# SOLVING A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS WITH SINC-COLLOCATION METHOD BASED ON DOUBLE EXPONENTIAL TRANSFORMATION

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*Sinc-collocation method based on double exponential transformation for solving nonlinear second order two-point boundary value problems has been developed with nonhomogeneous boundary conditions. Also we developed Sinc method based on single exponential transformations. These methods are tested on several problems. T he obtained results from Sinc collocation based on single and double exponential transformations are compared with each other, and with the existing methods too. The numerical result confirm that these methods are considerably efficient and accurate, and can be applied to singular and regular problems.*

**Keywords:** Sinc-collocation; singular nonlinear boundary value problem; single and double exponential transformation.

**MSC2010:** 65L 60; 34B 16

## 1. Introduction

Nonlinear two point boundary value problems arise in a variety of areas in applied mathematics, theoretical physics, engineering and chemical reaction. These categories of problems have been handled by a reasonable number of researches who are working both numerically and analytically. The majority of these problems cannot be solved analytically, so we have to use the numerical methods. Several techniques are available for approximating these problems [1–16], but there is not an unified method to handle all types of problems.

The Sinc method has been developed for solving linear boundary value problems by F.Stenger, more than thirty years ago [17]. It is well known that the approximation by Sinc methods has the order of accuracy $\mathrm{O}(\exp(-k\sqrt{n}))$, where $k$ is positive constant and $n$ is the number of nodes or bases functions [17–20].

In 1974, Takahasi and Mori proposed a new transformation for the efficient evaluation of integrals of an analytic function with end point singularity, which is called double exponential transformation [21]. Sugihara composed this transformation by Sinc method and he showed that the error in Sinc method based on double exponential transformation is $O(exp(-k'n/logn))$ with some positive $k'$ [22] [23]. After that, Mori, Sugihara and their co-workers extended the Sinc method based

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on double exponential transformation in several fields in applied mathematic such as integral equations [24] initial value problems with ordinary differential equations [25], linear boundary value problems in ordinary differential equations [26], [27], [28] and Green function [29]. Furthermore, they have studied the suitable choice of the classes of functions for double exponential transformation to evaluate the integral formulas [30] and also the choice of the classes of functions for the DE-Sinc approximations [31].

In this paper we consider a class of nonlinear two-point boundary value problem in general form:

$$(1)\qquad \begin{cases} y''(x)+p(x)y'(x)+q(x)y(x)+r(x)N(y(x))=f(x),\ \ a\le x\le b, \\ \\ y(a)=\alpha, \quad\ y(b)=\beta, \end{cases}$$

where $p, q, r$ and $f$ are analytic functions in an open interval $(a, b)$ and possibly singular in $a$ or $b$ or both, and $N(y)$ is analytic function of $y$.

A Special case of (1) when $N(y) = y^n$ with vanishing boundary conditions at $x = a$ and $x = b$ has been considered in [10] and solved by single exponential Sinc function. The Sinc method in [10] mainly based on transformation of the bases function from $(-\infty, \infty)$ to a finite interval $(a, b)$. In this study, we developed the Sinc-collocation methods based on double and single exponential transformations for problem (1). Our approach is based on transformation of problems from interval $[a, b]$ to interval $(-\infty, \infty)$ and used the bases functions on their natural domains $(-\infty, \infty)$.

The paper is organized as follows. In section (2), we introduced Sinc function and reviewed some definitions, theorems and notations. In section (3) we described transformation for converted nonhomogeneous conditions to homogeneous one [18]. In section (4), we developed Sinc-collocation method on the entire interval $(-\infty, \infty)$ for problem of (1) by homogeneous boundary conditions. In section (5) we developed Sinc-collocation method based on double and single exponential transformations on the general interval $[a, b]$. In section (6), we tested our methods on various kinds of boundary value problems. The obtained results were compered with each others and also with the results in the existing methods. Finally, in section (7) the conclusions of the study were given.

## 2. The Sinc function properties and double exponential transformation

In this section, some definitions, notations and theorems from [18] and [23] are recalling.

