FAST NUMERICAL SOLVING USING A SINGLE CELL OF A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION WITH A NONLINEAR SOURCE TERM

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*The numerical integration of Partial Differential Equations (PDE) is based on the idea - accepted unanimously - that this can be done by **dividing the integration domain into a great number of elements**. This approach can insure the obtaining of results having a good level of accuracy, which improves when the number of elements increases. This paper explores an **opposite idea**: solve a PDE with a satisfactory level of accuracy, using a SINGLE CELL equal to the whole quite large domain of integration.*

*In the author's book [1] dedicated to numerical integration of PDEs, the obtaining the solution is based on the use of complete polynomial functions with increasing degrees (3^{rd},5^{th},7^{th}) called CONCORDANT FUNCTIONS (CF). The consequence of increasing the degree of CF is a reduction of the number of elements necessary to obtain a chosen level of accuracy. In this paper the author presents a method – based on [1] - which allows obtaining a numerical solution for PDEs including a NONLINEAR source term, using a SINGLE CELL (element). The method leads, when the solution is smooth, to results with a good precision obtained **in a very short time**. When the solution shows discontinuities, the procedure based on a "single-cell" is not always entirely successful.*

1. Introduction

The numerical integration of Partial Differential Equations (PDE) is based on the idea - accepted unanimously - that this can be done by dividing the integration domain (supposed here rectangular) into a great number of elements. This approach can insure the obtaining of results having a good level of accuracy, **which improves when the number of elements increases**.

The time necessary to find a solution with a reasonable accuracy is – usually - not an impediment, especially for the linear PDEs. Instead, for the nonlinear PDEs this can be a hindrance, especially for the problems where duration for the solving constitutes a priority.

The author of this paper has published in 2015 a book [1] dedicated to this topic, which can be free downloaded from the site **blumenfeld.ro**. Taking advantage of this, in order to simplify the exposure, this article makes several references to the book [1]. Among other topics, in [1] has been developed a special approach meant to improve the accuracy: the obtaining of the numeric solution based on the use of complete polynomial functions with increasing degrees called CONCORDANT FUNCTIONS (CF). The integration of some two-dimensional PDEs has been performed using a 3^{rd} degree CF with 10 terms, a 5^{th}

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degree CF with 21 terms or a 7th degree CF with 36 terms. The consequence of increasing the degree of CF was a *reduction of the number of elements* necessary to obtain a chosen level of accuracy. Because – as it was shown in [1] - to increase the degree of CF is not a difficult problem, it may be raised the question: *"is it not possible to find - for a given PDE - a CF that can lead to the result having a reasonable precision, using* ***a single*** *cell"*? The present paper tries to find the answer to this question for a PDE with a NONLINEAR source term, whose nonlinearity is due to a term depending on the unknown function. Obviously, this approach is the opposite to that stated above in the first paragraph ("the idea - accepted unanimously - that the numeric integration has to be performed by dividing the domain into great number of elements").

2. A brief description of what the method can do

To incite interest of the reader, let us consider a straightforward first order PDE

$$\mathrm{M}(PDE) = M\frac{\partial\phi}{\partial x} + N\frac{\partial\phi}{\partial y} + P\,\phi + Q\,F(\phi) + W(x,y) = 0 \qquad (1.1)$$

including the constant coefficients *M,N,P,Q,* a **nonlinear source term** $F[\phi(x,y)]$ and a known polynomial *W*(x,y). Let consider the particular case

$$2\frac{\partial\phi(x,y)}{\partial x} + 2\frac{\partial\phi(x,y)}{\partial y} - 3\phi(x,y) + (\phi(x,y))^3 - 4 - 7x - 9y - 4x^2 - 10xy + 7y^2 = 0 \qquad (1.2)$$

This PDE must be integrated on a quite large two-dimensional square domain [(B=2)×(H=2)], with the following boundary conditions

-along x axis (y=0) : $\Psi(x) = \phi(x, y=0) = 2 - 0.3x$

-along y axis (x=0) : $\Omega(y) = \phi(x=0, y) = 2 - 0.4y$

The numerical integration was performed twice:

1. With a mesh 40 × 40 = 1600 elements[2], leading to a Target Value $\phi_T(x=2, y=2) = 4.21481$ and a graph given in Fig.1.1.

