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(F, g, η)-Convex submanifolds

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Rezultatele din această lucrare provin din studiul subvarietăţilor Riemanniene convexe, utilizând instrumente specifice invexităţii. Considerând o structură Riemanniană g, o aplicaţie de împerechere η şi un câmp vectorial F care satisface o anumită condiţie de ortogonalitate, definim şi studiem subvarietăţile Riemanniene (F, g, η)-convexe şi demonstrăm faptul ca acestea generalizează hipersuprafeţele convexe şi subvarietăţile H-convexe. Principalul rezultat obţinut de noi constă în demonstrarea existenţei unei schimbări de metrică şi a unei transformări a câmpului vectorial F induse de η, faţă de care subvarietatea devine convexă.

The results in this paper are coming from the study of convex Riemannian submanifolds using tools specific to invexity. Considering a Riemannian structure g on a differentiable manifold, a pairing map η and a vector field F satisfying a certain condition of orthogonality, we define and we study the (F, g, η)-convexity of the Riemannian submanifolds, and we prove that this kind of convexity is a generalization for the Riemannian convexity of hypersurfaces or for the Riemannian H-convexity of submanifolds. The main result of this study consists in proving that there is a change of metric and a change of the vector field F, induced by η, such that the submanifold becomes Riemannian convex, relative to the new objects.

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1. Tools for Riemannian convexity of submanifolds

The convexity of hypersurfaces in Riemannian setting has been defined and studied in [1]. Three mathematical ingredients were needed for this: the

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Riemannian structure of the manifold, the normal vector field of the hypersurface and the exponential map. At first sight, because of the connection between the metric and the exponential map, it seems redundant to make reference to both of them. In fact, we will prove that each of this two elements plays a specific role in convexity. In [2] and [3] the convexity is extended to arbitrary Riemannian submanifolds. The main idea was to replace the normal vector field from hypersurfaces with the mean curvature vector field. The conclusion is that we can study the (F, g)-convexity of a submanifold for each Riemannian structure g and each normal vector field F.

In the following two sections, we recall these definitions, together with some properties of the convex submanifolds.

The papers ([4]-[10]) deal with the concept of invexity. We take over the idea of replacing the local inverse of the exponential map by a vectorial map generated by pairs of points (called *pairing map*) and we use this substitute to define the (F, g, η)-convexity of a submanifold at a point x. We make an important assumption, that is, η must be locally antisymmetric via the parallel transport along geodesics. This time, $F(x)$ needs to be normal to a vectorial space induced by the given submanifold and by η. In Section 4, we prove that, by replacing the metric and the vector field F with some new elements controlled by η, we obtain an equivalence between the old concept of convexity (involving the exponential map) and the new one (involving the pairing map η). We also give some descriptions of the (F, g, η)-convex submanifolds, using the second fundamental form.

In the last section, we point out the main ideas.

2. Riemannian convex hypersurfaces

We recall the results related to convex hypersurfaces obtained in [1]. We consider an n-dimensional Riemannian manifold (M, g) and an oriented hypersurface $N \subset M$, endowed with the induced metric. We also denote by ξ the normal vector field on N. For a fixed point $x \in N$, we consider a neighborhood V_x of x in M such that the exponential map $exp_x : T_xM \to V_x$ is a diffeomorphism. The function

$$f : V_x \to R,\ f(y) = g(exp_x^{-1}(y), \xi(x)) \tag{1}$$

determines the totally geodesic hypersurface

$$TGH_x = \{y \in V_x |\ f(y) = 0\}, \tag{2}$$

and the subsets

$$TGH_x^- = \{y \in V_x |\ f(y) \leq 0\},\ TGH_x^+ = \{y \in V_x |\ f(y) \geq 0\}. \qquad (3)$$

We also consider the bilinear form

$$\Omega_x : T_xN \times T_xN \to R,\ \Omega_x(X, Y) = g(h(X, Y), \xi(x)), \qquad (4)$$

where h denotes the second fundamental form of the hypersurface.

We recall below the definition and the main properties of convex hypersurfaces [1].

Definition 1 *The hypersurface N is called (ξ, g)-convex at $x \in N$ if there is an open neighborhood $U \subseteq V_x \subset M$ of x such that $U \cap N \subseteq TGH_x^-$ or $U \cap N \subseteq TGH_x^+$.*

The hypersurface N is called strictly (ξ, g)-convex at x if it is convex and $N \cap U \cap TGH_x = \{x\}$.