The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

For $h > 0$ the translated Sinc function with evenly spaced nodes is given by

$$S(k,h)(x) \equiv Sinc(\frac{x-kh}{h}) \equiv \begin{cases} \frac{\sin((\pi/h)(x-kh))}{(\pi/h)(x-kh)}, & x \neq kh \\ 1, & x = kh, \end{cases}$$

where $k = 0, \pm 1, \pm 2, \ldots$ . The $S(k,h)(x)$ is the kth Sinc function with step size $h$ evaluated at $x$. If a function $f(x)$ is defined over the line, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)Sinc(\frac{x-kh}{h}), \tag{2}$$

is called the Whittaker cardinal expansion of $f$ whenever this series converges. The properties of Whittaker cardinal expansion have been extensively studied on [19].
**Definition 2.1** Let $D_d$ denote the infinite strip region with $2d(d > 0)$ in the complex plane:

$$D_d \equiv \{z\epsilon\mathbb{C} \mid |Imz| < d\}, \tag{3}$$

and for $0 < \varepsilon < 1$, let $D_d(\varepsilon)$ be defined by

$$D_d(\varepsilon) \equiv \{z\epsilon\mathbb{C} \mid |Rez| < 1/(\varepsilon), |Imz| < d(1-\varepsilon)\}. \tag{4}$$

Let $H^1(D_d)$ be the Hardy space over the region $D_d$, i.e., the set of functions $f$ analytic in $D_d$ such that

$$\lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(z)||dz| < \infty. \tag{5}$$

The following theorem, due to Sugihara [23], presents the convergence result of Sinc approximation on $(-\infty, \infty)$, which shows that the convergence rate is given by $O(exp(-kn/logn))$.
**Theorem 2.2** [23] Assume, with positive constants $\alpha, \beta, \gamma$ and $d$ that
1) $f$ belongs to $H^1(D_d)$
2) $f$ decays double exponentially on the real line, that is,

$$|f(x)| \leq \alpha \exp(-\beta \exp(\gamma |x|)), \;\; for\ all\ x\epsilon\mathbb{R}, \tag{6}$$

then we have

$$\sup_{-\infty < x < \infty} \Big| \, f(x) - \sum_{j=-N}^{N} f(jh)S(j,h)(x) \Big| \leqslant C \exp\Big[\frac{-\pi d\gamma N}{log(\pi d \gamma N/\beta)}\Big], \tag{7}$$

for some C, where the mesh size $h$ is taken as:

$$h = \frac{log(\pi d\gamma N/\beta)}{\gamma N}. \tag{8}$$

Many problems that arise in applied mathematics do not have the whole real line as their natural domain. There are two points of view. One is to change variables in the problem so that, in the new variables, the problem has a domain corresponding to that of the numerical process. A second procedure is to move the numerical process and to study it on the new domain. The former approach is the method chosen here. The development for transform Sinc method from one domain to another is accomplished via conformal mappings. Approximation can be constructed for infinite, semi-infinite and finite intervals. To construct the approximation on the interval $(a, b)$, for Sinc method, we may consider two maps:

Case 1

$$x = \psi_{SE}(t) = \frac{b-a}{2}\tanh(t/2) + \frac{b+a}{2}, \tag{9}$$

$$t = \phi_{SE}(x) = \psi_{SE}^{-1}(t) = \log(\frac{x-a}{b-x}), \tag{10}$$

(9) is called single exponential (SE) transformation, and when combined by Sinc function we have SE-Sinc approximation. The SE transformation map $\mathbb{R}$ onto $(a, b)$ and map $D_d$ onto the domain

$$\psi_{SE}(D_d) = \{z\epsilon\mathbb{C} : |\arg(\frac{z-a}{b-z})| < d\}, \tag{11}$$

Case 2

$$x = \psi_{DE}(t) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh(t)) + (\frac{b+a}{2}), \tag{12}$$

$$t = \phi_{DE}(x) = \psi_{DE}^{-1}(t) = \log\big[\frac{1}{\pi}\log(\frac{x-a}{b-x}) + \sqrt{1 + \{\frac{1}{\pi}\log(\frac{x-a}{b-x})\}^2}\,\big], \tag{13}$$

which is called double exponential (DE) transformation and the Sinc approximation with DE transformation is called DE-Sinc approximation. Also the DE transformation maps $\mathbb{R}$ onto $(a, b)$ and map $D_d$ onto the domain

$$\psi_{DE}(D_d) = \{z\epsilon\mathbb{C} : \big|\arg(\frac{1}{\pi}\log(\frac{z-a}{b-z}) + \sqrt{1 + \{\frac{1}{\pi}\log(\frac{z-a}{b-z})\}^2})\big| < d\}, \tag{14}$$

By DE transformation, the convergence of the Sinc approximation on the interval $(a, b)$ is guaranteed by the following theorem

**Theorem 2.3** [23] Assume that, for a variable transformation $x = \psi(t)$, the transformed function $f(\psi(t))$ satisfies assumptions 1 and 2 in Theorem 2.2 with some positive constants $\alpha, \beta, \gamma$ and $d$. Then we have:

$$\sup_{a<x<b} \big|\, f(x) - \sum_{j=-N}^{N} f(\psi(jh))S(j,h)(\psi^{-1}(x))\big| \leqslant C\exp\big[\frac{-\pi d\gamma N}{log(\pi d\gamma N/\beta)}\big], \tag{15}$$

for some C, where the mesh size $h$ is taken as:

$$h = \frac{log(\pi d\gamma N/\beta)}{\gamma N}. \tag{16}$$

The collocation method requires derivatives of Sinc function evaluated at the node so we need to use the following lemma.