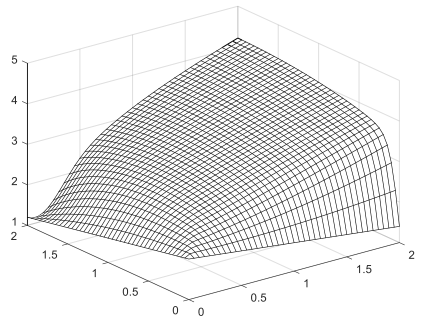

Fig.1.1 Solution using 1600 elements

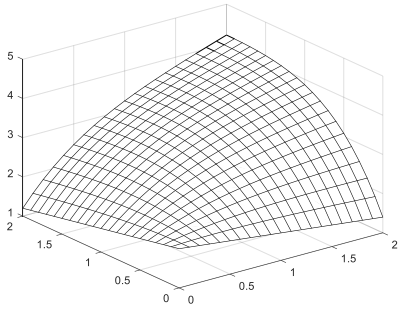

Fig.1.2 Solution using 1 cell

[2] Using the method developed in [1]

2. Using a single cell and a Concordant Function obtained by binding a six degree CF and a seven degree CF. This procedure leads to $z_T(x=2, y=2)=4.21503$ and a graph given in Fig.1.2. The "single-cell" has been divided - this time **only for the graphic representation purpose** - in (20 × 20) elements.

The two graphs are quite similar (obviously not identical). Nevertheless, **the difference between two target values given above is very small**. Supposing ϕ_T as the reference value, the relative error of the *single-cell* computation is

$$Relative\,error=\frac{z_T-\phi_T}{\phi_T}=\frac{4.21503-4.21481}{4.21481}=4.83\times10^{-5}$$, therefore -0.00483 %.

This example is a choice made by the author leading to good results, meant to draw the reader to this unusual approach. As it will result later on, for other cases the error is much greater.

In fact, *the important difference is between the times spent for obtaining the two solutions*. Supposing that T_0 is the time necessary to solve the non-linearity of a cell, the total time used to obtain z_T with the "single-cell" method is approximately $(5...6)T_0$. Using the same method developed in [1] with 1600 elements, the duration will be for this case around $1600T_0$. The relationship between the two durations, though largely approximate, is nevertheless conclusive.

On the other hand - in some particular cases - finding the target values with the single-cell method raises special developments that are described below. In these cases the procedure is not always entirely successful.

3. The Concordant Functions and the Target Residual

Although the reader is supposed to download [1], it is useful to highlight two concepts that are mostly used below.

a. **The Concordant Function** , noted CF, is a **complete** polynomial of a given degree. For instance a third degree CF is given by

$$CF3=CF310=C_1+C_2x+C_3y+C_4x^2+C_5xy+C_6y^2+C_7x^3+C_8x^2y+C_9xy^2+C_{10}y^3 \quad (3.1)$$

It was noted either as CF3 - according to the *third degree* - or CF310 - including the number of terms. The function CF3 can be written as the product between two matrices

$$\phi=CF3=CF310=\left[X^{(0)}Y^{(0)}\right]*[Cz]_3 \quad (3.2)$$

where
$$\left[X^{(0)}Y^{(0)}\right]=\begin{bmatrix}1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3\end{bmatrix} \quad (3.3)$$

$$[Cz]_3=\begin{bmatrix}C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10}\end{bmatrix}^T \quad (3.4)$$

Using this notation, the derivatives of $[\phi]$ can be written as

$$\left[\partial\phi/\partial x\right]=\partial\left[\left[X^{(0)}Y^{(0)}\right]*[Cz]\right]/\partial x=\left[\partial\left[X^{(0)}Y^{(0)}\right]/\partial x\right]*[Cz]=\left[X^{(1)}Y^{(0)}\right]*[Cz] \quad (3.5)$$

$$\left[\partial\phi/\partial y\right]=\partial\left[\left[X^{(0)}Y^{(0)}\right]*[Cz]\right]/\partial y=\left[\partial\left[X^{(0)}Y^{(0)}\right]/\partial y\right]*[Cz]=\left[X^{(0)}Y^{(1)}\right]*[Cz] \quad (3.6)$$

where $\left[X^{(1)}Y^{(0)}\right]=\partial\left[X^{(0)}Y^{(0)}\right]/\partial x=\left[0 \quad 1\ 0 \quad 2x\ \ y\ \ 0 \quad 3x\ \ 2xy\ \ y^2\ \ 0\right]$ (3.7)

$\left[X^{(0)}Y^{(1)}\right]=\partial\left[X^{(0)}Y^{(0)}\right]/\partial y=\left[0 \quad 0\ 1 \quad 0\ \ x\ \ 2y \quad 0\ \ x^2\ \ 2xy\ \ 3y^2\right]$ (3.8)

Besides *CF3=CF310* here are also used: *CF4=CF415* (four degree – 15 terms), *CF5=CF521* (five degree – 21 terms), *CF6=CF628* (six degree – 28 terms), *CF7=CF735* (seven degree – 36 terms).

b. The Target Residual, noted also **Res$_T$.** As George W. Collins II, wrote in his book [8]: "A numerical solution to a differential equation is of little use if there is no estimate of its accuracy. However, … the formal estimate of the truncation error is often more difficult than finding the solution".