Theorem 1 *If N is an oriented hypersurface in (M, g), (ξ, g)-convex at x, then the bilinear form Ω is positive or negative semidefinite. If Ω is positive or negative definite, then N is strictly (ξ, g)-convex at x.*

Theorem 2 *If the Riemannian manifold (M, g) supports a function with positive definite Hessian, then, for each compact and oriented hypersurface N of M, there is a point $x \in N$ such that Ω_x is definite.*

Theorem 3 *If the Riemannian manifold (M, g) supports a function with positive definite Hessian, then*

1. *there are no minimal compact hipersurfaces in M;*

2. *if a hypersurface N is connected and compact, with the Gauss curvature nowhere zero, then N is strictly (ξ, g)-convex.*

Theorem 4 *If $\varphi : M \to R$ is a differentiable function with positive definite Hessian, then its constant level hypersurfaces, which do not contain any critical points of φ, are strictly $(\operatorname{grad} \varphi, g)$-convex.*

3. Riemannian H-convex submanifolds

In this section, we recall the theory of Riemannian H-convex submanifolds (see [2], [3]). Again, (M,g) is an n-dimensional Riemannian manifold, N is an m-dimensional submanifold endowed with the induced metric and H denotes the mean curvature vector field on N. If $x \in M$ is a fixed point such that $H(x) \neq 0$ and V_x is a neighborhood of x as above, we define the function

$$f : V_x \to R,\ f(y) = g(exp_x^{-1}(y), H(x)) \tag{5}$$

and the bilinear form

$$\Omega_x : T_xN \times T_xN \to R,\ \Omega_x(X,Y) = g(h(X,Y), H(x)), \tag{6}$$

where h denotes again the second fundamental form of N.

We add the subsets

$$TGH_x^- = \{y \in V_x|\ f(y) \leq 0\},\ TGH_x^+ = \{y \in V_x|\ f(y) \geq 0\}$$

and

$$TGH_x = \{y \in V_x|\ f(y) = 0\}.$$

Definition 2 *The submanifold N is called (H,g)-convex at $x \in N$ if there is an open neighborhood $U \subset V_x$ of x in M such that $U \cap N \subseteq TGH_x^-$ or $U \cap N \subseteq TGH_x^+$.*

The Riemannian submanifold N is called strictly (H,g)-convex at x if N is (H,g)-convex and $N \cap U \cap TGH_x = \{x\}$.

Theorem 5 *If the submanifold N is (H,g)-convex at $x \in N$, then the bilinear form Ω_x is positive or negative semidefinite. If the bilinear form Ω_x is definite, then N is strictly (H,g)-convex at x.*

Theorem 6 *If (M,g) is a Riemannian manifold and $c : I \to M$ is a differentiable curve having nonzero mean curvature, then c is a strictly (H,g)-convex submanifold of M.*

Remark. We can define and study in a similar way the (F,g)-*convex submanifolds*, where F denotes an arbitrary normal vector field of the investigated submanifold. The following result holds.

Theorem 7 *If F is a normal vector field of a submanifold N such that N is (F,g)-convex at $x \in N$, then the bilinear form*

$$\Omega_x : T_xN \times T_xN \to R,\ \Omega_x(X,Y) = g(h(X,Y), F_x), \tag{7}$$

is semidefinite. If the bilinear form Ω_x is definite, then N is strictly (F,g)-convex at x.

Remark. The normal vector field F and the Riemannian structure play an equally important role in convexity. In order to enforce this idea we use the term of (F,g)-convexity, especially if we need to specify the Riemannian structure that creates the convexity. Furthermore, in the following section we prove that there is also a third parameter involved in convexity, that is, the local inverse of the exponential map. By taking variations for this third geometric parameter, we define and study next the (F,g,η)-convexity.

4. (F,g,η)-Convex submanifolds

In this section (M,g) continues to be an n-dimensional Riemannian manifold, N is an m-dimensional Riemannian submanifold, and V_x is a neighborhood of x such that $exp_x : T_xM \to V_x$ is a diffeomorphism.

Definition 3 *A vectorial map*

$$\eta : M \times M \to TM,\ \eta(y,x) \in T_xM \tag{8}$$

is called pairing map.