**Lemma 2.4** [18] Let $S(j, h)(x)$ is the kth Sinc function with step size $h$, so

$$\delta_{jk}^{(0)} = S(j,h)(kh) = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{dz}[S(j,h)(z)](kh) = \begin{cases} 0, & j = k \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{dz^2}[S(j,h)(z)](kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of the discrete system, it is convenient to define the following matrices:

$$I^{(l)} = [\delta_{jk}^{(l)}] \; l = 0, 1, 2, \tag{17}$$

the matrix $I^{(0)}$ is the $m \times m$ identity matrix, the matrix $I^{(1)}$ is the $m \times m$ and skew symmetric Toeplitz matrix, and $I^{(2)}$ is the $m \times m$ and symmetric Toeplitz matrix.

## 3. **Treatment of boundary conditions**

Before illustrating Sinc-collocation methods, we have to emphasize the treatment of nonhomogeneous boundary conditions. The nonhomogeneous conditions require a slight modification in the form of approximate solutions, so we can proceed in one of two ways

1-Converting the differential equation (1) to the new one with homogeneous conditions by changing the variable.

2-Proceeding directly from the approximate solution to another one, which satisfies the nonhomogeneous conditions.

In this paper we prefer the choice 1, so we define the following function :

$$\Gamma(x) = Ax + B, \tag{18}$$

where A and B are real constants

$$A = \frac{\beta - \alpha}{b - a} \; , \; B = \frac{b\alpha - a\beta}{b - a}, \tag{19}$$

now, by using the following change of variable:

$$v(x) = y(x) - \Gamma(x). \tag{20}$$

The problem (1) can be converted as the following form:

$$\begin{cases} v''(x) + p(x)v'(x) + q(x)v(x) + r(x)N(v(x) + \Gamma(x)) = \widehat{f}(x), \\ v(a) = 0, \; v(b) = 0, \end{cases} \tag{21}$$

where $\widehat{f} = f - (\Gamma'' + p\Gamma' + q\Gamma)$.

## 4. **Sinc-collocation method on the real line**

Consider the following nonlinear two-point boundary value problem on the interval $(-\infty, \infty)$

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) + r(x)N(u(x) + \Gamma(x)) = f(x), \\ \lim_{x \longrightarrow \pm\infty} u(x) = 0. \end{cases} \tag{22}$$

To solve this problem, we assume the approximate solution $u_m(x)$ of the form

$$u_m(x) = \sum_{k=-M}^{M} c_k S_k(x), \; m = 2M + 1, \tag{23}$$

where the $2M + 1$ coefficients $\{c_k\}_{k=-M}^{M}$ are unknowns. Notice that $u_m(x)$ satisfies the boundary conditions in (22).

To determine the coefficients of $c_k$'s in (23), we have to substituting $u_m(x)$, and its first and second derivatives into (22) and then by replacing $x$ by the collocation points $x_j = jh$, $j = -M, -M+1, ..., M$ in which $h$ is defined in Theorem 2.3, we obtain the following nonlinear system:

$$\sum_{k=-M}^{M} c_k\big(\frac{d^2}{dx^2}S_k(x_j) + p(x_j)\frac{d}{dx}S_k(x_j) + q(x_j)S_k(x_j)\big)$$

$$+\, r(x_j)N(\sum_{k=-M}^{M} c_k S_k(x_j) + \Gamma(x_j)) = f(x_j), \quad j = -M, \cdots, M \tag{24}$$

by using lemma 2.4 and we know that $\delta^{(0)}_{jk} = \delta^{(0)}_{kj}$, $\delta^{(1)}_{jk} = -\delta^{(1)}_{kj}$ and $\delta^{(2)}_{jk} = \delta^{(1)}_{kj}$, we can rewrite the above system in the following form

$$\sum_{k=-M}^{M} c_k\big(\frac{1}{h^2}\delta^{(2)}_{jk} - p(x_j)\frac{1}{h}\delta^{(1)}_{jk} + q(x_j)\delta^{(0)}_{jk}\big)$$

$$+\, r(x_j)N(c_j + \Gamma(x_j)) = f(x_j), \quad j = -M, \cdots, M. \tag{25}$$

To write down the above system in Matrix-vector form, we need the notation of 'D' which is defined as follows. For any function of $g(x)$ we define the $m \times m$ diagonal matrix:

$$D(g) = diag\big(g(x_{-M}), \cdots, g(x_0), \cdots, g(x_M)\big). \tag{26}$$

By this notation and notice to the notation of $I^{(l)}$ in lemma 2.4 we may write the system of (25) in the following Matrix-vector form system

$$A\ C\ +\ B\ N(C + \Gamma) =\ H, \tag{27}$$

where

$$A = \frac{1}{h^2}I^{(2)} - \frac{1}{h}D(p)I^{(1)} + D(q)I^{(0)}, \tag{28}$$

$$B = D(r), \tag{29}$$

$$N(C + \Gamma) = (N(c_{-M} + \Gamma(x_{-M})), ..., N(c_M + \Gamma(x_M)))^T, \tag{30}$$

$$H = (f(x_{-M}), f(x_{-M+1}), ..., f(x_M))^T, \tag{31}$$

$$C = (c_{-M}, c_{-M+1}, .., c_M)^T, \tag{32}$$

we can solve (27) by using Newton's method.

## 5. **Sinc-collocation method on the interval** $(a, b)$

To applied Sinc-collocation method in the problem on interval $(a, b)$, first of all we translate the problem onto interval $(-\infty, \infty)$ by SE or DE transformation, described in pervious section.

Consider the problem (21) as follow:

$$\begin{cases} v''(x) + p(x)v'(x) + q(x)v(x) + r(x)N(v(x) + \Gamma(x)) = \widehat{f}(x), \\ v(a) = 0, \quad v(b) = 0, \end{cases} \tag{33}$$

suppose an appropriately selected variable transformation $x = \psi(t)$ so that

$$\psi : (-\infty, \infty) \longrightarrow (a, b). \tag{34}$$

By this transformation the problem (33) is transformed into following problem on $(-\infty, \infty)$

$$(35) \qquad \begin{cases} u''(t) + (p(\psi(t))\psi'(t) - \frac{\psi''(t)}{\psi'(t)})u'(t) + (\psi'(t))^2 q(\psi(t))u(t) \\ +(\psi'(t))^2 r(\psi(t))N(u(t) + \Gamma(\psi(t))) = \psi'(t)^2 \widehat{f}(\psi(t)), \\ \lim_{t\longrightarrow\pm\infty} u(t) = 0, \end{cases}$$

where $v(\psi(t)) = u(t)$.

Now we have a problem which is defined on the interval $(-\infty, \infty)$. To solve this problem, by considering the previous section 4, and the Sinc-collocation method to the problem (35), the arising nonlinear system can be obtained as

$$(36) \qquad A\ C \ + \ B\ N(C + \Gamma) = \ H,$$

where

$$(37) \qquad A = \frac{1}{h^2} I^{(2)} - \frac{1}{h} D\Big(p(\psi)\psi' - \frac{\psi''}{\psi'}\Big) I^{(1)} + D\Big(q(\psi)(\psi')^2\Big) I^{(0)},$$

$$(38) \qquad B = D\Big((\psi')^2 r(\psi)\Big),$$

$$(39) \qquad N(C + \Gamma) = \Big(N(c_{-M} + \Gamma(\psi(-Mh))), ..., N(c_M + \Gamma(\psi(Mh)))\Big)^T,$$

$$(40) \qquad H = D\Big((\psi')^2)(\widehat{f}(\psi(-Mh)), ..., \widehat{f}(\psi(Mh))\Big)^T,$$

$$(41) \qquad C = \Big(c_{-M}, c_{-M+1}, .., c_M\Big)^T,$$

by solving the above system and determine the coefficients $C$, we can find the approximate solution $u_m(x)$ defined in (23).

In the transformed problem (35), if we use $\psi_{SE}$ from (9) the obtained method is SE-Sinc collocation, also if we choose $\psi_{DE}$ from (12) the resulting method is DE-Sinc collocation. The formulas required for solving the transformed differential equation (35) can be defined as follow:

$$(42) \qquad \psi'_{SE}(t) = \Big(\psi_{SE}(t) - a\Big)\Big(b - \psi_{SE}(t)\Big),$$

$$(43) \qquad \psi'_{DE}(t) = \frac{\pi}{b-a}\cosh(t)\Big(\psi_{DE}(t) - a\Big)\Big(b - \psi_{DE}(t)\Big),$$

$$(44) \qquad \frac{\psi''_{SE}(t)}{\psi'_{SE}(t)} = -2\psi_{SE}(t) + (a + b),$$

$$(45) \qquad \frac{\psi''_{DE}(t)}{\psi'_{DE}(t)} = \frac{\pi}{b-a}\cosh(t)\Big(-2\psi_{DE}(t) + (a + b)\Big) + \tanh(t).$$