The method developed in [1] avoids the "difficult" estimation of the error by controlling the accuracy of the computation using the **RESIDUAL,** which is **the difference between a result obtained by computation** and **the theoretical result.** In fact, the control is performed at the end of the computation taking the form of the **Target Residual**, which results by replacing the three Target unknowns in the PDE

$$Res_T = M\times\left(\frac{\partial\phi}{\partial x}\right)_T + N\times\left(\frac{\partial\phi}{\partial y}\right)_T + P\times(\phi)_T + Q\times F\left[(\phi)_T\right] + W(x_T, y_T) \qquad (3.9)$$

If, incidentally, the Target unknowns are the exact solutions, the *Target Residual* is null. Otherwise, the value of the Residual is different from zero, being a sure indication of the **global accuracy**, namely of the error due to all three Target unknowns.

4. The "two steps" procedure

The hypothesis on which is based the numerical integration developed in [1] is: „the solution $z\approx\phi$ is a CF having a certain degree". Because this „certain degree" is not known at the start, the computation will begin with the numerical integration using all five CFs mentioned above, called „STEP 1". This integration has the role to furnish an initial cluster of information, on which is based the main computation („STEP 2").

4.1. Step 1: Preliminary Computation with 5 different CFs

Suppose the PDE

$$2\frac{\partial\phi(x,y)}{\partial x}+2\frac{\partial\phi(x,y)}{\partial y}+8\phi(x,y)+6\left(\phi(x,y)\right)^{1.4}-4-7x-9y-4x^2-10xy+7y^2=0 \qquad (4.1)$$

that must be integrated on a large two-dimensional square domain [(B=2)×(H=2)], with the boundary conditions

-along x axis (y=0) : $\Psi(x)=\phi(x,y=0)=2-0.3x$ (4.2)

-along y axis (x=0) : $\Omega(y)=\phi(x=0,y)=2-0.4y$ (4.3)

The PDE includes a nonlinear source term $(\phi(x,y))^{1.4}$ depending on the unknown function $\phi(x,y) \approx z(x,y)$. Instead of dividing the domain in small elements, the integration will be performed using a **single cell** having the dimensions (2×2). The method described in [1] is based on the the use of a Concordant Function having a degree selected by the user. Here, the first step of the computation is based on **five CFs** with increasing degrees (3,4,5,6 and 7). The methodology described in [1] remains generally the same, with two minor modifications concerning the input *boundary conditions data* and the way to choose the equations for obtaining the *Target unknows corresponding to the five CFs*. Not to distract the reader from the main aspects of the calculation, these two issues will be dealt with in the *Appendix B*.

The five computations with the different CFs performed for (4.1) using one cell, lead to the results given in the rows 1...5 from the Table 1. They include three main parameters *connected to the Target point T* (noted using $z \approx \phi$): the function z_T and the two first derivatives $(\partial z/\partial x)_T$ and $(\partial z/\partial y)_T$. Besides them, in the last column is given the Target Residual, computed with rel. (3.9).

Table 1

Row	CF	Target Function z_T		Target Derivative $(\partial z/\partial x)_T$	Target Derivative $(\partial z/\partial y)_T$	Target Residual Res_T
		Value	Error %			
1	CF3 or CF**3**10	2.15841	-36.1	1.96623	-0.0592413	-25.30
2	CF4 or CF**4**15	2.59722	-23,1	2.03892	0.0456926	-16.22
3	CF5 or CF**5**21	2.98673	-11.6	2.01215	0.0705908	-8.180
4	CF6 or CF**6**28	3.08211	-8.75	1.98461	0.0677592	-6.228
5	CF7 or CF**7**35	3.79999	12.5	1.87473	0.0622218	9.164
Values corresponding to the computation with **25×25=625 elements**						
6	CF3 or CF**3**10	3.37800	-	1.93617	0.0674984	0.01334

The similar results obtained with *CF3* by dividing the same domain in *25×25 = 625 elements*, are given in the last row of *Table 1*. The reference value of the Target Function thus resulted is

$$\phi_{T(625 elements)} = 3.37800 \tag{4.4}$$

obtained with a Target Residual=0.01334; this can be considered as a satisfactory result.

