongly prime proper bi-Γ-hyperideals of S: Define for each $B \in \mathsf{B}$

$$\Theta_B = \{J \in \mathsf{P} : B\text{Ú}J\}$$
$$\Im(\mathsf{P}) = \{\Theta_B : B \quad \text{isabi} - \Gamma - \text{hyperidealof} \quad S\}$$

Theorem 3.14 *If S is a regular and intra-regular semigroup. Then $\Im(\mathsf{P})$ forms a topology on the set P.*

Proof. Since $\{0\}$ is a bi-Γ-hyperideal of S,

$$\Theta_0 = \{J \in \mathsf{P} : \{0\}\text{Ú}J\}$$
$$= \phi \quad \text{because0belongstoeverybi} - \Gamma - \text{hyperidealof} \quad S$$

Also, since S is a bi-Γ-hyperideal of S,

$$\Theta_S = \{J \in \mathsf{P} : S\text{Ú}J\} = \mathsf{P}$$

because P is the collection of all strongly prime proper bi-Γ-hyperideal os S.

Let $\{\Theta_{B_i} : i \in I\} \subseteq \Im(\mathsf{P})$. Then $\bigcup_{i \in I} \Theta_{B_i} = \{J \in \mathsf{P} : B_i\text{Ú}J$ for some $i \in I\} = \{J \in \mathsf{P} : \overset{\wedge}{\bigcup_{i \in I}} B_i \text{Ú} J\} = \Theta_{\overset{\wedge}{\bigcup_{i \in I}} B_i}$, where $\overset{\wedge}{\bigcup_{i \in I}} B_i$ is the bi-Γ-hyperideal of S generated by $\bigcup_{i \in I} B_i$.

Now let Θ_{B_1} and $\Theta_{B_2} \in \Im(\mathrm{P})$. If $J \in \Theta_{B_1} \cap \Theta_{B_2}$, then $J \in \mathrm{P}$ and $B_1 \nsubseteq J$, $B_2 \nsubseteq J$. Suppose $B_1 \cap B_2 \subseteq J$. Since S is both regular and intra-regular, $B_1 \cap B_2 = B_1 \Gamma B_2 \cap B_2 \Gamma B_1$. Hence, $B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq J$. This implies either $B_1 \subseteq J$ or $B_2 \subseteq J$, a contradiction. Consequently, $B_1 \cap B_2 \nsubseteq J$, which implies that $J \in \Theta_{B_1 \cap B_2}$. Thus $\Theta_{B_1} \cap \Theta_{B_2} \subseteq \Theta_{B_1 \cap B_2}$. If $J \in \Theta_{B_1 \cap B_2}$, then we have

$J \in \mathrm{P}$ and $B_1 \cap B_2 \nsubseteq J$

This implies that $B_1 \nsubseteq J, B_2 \nsubseteq J$. Thus $J \in \Theta_{B_1}$ and $J \in \Theta_{B_2}$, and therefore $J \in \Theta_{B_1} \cap \Theta_{B_2}$. Hence $\Theta_{B_1 \cap B_2} \subseteq \Theta_{B_1} \cap \Theta_{B_2}$.

Consequently, $\Theta_{B_1 \cap B_2} = \Theta_{B_1} \cap \Theta_{B_2}$.

This shows that $\Im(\mathrm{P})$ is a topology on P.

R E F E R E N C E S

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