## 6. **Numerical Results**

In this section, we consider five test problems. We applied the DE-Sinc-collocation problems with $\beta = \frac{\pi}{2}$, $\gamma = 1$ and $d = \frac{\pi}{4}$ on problems of 1-4, and $\beta = \pi$, $\gamma = 1$ and $d = \frac{\pi}{4}$ on problem 5, different $Ms$ are chosen and $h$ can be determined by $h = \frac{log(\pi d\gamma M/\beta)}{\gamma M}$, also SE-Sinc-collocation is applied with $\gamma = \frac{1}{2}$, $d = \frac{\pi}{2}$ on problems of 1-4, and $\gamma = 1$, $d = \frac{\pi}{2}$ on problem 5 with $h = \sqrt{\frac{\pi d}{\gamma M}}$ [19].

The absolute numerical errors are checked on 999 equally-spaced points $U$ defined as:

$$U = \{z_1 = a, z_2, ..., z_{999} = b\}, \tag{46}$$

$$z_k = a + kh_u, \qquad h_u = \frac{b-a}{1000}, \qquad k = 1, 2, ..., 999. \tag{47}$$

For solving the nonlinear system of (36) by Newton's method, we start with an initial guess $C_0 = \overrightarrow{0}$ and use the Newton iteration as follow:

$$C_{j+1} = C_j - J^{-1}(C_j)\ F(C_j), \tag{48}$$

where

$$F(C_j) = A\ C_j\ + B\ N(C_j + \Gamma) -\ H, \tag{49}$$

$$J(C_j) = A +\ B\ D(N'(C_j + \Gamma)), \tag{50}$$

where $A, B, H, C$ and $N$ are defined in $(37) - (41)$ and $N'$ is $\frac{\partial}{\partial C}N$. Here, $C_j$ is the current iterate and $C_{j+1}$ is the new iterate. A common numerical practice is to stop the Newton iteration wherever the distance between two successive iterates is less than a given tolerance , i.e., when $\|C_{j+1} - C_j\| \leq \varepsilon$ where the Euclidean norm is used.

The computation developed on personal computer with 1 GIG memory by MATLAB.
At first, in all examples we used transformation of (20) to convert the nonhomogeneous boundary conditions to the homogeneous one and then used the DE-Sinc-collocation method based on Section5 .

**Problem 1** Consider the following boundary value problem [6]:

$$\begin{cases} y''(x) + \frac{1}{x}y'(x) + \frac{1}{x(1-x)}y(x) + x\sin\sqrt{y(x)} = f(x), \\ y(0) = 1\ , y(1) = e, \end{cases} \tag{51}$$

in which $f(x) = \frac{1}{x(1-x)}[-(\exp(x)(-2 + x^2)) - (-1 + x)x^2 \sin\sqrt{\exp(x)}\ ]$ , with the exact solution $y(x) = e^x$.

By using the transformation of (21) where $\Gamma(x) = (e-1)x + 1$, the above problem has been transformed to the following homogeneous BVP:

$$\begin{cases} v''(x) + \frac{1}{x}v'(x) + \frac{1}{x(1-x)}v(x) + x\sin\sqrt{v(x) + (e-1)x + 1} = \widehat{f}(x), \\ v(0) = 0\ , v(1) = 0, \end{cases} \tag{52}$$

where
$\widehat{f}(x) = \frac{1}{x(1-x)}[-(\exp(x)(-2+x^2)) - (-1+x)x^2 \sin\sqrt{\exp(x)}\ ] - \frac{1}{x}(e-1) - \frac{1}{x(1-x)}((e-1)x + 1)$, and The exact solution is $v(x) = e^x - (e-1)x - 1$. This example has been solved by Sinc collocation method based on DE and SE transformations. The absolute errors in the solution with our method for $M = 30, M = 50$, method of Sinc collocation based on single exponential transformation and method of combining the homotopy and reproducing Kernel Hilbert space (HRKH) [6] at specified points from [6] are tabulated in Table (1). Also Table (2) shows the maximum absolute error in the solution of Sinc collocation based on DE and Sinc collocation based on SE for different $Ms$ on uniform points $U$ which has been introduced in (46).