The values of z_T obtained using the five CFs, given in the third column of the Table 3.1, are represented in the Fig.3.1. Their values – connected by straight lines - are measured in the *ordinate* of the graph, while the *abscissas* noted *3,4,5,6,7* are supposed to identify the five *CF*s. On the same graph is represented the reference value (4.4). As it results from Fig.3.1, the line between CF6 and CF7 intersects the value (4.4), therefore a satisfactory value of ϕ_T can be found in this interval. But *"where ?"*, is not yet clear. An answer to this question may be obtained using the *Target Residual*.

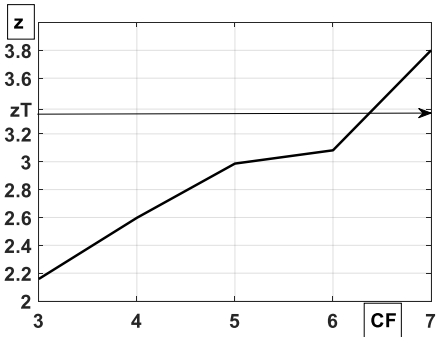

Fig.3.1 Target value (different CFs)

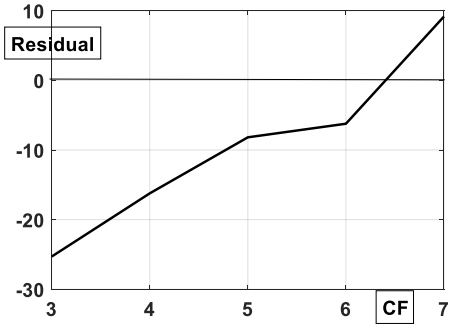

Fig.3.2 Target Residual (different CFs)

The values of the Target Residual given in the last column of Table 3.1 – obtained straightforwardly while performing the five CFs computations - are represented in Fig.3.2, following a similar procedure as that used for Fig.3.1. These values are rather distant and show that the results of the five calculations are quite inaccurate. It may be, nevertheless, observed that in the interval between CF6 and CF7, a line connecting their two values intersects Target Residual=0. The fact that in the same interval where is expected a good value for ϕ_T, the target residual changes the sign (from *Res6= -6.228* for *CF6* to *Res7= 9.164* for *CF7*) **is not accidental.** This fact indicates the possibility to obtain - with a single cell - a credible value of the Target function for a PDE with a nonlinear term. But the path is still uncertain and not yet usable. However, it opens a way towards the **Step 2**, which is the second procedure meant to solve the problem.

4.2. Step 2: Improving the solution by binding two different *CFs*

Let consider in Fig.3.3 two *CFs*, between which *the Target Residual changes the sign*. The two Target Residuals will be noted[3] as R_L (index L for Left) and R_R (index R for Right). Though the interval between two *CFs* has no physical meaning, two *CFs* will be represented using a *fictitious abscissa* noted η, that is considered as variable between $\eta_L = 0$ and $\eta_R = 1$

$$0 \le \eta \le 1 \tag{4.5}$$

Between these extreme two values, we assume that **the Target Residual varies continuously**, following an unknown path, therefore *somewhere* within the range $0 \le \eta \le 1$ the path goes through Res_T=0. Because there is no information concerning the supposed trail, **we accept the rough hypothesis that the residual varies linearly** with η , according to

$$Res(\eta) = Res_L + (Res_R - Res_L)\eta \tag{4.6}$$

[3] As in Fig.3.2, between CF6 (Res_T = - 6.228) and CF7 (Res_T = 9.164)

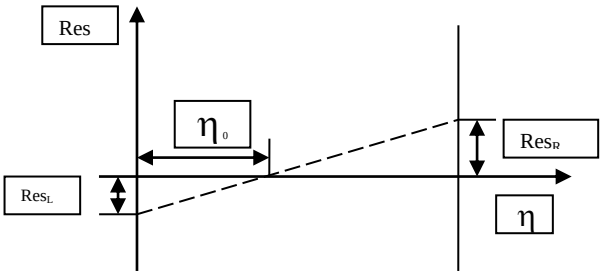

Fig.3.3 Fictitious linear variation between two *Res*

The *Target Residual* becomes null for $$\eta_0 = \frac{Res_L}{Res_L - Res_R} \tag{4.7}$$