From now on, we suppose that the restriction

$$\eta^{(x)} : V_x \to T_xM,\ \eta^{(x)}(y) = \eta(y,x)$$

is a diffeomorphism.

Definition 4 *The pairing map $\eta : M \times M \to TM$ is called locally antisymmetric via the parallel transport if, for each point $x \in M$ there is a neighborhood V such that, for each point $y \in V$ and each geodesic connecting x and y, we have*

$$T_y\eta(y,x) = -\eta(x,y),\ \forall x,y \in M, \tag{9}$$

where $T_y\eta(y,x)$ is the parallel transport of $\eta(y,x)$ from x to y along that geodesic.

Remark. For each point $x \in M$ and a neighborhood V_x of x such that $exp_x : T_xM \to V_x$ is a diffeomorphism, we denote by $\eta_0^{(x)}$ the map defined by

$$\eta_0^{(x)} : V_x \to T_xM,\ \eta_0^{(x)}(y) = exp_x^{-1}(y). \tag{10}$$

Furthermore, we consider an arbitrary extension $\eta_0 : M \times M \to TM$ of the previous map, satisfying $\eta_0(y,x) = \eta_0^{(x)}(y)$, $\forall x \in M,\ \forall y \in V_x$. Then η_0 is an example of a locally antisymmetric pairing map.

If $\eta : M \times M \to TM$ is a locally antisymmetric pairing map on M and $x \in M$ is a fixed point, we consider the vector field

$$U^{(x)}(y) = -\eta(x,y).$$

In the following, we consider $x \in N$ a fixed point. We know that both $\eta^{(x)} : V_x \to T_xM$ and $exp_x : T_xM \to V_x$ are diffeomorphisms. It follows that the map

$$\varphi^{(x)} : V_x \to V_x,\ \varphi^{(x)}(y) = exp_x(\eta^{(x)}(y)) \tag{11}$$

is also a diffeomorphism on V_x having x as fixed point (i.e. $\varphi^{(x)}(x) = x$).

We consider the submanifold $\tilde{N} = \varphi^{(x)}(N \cap V_x)$ with the induced metric g.

Lemma 8 *If $\eta : M \times M \to TM$ is a locally antisymmetric pairing map, differentiable in the first argument, then*

$$\varphi_*^{(x)} X = \nabla_X U^{(x)},\ \forall X \in T_xM. \tag{12}$$

Proof. Let us consider $x \in M$ and $X \in T_xM$ and let $\alpha : I \to M$ be a geodesic such that $\alpha(0) = x$ and $\dot{\alpha}(0) = X$. We know that there is some open neighborhood V of x where η is antisymmetric (as in definition). We suppose that $Im(\alpha) \subset V$ and, if $Y(x) \in T_xM$ and $Y \in \mathcal{X}(\mathcal{M})$ is a parallel extension of $Y(x)$ along the geodesic α, we have

$$g(\eta(\alpha(t),x),Y(x)) = -g(\eta(x,\alpha(t)),Y(\alpha(t))) = g(U^{(x)}(\alpha(t)),Y(\alpha(t))) \Rightarrow$$

$$\frac{d}{dt}[g(\eta(\alpha(t),x),Y(x)) - g(U^{(x)}(\alpha(t)),Y(\alpha(t)))]|_{t=0} = 0 \Rightarrow$$

$$g(Z(x),Y(x)) = 0, \forall Y \in \mathcal{X}(\mathcal{M}),$$

where

$$Z(x) = [\frac{d}{dt}\eta^i(\alpha(t),x)|_{t=0} - (\nabla_X U^{(x)})^i]\frac{\partial}{\partial x^i}|_x.$$

It follows that

$$\frac{d}{dt}\eta(\alpha(t),x)|_{t=0} = \nabla_X U^{(x)}$$

and

$$\varphi_*^{(x)} X = \frac{d}{dt} exp_x(\eta(\alpha(t),x))|_{t=0} = \frac{d}{dt}\eta(\alpha(t),x)|_{t=0} = \nabla_X U^{(x)}.$$

The immediate consequence of the previous Lemma is that

$$T_x\tilde{N} = \{\nabla_X U^{(x)} |\ X \in T_x M\}.$$

From now on, $F \in \mathcal{X}(\mathcal{M})$ is a vector field such that $F_x \in T_x^{\perp}\tilde{N}$, that is

$$F(x) \perp^g T_x\tilde{N}, \tag{13}$$

where $\perp^g$ denotes the orthogonality with respect to the metric g.