**Problem 2** Consider the following singular nonlinear BVP:

Table 1. Absolute error in the solution of Problem 1

| | | SE-Sinc collocation | | DE-Sinc collocation | |
|---|---|---|---|---|---|
| $x$ | HRKH [6] | $M = 30$ | $M = 50$ | $M = 30$ | $M = 50$ |
| 0.001 | $4.04E-07$ | $2.28E-07$ | $2.21E-09$ | $2.58E-11$ | $3.32E-13$ |
| 0.08 | $2.39E-06$ | $1.35E-07$ | $4.19E-09$ | $2.75E-11$ | $3.16E-13$ |
| 0.16 | $2.91E-06$ | $1.53E-07$ | $4.02E-09$ | $2.56E-11$ | $2.90E-13$ |
| 0.32 | $1.84E-06$ | $3.64E-07$ | $3.92E-09$ | $2.09E-11$ | $2.36E-13$ |
| 0.48 | $3.10E-06$ | $8.39E-08$ | $1.01E-09$ | $1.65E-11$ | $1.82E-13$ |
| 0.64 | $5.53E-06$ | $1.22E-07$ | $2.11E-11$ | $1.18E-11$ | $1.26E-13$ |
| 0.80 | $4.03E-06$ | $1.82E-07$ | $3.37E-10$ | $6.78E-12$ | $7.05E-14$ |
| 0.96 | $7.46E-07$ | $4.10E-09$ | $4.25E-11$ | $1.50E-12$ | $1.40E-14$ |

Table 2. Maximum absolute error in the solution of Problem 1

| $M$ | SE-Sinc collocation | DE-Sinc collocation |
|---|---|---|
| 2 | $1.4328E-01$ | $3.3225E-02$ |
| 5 | $5.0222E-03$ | $5.4528E-03$ |
| 10 | $1.6886E-04$ | $1.2663E-04$ |
| 20 | $6.6625E-06$ | $1.8183E-07$ |
| 30 | $4.1274E-07$ | $2.9494E-11$ |
| 40 | $3.7595E-08$ | $5.6031E-13$ |
| 50 | $4.3333E-09$ | $3.1922E-13$ |

$$\left\{ \begin{array}{l} y''(x) + \frac{1}{x}y'(x) + \frac{1}{1-x}y(x) + y^2 = f(x), \\ y(0) = 0\ , y(1) = 1, \end{array} \right. \tag{53}$$

where $f(x) = \frac{1}{(1-x)}[\ 4 - 4x + x^2 + x^4 - x^5\ ]$, and the exact solution is $y(x) = x^2$.

By applying the transformation of (21) it is easy to see that $\Gamma(x) = x$ and we will obtain the following homogeneous BVP:

$$\left\{ \begin{array}{l} v''(x) + \frac{1}{x}v'(x) + \frac{1}{(1-x)}v(x) + (v(x) + x)^2 = \widehat{f}(x), \\ v(0) = 0\ , v(1) = 0, \end{array} \right. \tag{54}$$

in which $\widehat{f}(x) = \frac{1}{(1-x)}[\ 4 - 4x + x^2 + x^4 - x^5\ ] - \frac{1}{x} - \frac{x}{(1-x)}$ , and the exact solution is $v(x) = x^2 - x$.

The absolute error in the solution of this problem by method of Sinc collocation based on double and single exponential with $M = 30, M = 50$, method of combining the homotopy and reproducing Kernel Hilbert space (HRKH) [6], and method of reproducing Kernel space (RKS) [3] at specified points from [6] are tabulated in Table (3), also Table (4) shows the maximum absolute error of DE-Sinc collocation

Table 3. Absolute error in the solution of Problem 2

| | | | SE-Sinc collocation | | DE-Sinc collocation | |
|---|---|---|---|---|---|---|
| $x$ | RKS | HRKH | $M=30$ | $M=50$ | $M=30$ | $M=50$ |
| 0.08 | 0.004 | $1.91E-06$ | $9.10E-08$ | $1.41E-09$ | $2.90E-13$ | $2.65E-15$ |
| 0.16 | 0.004 | $1.54E-06$ | $2.49E-08$ | $9.90E-10$ | $3.16E-13$ | $2.55E-15$ |
| 0.24 | 0.009 | $1.58E-06$ | $2.28E-09$ | $6.73E-10$ | $3.29E-13$ | $2.60E-15$ |
| 0.32 | 0.014 | $1.59E-06$ | $1.06E-07$ | $6.91E-10$ | $3.09E-13$ | $2.33E-15$ |
| 0.48 | 0.020 | $1.20E-06$ | $3.02E-08$ | $1.19E-10$ | $3.07E-13$ | $2.08E-15$ |
| 0.64 | 0.021 | $3.96E-07$ | $5.09E-08$ | $5.43E-10$ | $2.72E-13$ | $1.47E-15$ |
| 0.80 | 0.017 | $6.07E-08$ | $8.07E-08$ | $6.24E-10$ | $2.20E-13$ | $1.13E-15$ |
| 0.96 | 0.004 | $8.13E-09$ | $1.31E-07$ | $7.23E-10$ | $1.57E-13$ | $2.70E-16$ |