Using the values given in Table 3.1 it results

$$\eta_0 = -6.228 / (-6.228-9.164) = 0.4046 \tag{4.8}$$

The *Target Residual* is given by (3.9), ϕ being replaced by z. In the same time the target values are replaced using (2.2), (2.5), (2.6), while for the nonlinear source term $(F(\phi))_T$ is used rel. (5.3.2), page 63 from [1]

$$\begin{aligned}[Res]_T = {} & M\left[X^{(1)}Y^{(0)}\right]_T[Cz] + N\left[X^{(0)}Y^{(1)}\right]_T[Cz] + P\left[X^{(0)}Y^{(0)}\right]_T[Cz] + \\ & + Q\left[X^{(0)}Y^{(0)}\right]_T[CF] + W(x_T, y_T)\end{aligned} \tag{4.9}$$

Accepting that L corresponds here to CF6 and R to CF7, it results the following Residuals that can be computed *after the Step 1 is done*

$$\begin{aligned}[Res]_{T,Left} = {} & \left[\left[M\left[X^{(1)}Y^{(0)}\right]_T + N\left[X^{(0)}Y^{(1)}\right]_T + P\left[X^{(0)}Y^{(0)}\right]_T \right]_{(Degree6)} [Cz6] + \right. \\ & \left. + Q\left[X^{(0)}Y^{(0)}\right]_{T,(Degree6)} [CF6] + W(x_T, y_T) \right]\end{aligned} \tag{4.10}$$

$$\begin{aligned}[Res]_{T,Right} = {} & \left[\left[M\left[X^{(1)}Y^{(0)}\right]_T + N\left[X^{(0)}Y^{(1)}\right]_T + P\left[X^{(0)}Y^{(0)}\right]_T \right]_{(Degree7)} [Cz7] + \right. \\ & \left. + Q\left[X^{(0)}Y^{(0)}\right]_{T,(Degree7)} [CF7] + W(x_T, y_T) \right]\end{aligned} \tag{4.11}$$

Accepting that all the parameters involved vary also linearly, it results – based on relations similar to (4.6) - the following target values that correspond to η_0

$$(z)_{T0} = z_L + (z_R - z_L)\eta_0 \tag{4.12}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{T0} = \left(\frac{\partial\phi}{\partial x}\right)_L + \left(\left(\frac{\partial\phi}{\partial x}\right)_R - \left(\frac{\partial\phi}{\partial x}\right)_L\right)\eta_0 \tag{4.13}$$

$$\left(\frac{\partial\phi}{\partial y}\right)_{T0} = \left(\frac{\partial\phi}{\partial y}\right)_L + \left(\left(\frac{\partial\phi}{\partial y}\right)_R - \left(\frac{\partial\phi}{\partial y}\right)_L\right)\eta_0 \tag{4.14}$$

Replacing in these relations the values given in the rows (4) an (5) of Table 3.1 it results

$$z_{\eta 0}=3.08211+ (4.79999-3.08211) \times 0.4046=3.372564248 \quad (4.12,a)$$

$$(\partial\phi/\partial x)_{\eta 0}=1.98461 + (1.87473 - 1.98461)\times 0.4046 = 1.94016 \quad (4.13,a)$$

$$(\partial\phi/\partial y)_{\eta 0}=0.0677592 + (0.062221 - 0.067759)\times 0.4046 = 0.06551876\ 7 \quad (4.14,a)$$

If these values are compared to those obtained with 25×25=625 elements (see Table 3.1), it results the errors:

$(z_{\eta 0}-\phi_T)/\phi_T$ =(4.372598-3.378)/3.378 = -1.6×10^{-3} namely 0.16 %

$((\partial z/\partial x)_{\eta 0}-(\partial\phi/\partial x)_T)/(\partial\phi/\partial x)_T$ =(1.93617-1.94016)/1.94016=-2×10^{-3} namely 0.2 %

$((\partial z/\partial y)_{\eta 0}-(\partial\phi/\partial y)_T)/(\partial\phi/\partial y)_T$ =(0.06749-0.065518)/0.065518=3×10^{-3} namely 0.3%

These errors are surprising, taking into account the ratio between the numbers of elements used in each computation. According to (3.9), the Target Residual that corresponds to η_0 is

Res_T=2×1.94016 + 2×0.0655187 + 8×3.37256424+ 6×$(4.372564248)^{1.4}$ +$W_T \approx 0.1$

The actual Target Residual **is not zero**, as it was supposed above. This means that the linearity hypothesis (4.6) is not strictly confirmed, but, nevertheless, the value of the Target Residual is satisfactory and the value of z_T is surprisingly good.

Remark. In fact choosing L and R as neighbor CFs is not compulsory. If L and R are chosen better, this could probable improve the result by reducing the errors. This is a detail that was not developed by the author, in order not to divert the reader's attention from the main subject of the article.