For a fixed point $x \in M$, we introduce the function

$$f^{(x)} : V_x \to R,\ f^{(x)}(y) = g(\eta(y, x), F(x)) \tag{14}$$

on M, similar to the height function used in the study of convex hypersurfaces or H-convex submanifolds. Also we use the subsets TGH_x, TGH_x^- and TGH_x^+ associated to $f^{(x)}$.

Definition 5 *The submanifold N is called (F, g, η)-convex at $x \in N$ if there is an open neighborhood $U \subset V_x$ of x in M such that $U \cap N \subseteq TGH_x^-$ or $U \cap N \subseteq TGH_x^+$.*

The submanifold N is called strictly (F, g, η)-convex at x if N is (F, g, η)-convex at x and $N \cap U \cap TGH_x = \{x\}$.

Remark. If N is an oriented hypersurface of (M, g), if F is the normal vector field and $\eta = \eta_0$, we regain the definition of convex hypersurfaces. Moreover, if $\eta = \eta_0$ and $F = H$, then we obtain the definition of H-convex submanifolds. Generally, if $\eta = \eta_0$ and F is a normal vector field of the submanifold N, we regain the (F, g)-convexity.

The function $f^{(x)}$ can also be expressed as

$$f^{(x)}(y) = g(exp_x^{-1}(\varphi^{(x)}(y)), F(x)).$$

Let $\overline{g} = \varphi^{(x)*}g$ be the Riemannian metric on V_x induced by $\varphi^{(x)}$ and let $\overline{\nabla}$ be the associated Levi-Civita connection. Then

$$\varphi_*^{(x)}(\overline{\nabla}_X Y) = \nabla_{\varphi_*^{(x)} X} \varphi_*^{(x)} Y,\ \forall X, Y \in \mathcal{X}(V_x).$$

We also consider

$$\overline{F}(x) = \varphi_{*,x}^{(x)-1}(F(x)).$$

Because $F(x) \in T_x\tilde{N}$, it follows that $g(F(x), \nabla_X U^{(x)}) = \overline{g}(\overline{F}(x), X) = 0,\ \forall X \in T_x N$, that is

$$\overline{F}(x) \perp^{\overline{g}} T_x N. \tag{15}$$

At this moment, we have two Riemannian submanifolds: $(N, \overline{g})$ and $(\tilde{N}, g)$, both containing x, and two normal vectors

$$\overline{F}(x) \perp^{\overline{g}} T_x N \text{ and } F(x) \perp^{g} T_x \tilde{N}.$$

We prove below that there is some relation between the convexities of those two structures.

Theorem 9 *The following statements are equivalent:*

1. *the submanifold N is (strictly) (F, g, η)-convex at x;*
2. *the submanifold $\tilde{N}$ is (strictly) (F, g)-convex at x;*
3. *the submanifold N is (strictly) $(\overline{F}, \overline{g})$-convex at x.*

Proof. Because $F(x) \perp^{g} T_x \tilde{N}$ and $\overline{F}(x) \perp^{\overline{g}} T_x N$, it makes sense to talk about the (F, g)-convexity of $\tilde{N}$, respectively the $(\overline{F}, \overline{g})$-convexity of N. Furthermore, the submanifold N is (F, g, η)-convex at x if there is a neighborhood U of x in V_x such that $f^{(x)}(y) \geq 0$ or $f^{(x)}(y) \leq 0$, $\forall x \in U \cap N$. Now, if $\tilde{U} = \varphi^{(x)}(U)$ and $z \in \tilde{U} \cap \tilde{N}$, $z = \varphi^{(x)}(y)$, where $y \in U \cap N$, then the previous inequalities become $g(exp_x^{-1}(z), F(x)) \geq 0$, respectively $g(exp_x^{-1}(z), F(x)) \leq 0$, obtaining precisely the condition for $\tilde{N}$ to be (F, g)-convex.