Table 4. Maximum absolute error in the solution of Problem 2

| $M$ | SE-Sinc collocation | DE-Sinc collocation |
|---|---|---|
| 2 | $9.2150E-02$ | $2.2218E-02$ |
| 5 | $3.3927E-03$ | $2.2740E-03$ |
| 10 | $1.6926E-04$ | $2.2252E-05$ |
| 20 | $3.1596E-06$ | $2.3175E-09$ |
| 30 | $1.6245E-07$ | $3.4868E-13$ |
| 40 | $1.3287E-08$ | $9.3536E-15$ |
| 50 | $1.4108E-09$ | $2.0817E-15$ |

method and SE-Sinc collocation for different $Ms$ on uniform points $U$ which has been introduced in (46).

**Problem 3** We consider following two-point singular B.V.P arising in Astronomy:

$$\begin{cases} y''(x)+\frac{2}{x}y'(x)+y^5=0, \\ y(0)=1\ ,y(1)=\frac{\sqrt{3}}{2}, \end{cases} \tag{55}$$

with exact solution
$y(x)=\frac{1}{\sqrt{1+\frac{x^2}{3}}}$.

By using the transformation of (21) with $\Gamma(x)=(\frac{\sqrt{3}}{2}-1)x+1$, can be converted to the original problem as :

$$\begin{cases} v''(x)+\frac{2}{x}v'(x)+(v(x)+(\frac{\sqrt{3}}{2}-1)x+1)^5=-\frac{2}{x}(\frac{\sqrt{3}}{2}-1), \\ v(0)=0\ ,v(1)=0, \end{cases} \tag{56}$$

This problem has been considered by several authors [2], [4], [8], [10], [11], [12], [13]. The Absolute error in the solution for several method is reported in Table

TABLE 5. Absolute error in the solution of Problem 3

| | | | SE-Sinc collocation | | DE-Sinc collocation | |
|---|---|---|---|---|---|---|
| $x$ | Spline[11] | FD[11] | $M=30$ | $M=50$ | $M=30$ | $M=50$ |
| 0.125 | $1.29E-06$ | $9.59E-05$ | $1.43E-08$ | $4.25E-10$ | $1.20E-11$ | 0.00 |
| 0.250 | $1.10E-06$ | $7.33E-05$ | $3.45E-08$ | $8.71E-10$ | $1.17E-11$ | $1.11E-16$ |
| 0.375 | $8.26E-07$ | $5.55E-05$ | $4.35E-08$ | $2.95E-10$ | $3.87E-13$ | $1.11E-16$ |
| 0.500 | $5.28E-07$ | $3.89E-05$ | $2.70E-08$ | $2.48E-10$ | $4.09E-12$ | 0.00 |
| 0.625 | $2.61E-07$ | $3.41E-05$ | $1.57E-08$ | $7.64E-10$ | $7.75E-12$ | $1.11E-16$ |
| 0.75 | $7.00E-08$ | $1.29E-05$ | $9.83E-09$ | $1.27E-09$ | $4.88E-14$ | 0.00 |
| 0.875 | $1.93E-08$ | $9.60E-06$ | $7.15E-08$ | $9.69E-10$ | $4.39E-13$ | 0.00 |

TABLE 6. Maximum absolute error in the solution of Problem 3

| $M$ | SE-Sinc collocation | DE-Sinc collocation |
|---|---|---|
| 2 | $2.2111E-02$ | $6.4439E-03$ |
| 5 | $4.8680E-04$ | $4.3040E-04$ |
| 10 | $4.9347E-05$ | $1.2318E-05$ |
| 20 | $1.3314E-06$ | $8.5545E-09$ |
| 30 | $9.3387E-08$ | $1.2231E-11$ |
| 40 | $1.0012E-08$ | $7.0676E-14$ |
| 50 | $1.3924E-09$ | $4.9960E-16$ |

(5) and the maximum absolute error in the solution by DE-Sinc collocation method and SE-Sinc collocation for different $Ms$ on uniform points $U$ are summarized in Table (6).