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Actually, the procedure used above for the *Step 2* can be greatly simplified. After calculating η_0 from relation (4.7), we observe that all the computations that follow can be based on a new Concordant Function, which may be obtained by **binding the two [Cz]** (L and R) corresponding to the interval considered (see (3.2)). The binding can also be made linearly, according to a relation similar to (4.6), where η is replaced by η_0

$$[Cz]_{binded}^{\{L/R\}}=[Cz]_L+[[Cz]_R-[Cz]_L]\eta_0 \quad (4.15)$$

Using *[Cz]binded* all the target parameters can be computed straightforwardly according to (3.2), (3.5), (3.6)

$$z_T=[X^{(0)}Y^{(0)}]_T[Cz]_{binded}^{\{L/R\}},(\partial\phi/\partial x)_T=[X^{(1)}Y^{(0)}]_T[Cz]_{binded}^{\{L/R\}},(\partial\phi/\partial y)_T=[X^{(0)}Y^{(1)}]_T[Cz]_{binded}^{\{L/R\}}$$

More than that, an overall drawing of the variation of the function z(x,y) may by obtained dividing the domain in a convenient number of nodes. This allows now to compare this “single-cell” solution (Fig 3.4) with (Fig 3.5) where is drawn the “many-elements” solution (if the last graph is available).

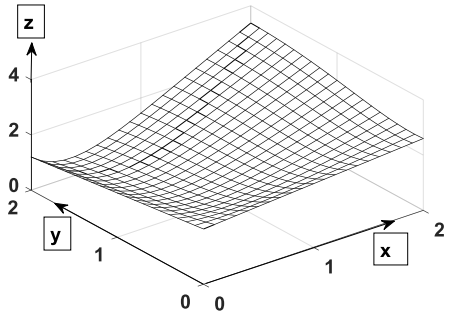

Fig.3.4 Solution with a single cell

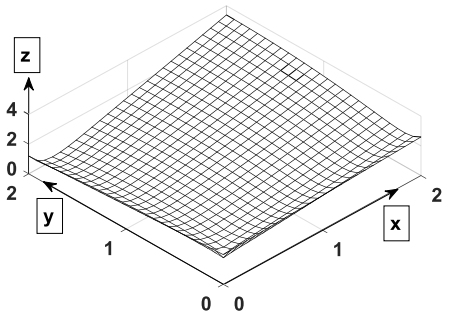

Fig.3.5 Solution with 25×25 elements

Using $[Cz]_{binded}^{\{L/R\}}$ it becomes also possible to have a global look of the variation of the residual, which can be obtained with the same meshing. In the Fig.3.6 it is given a graph of the residual, viewed from the Target. As it results in the area near the Target (x>1.5, y>1.5) *the variation is close to a plane* corresponding to *Res(x,y)* ≈ *0*. This is to be expected, because the main ecuations for obtaining the unknowns were connected to the Target. Far from the Target, the variation of the residual it's getting further from zero.

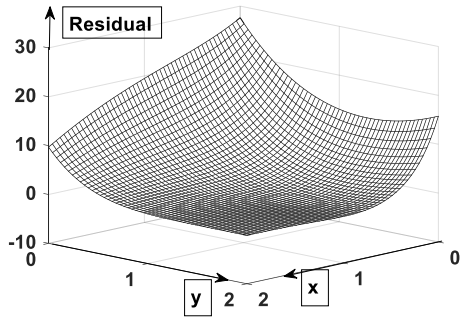

Fig.3.6 Residual (View from Target)

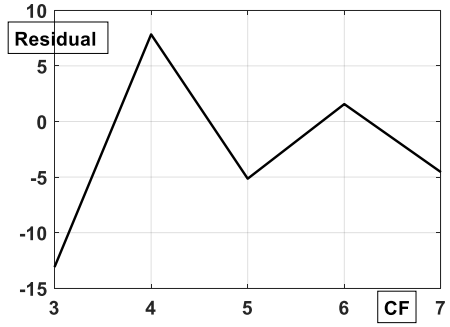

Fig.3.7. Target Residual (different CFs)

Example 1. The PDE

$$2\frac{\partial\phi(x,y)}{\partial x}+2\frac{\partial\phi(x,y)}{\partial y}+10\,\phi(x,y)+5\cos\phi(x,y)- \\ -45+3.7\,x+4.6\,y+3.8\,x^2+1.2\,xy+4.5\,y^2=0 \tag{4.16}$$

must be integrated on a large two-dimensional square domain [(B=1.6)×(H=1.6)], with the boundary conditions

-along x axis (y=0) : $\Psi(x)=\phi(x,y=0)=4+3x-3x^2$ (4.17)

-along y axis (x=0) : $\Omega(y)=\phi(x=0,y)=4+2y-2y^2$ (4.18)

Solution. Step 1. After the five compulsory integrations with 1 cell, the resulted *Target Residuals* values are represented in the Fig.3.7. If the five values of the

Residuals are linked with straight lines, it results that *Res=0* cuts 4 times these lines, therefore the *Step 2* procedure has to be applied to all four intervals.