Furthermore, because $\varphi_{*,x}^{(x)-1} \circ exp_x^{-1} = (\overline{exp}_x)^{-1} \circ \varphi^{(x)-1}$, where $\overline{exp}$ is the exponential map induced by $\overline{g}$, the function $f^{(x)}$ can also be expressed by the following relation

$$f^{(x)}(y) = \overline{g}(\overline{exp}_x^{-1}(y), \overline{F}(x))$$

and the above inequalities also express the fact that N is $(\overline{F}, \overline{g})$-convex at x. Similar arguments can be used to prove the equivalences in the strictly convex case. $\diamond$

Corollary 10 *The following statements are true.*

1. *If N is (F, g, η)-convex at $x \in N$, then the bilinear form*

$$\Omega_1^{(x)} : T_x N \times T_x N \to R,\ \Omega_1^{(x)}(X, Y) = g(h(\nabla_X U^{(x)}, \nabla_Y U^{(x)}), F(x)) \tag{16}$$

is semidefinite, where h denotes the second fundamental form of the submanifold $(\tilde{N}, g)$.

2. *If N is (F, g, η)-convex at $x \in N$, then the bilinear form*

$$\Omega_2^{(x)} : T_xN \times T_xN \to R,\ \Omega_2^{(x)}(X, Y) = \overline{g}(\overline{h}(X, Y), \overline{F}(x)) \tag{17}$$

is semidefinite, where $\overline{h}$ denotes the second fundamental form of the submanifold $(N, \overline{g})$.

3. *If one of the previous two bilinear forms is definite, then the other one is also definite and the submanifold N is strictly (F, g, η)-convex at x.*

Proof. We suppose that N is (F, g, η)-convex at $x \in N$ and, moreover, $U \cap N \subseteq TGH_x^+$. An immediate consequence is that $\tilde{N}$ is (F, g)-convex and, similarly to the results from the previous paragraphs, it follows that

$$g(h(\tilde{X}, \tilde{X}), F(x)) \geq 0,\ \forall \tilde{X} \in T_x(\tilde{N}),$$

or, equivalent

$$g(h(\nabla_X U^{(x)}, \nabla_X U^{(x)}), F(x)) \geq 0,\ \forall X \in T_xN.$$

Furthermore, because N is also $(\overline{F}, \overline{g})$-convex, we also have

$$\overline{g}(\overline{h}(X, X), \overline{F}(x)) \geq 0,\ \forall X \in T_xN.$$

Conversely, we know that

$$\overline{g}(\overline{h}(X, X), \overline{F}(x)) = g(h(\nabla_X U^{(x)}, \nabla_X U^{(x)}), F(x))$$

and we suppose that one of them is strictly positive. With the same arguments as in the previous paragraphs, it follows that $\tilde{N}$ is strictly (F, g)-convex at x and N is strictly $(\overline{F}, \overline{g})$-convex at x, therefore N is also (F, g, η)-convex at x.

Remark. When considering $\eta = \eta_0$ in the previous theorem, we found, as expected, the descriptions for the (F, g)-convex submanifolds. Indeed, when considering $\eta_0(y, x) = exp_x^{-1}(y)$ (locally), we have

$$\varphi^{(x)} = Id_{V_x},\ \tilde{N} = N \cap V_x,\ \nabla_X U^{(x)} = X$$

and we obtain $g(h(\nabla_X U^{(x)}, \nabla_X U^{(x)}), F(x)) = g(h(X, X), F(x))$.

5. Conclusions

The novelty of this paper consists in the study of convexity of Riemannian submanifolds with respect to a pairing map η. The most relevant outcomes of this analyze are:

(1) The convexity of a hypersurface is a particular case of (F, g, η)-convexity, when F is the normal vector field, g is the Riemannian structure of the manifold and η is locally defined by the inverse of the exponential map. Similarly, when F is the mean curvature vector field, the (F, g, η)-convexity is equivalent to the H-convexity.

(2) Every (F, g, η)-convex submanifold is also $(\overline{F}, \overline{g})$-convex, where $\overline{F}$ and $\overline{g}$ are deformations induced by η. More precisely, η generates a local diffeomorphism of the manifold which transforms η into the inverse of the exponential map.

(3) There are two descriptions for a (F, g, η)-convex submanifold, involving the second fundamental form of the given submanifold, respectively the second fundamental form of the image of the given submanifold under the local diffeomorphism induced by η.

R E F E R E N C E S

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