**Problem 4** We consider the following singular nonlinear boundary value problem:

$$\begin{cases} y''(x) + \frac{x^2}{1-x}y'(x) + xy(x) + \frac{x^2}{1-x}\frac{Ln(y)}{y} = f(x), \\ y(0) = 1 \ , y(1) = e, \end{cases} \tag{57}$$

where $f(x) = \frac{1}{1-x}[\exp(x) + x^3\exp(-x)]$, and the exact solution is $y(x) = e^x$. In this case $\Gamma(x) = (e-1)x + 1$, so by using the transformation of (20) we have following homogenous BVP:

$$\begin{cases} v''(x) + \frac{x^2}{1-x}v'(x) + xv(x) + \frac{x^2}{1-x}\frac{Ln(v(x)+(e-1)x+1)}{v(x)+(e-1)x+1} = \widehat{f}(x), \\ v(0) = 0 \ , v(1) = 0, \end{cases} \tag{58}$$

which is $\widehat{f}(x) = \frac{1}{1-x}[\exp(x) + x^3\exp(-x)] - \frac{x^2}{1-x}(e-1) - x + (1-e)x^2$, The maximum absolute errors in the solution by DE-Sinc collocation method and SE-Sinc collocation for different $Ms$ on uniform points $U$ are reported in Table (7).

**Problem 5** Consider another nonlinear boundary value problem:

Table 7. Maximum absolute error in the solution of Problem 4

| $M$ | SE-Sinc collocation | DE-Sinc collocation |
|---|---|---|
| 2 | $4.4748E-02$ | $1.2241E-02$ |
| 5 | $3.1501E-03$ | $1.1947E-03$ |
| 10 | $3.1409E-03$ | $8.4486E-06$ |
| 20 | $3.4273E-06$ | $4.9487E-09$ |
| 30 | $1.8878E-07$ | $1.0772E-12$ |
| 40 | $1.5894E-08$ | $1.6653E-15$ |
| 50 | $1.7452E-09$ | $4.9960E-16$ |

Table 8. Absolute error in the solution of Problem 5

| | | | | SE-Sinc-collocation | | DE-Sinc collocation | |
|---|---|---|---|---|---|---|---|
| x | HPM[1] | EADM[5] | SHM[7] | $M=30$ | $M=50$ | $M=30$ | $M=50$ |
| 0.1 | $0.1E-09$ | $6.9E-07$ | $0.8E-06$ | $4.0E-07$ | $4.2E-11$ | $7.0E-10$ | $1.1E-15$ |
| 0.2 | $0.5E-09$ | $1.3E-06$ | $2.8E-06$ | $3.6E-07$ | $1.8E-09$ | $5.8E-10$ | $4.4E-16$ |
| 0.3 | $0.5E-09$ | $1.9E-06$ | $5.4E-06$ | $1.8E-07$ | $9.7E-10$ | $6.3E-10$ | $1.1E-15$ |
| 0.4 | $0.1E-09$ | $2.3E-06$ | $7.5E-06$ | $6.7E-08$ | $6.5E-10$ | $3.9E-10$ | $1.2E-15$ |
| 0.5 | $0.1E-09$ | $2.5E-06$ | $8.3E-06$ | $1.2E-07$ | $4.2E-10$ | $5.1E-10$ | $1.4E-15$ |
| 0.6 | $0.6E-09$ | $2.3E-06$ | $7.5E-06$ | $6.7E-08$ | $6.5E-10$ | $3.9E-10$ | $1.1E-15$ |
| 0.7 | $0.6E-09$ | $1.9E-06$ | $5.4E-06$ | $1.8E-07$ | $9.4E-10$ | $6.3E-10$ | $7.7E-15$ |
| 0.8 | $0.7E-09$ | $1.3E-06$ | $2.7E-06$ | $3.6E-07$ | $1.8E-09$ | $5.8E-10$ | $2.7E-16$ |
| 0.9 | $0.9E-09$ | $6.9E-06$ | $0.6E-06$ | $4.0E-07$ | $4.2E-11$ | $7.0E-10$ | $1.4E-15$ |

$$\begin{cases} y''(x) - y^2 = f(x), \\ y(0) = 0\ , y(1) = 0, \end{cases} \tag{59}$$

in which $f(x) = 2\pi^2 \cos(2\pi x) - \sin^4(\pi x)$, and the exact solution is $y(x) = \sin^2(\pi x)$.

This problem has been solved by Homotopy perturbation method (HPM) [1], the extended Adomian decomposition method (EADM) [5] and nonlinear Shooting method (SHM) [7]. We solved this problem by SE-Sinc collocation and DE-Sinc collocation methods and compared our numerical result with $M = 30, M = 50$ by results of others methods. The numerical results are tabulated In Table (8).

## 7. Conclusions

We developed Sinc collocation methods based on SE and DE transformation which are rapidly convergent. We applied these methods to solve regular and singular nonlinear boundary value problems subjected to the homogeneous and nonhomogeneous boundary conditions. We tested DE and SE sinc collocation methods on the five different nonlinear problems. The absolute errors in the solution are compared with the methods in [1, 3, 5, 6, 11], and tabulated in the tables. The results in these tables verified that our methods are more accurate in comparison with the methods in [1, 3, 5, 6, 11].

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