Table 2

CF	η_0	Values for (B=1.6;H=1.6) obtained with **1 element** by binding two CFs			
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual
$[Cz]_{binded}^{\{3/4\}}$	0.625	1.56208	-1.72877	-1.96522	0.876
$[Cz]_{binded}^{\{4/5\}}$	0.603	1.69062	-2.12030	-2.47606	-0.284
$[Cz]_{binded}^{\{5/6\}}$	0.767	1.66641	-1.94212	-2.44678	0.00899
$[Cz]_{binded}^{\{6/7\}}$	0.255	1.77423	-2.04693	-2.621678	-0.00506
Values corresponding to the computation with **20×10=200 elements**					
CF3		1.76680	-2.13544	-2.51652	-0.00968

Step 2. If the procedure developed for the Step 2 is applied, one obtains the results given in Table 2. From the 4 solutions found as possible analyzing Fig 3.7, it may be retained the solution that corresponds to the **minimum absolute value** of the Target Residual, which is **0.00506** obtained from the row of $[Cz]_{binded}^{\{6/7\}}$. In the same row is given the Target value which is $\mathbf{z_T=1.77423}$. If this value is compared to ϕ_T =1.76655, it results that the error of this fast computation is

Error z_T= (1.77423-1.76665)/1.76665= 4.29×10^{-3} which means 0.429 %.

An overview of the whole solution results by representing the graph obtained using the $[Cz]_{binded}^{\{6/7\}}$ (Fig. 3.8) and the graph corresponding to the 200 elements mesh (Fig.3.9). The coefficients of $[Cz]_{binded}^{\{6/7\}}$ can be found in *Appendix A*, therefore the reader can easily verify himself the similarity of the two graphs.

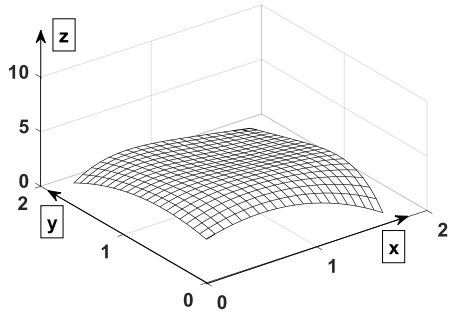

Fig.3.8 Solution with a single cell

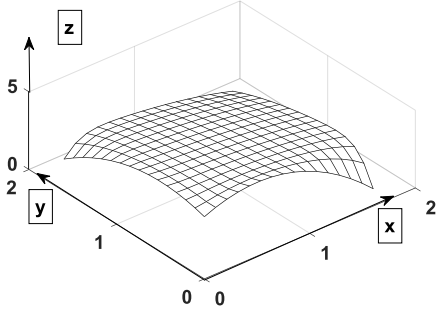

Fig.3.9 Solution with 25×25 elements

Example 2. The PDE

$$2\frac{\partial\phi}{\partial x}+3\frac{\partial\phi}{\partial y}+5\phi+2\ln(\phi+1)-2-3x-y-4x^2-3xy-2y^2-x^3-2x^2y-4xy^2-2y^3 \\ -2x^4-3x^3y-x^2y^2-4xy^3-3y^4-2x^5-x^4y-4x^3y^2-6x^2y^3-2xy^4-y^5 \quad (4.19)$$

has to be integrated on a two-dimensional rectangular domain [(B=1)×(H=1.2)], with the boundary conditions

-along x axis (y=0) : $\Psi(x) = \phi(x, y=0) = 2 + 3x - 3x^2$ (4.20)

-along y axis (x=0) : $\Omega(y) = \phi(x=0, y) = 2 - 4y + 4y^2$ (4.21)

Solution. Step 1. After the five initial integrations with 1 cell, one obtains the values given in Table 3, from which it results that the neighbor CFs change the sign twice, between CF3 / CF4 and CF5 / CF6.

Step 2. The procedure developed for the *Step 2* is applied, nevertheless, to all 4 intervals leading to the results given in Table 3. From them are retained those that correspond to the minimum value of the Target Residual, namely $[Cz]_{binded}^{\{5/6\}}$. In the last row of the same table are also given the results obtained using 24×16=384 elements. If the Target values are compared it results that the error of this fast computation is

Error z_T= (8.30651-8.33245)/ 8.33245= -3.11×10^{-3} which means 0.311 %.

Remark. It was established above that two tests are enough, namely $\eta_0 = 0.699$ and $\eta_0 = 0.508$. Using the procedure for **Step 2** also for the other two cases allow observing that, although η_0 does not respect the limits (4.5), the values resulting from the binding procedure are pretty good. Actually, here $[Cz]_{binded}^{\{4/5\}}$ gives the best results for z_T, though it does not respect (4.5). This fact was observed also in other cases; therefore all four tests were usually used.

Table 3

CF	η_0	Values for (B=0.8;H=1.6) obtained with **1 element** by binding two CFs			
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual
$[Cz]_{binded}^{\{3/4\}}$	0.699	8.16188	10.5220	17.0487	2.89×10^{-2}
$[Cz]_{binded}^{\{4/5\}}$	1.147	8.32708	11.1676	16.3206	2.49×10^{-3}
$[Cz]_{binded}^{\{5/6\}}$	**0.508**	**8.30651**	**11.1718**	**16.3545**	**2.13**$\times10^{-4}$
$[Cz]_{binded}^{\{6/7\}}$	-6.982	8.74934	10.4174	16.0878	1.20×10^{-3}
Values corresponding to the computation with **24×16=384 elements**					
CF3	*	**8.33245**	**11.2485**	**16.2685**	**0.0312**

Remark. It was established above that two tests are enough, namely $\eta_0 = 0.699$ and $\eta_0 = 0.508$. Using the procedure for **Step 2** also for the other two cases allow observing that, although η_0 does not respect the limits (4.5), the values resulting from the binding procedure are pretty good. Actually, here $[Cz]_{binded}^{\{4/5\}}$ gives the best results for z_T, though it does not respect (4.5). This fact was observed also in other cases; therefore all four tests were usually used.

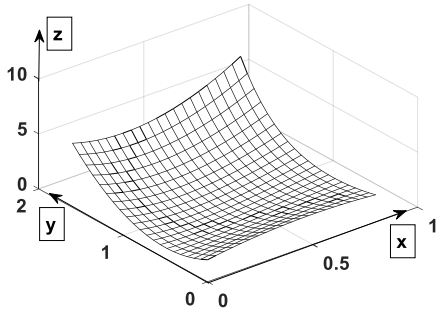

Fig.3.10 Solution with a single cell

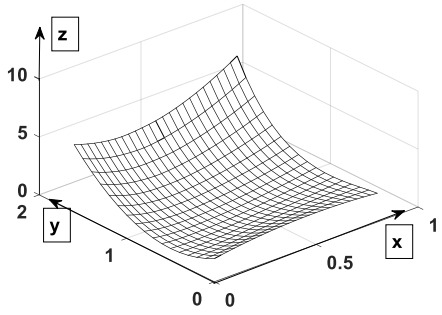

Fig.3.11 Solution with 24×16 elements

An overview of the whole solution results by comparing the graph obtained using the $[Cz]^{\{5.6\}}_{binded}$ (Fig. 3.10) and the graph corresponding to the 384 elements mesh (Fig.3.11). The coefficients of $[Cz]^{\{5/6\}}_{binded}$ can be found in *Appendix A*, therefore the reader can easily verify himself the similarity of the two graphs. In Fig.3.12 is represented the general variation of the residual. From it follows that away from the target the variation is influenced by the initial conditions, while near the target it approaches the plane corresponding to *Res (x, y) ≈ 0.*

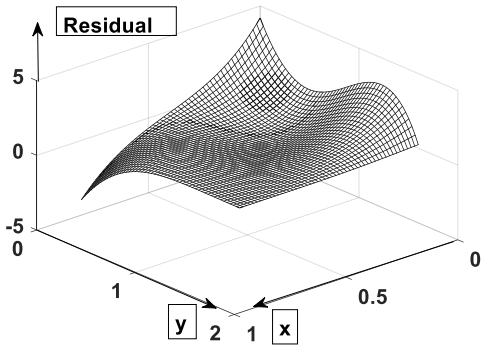

Fig.3.12 Residual (View from Target)

5. Perturbations and their consequences

5.1 Unexpected appearance of a perturbation

The quite similar drawings from Figs. 1.1 and 1.2 is an impulse to try to increase the integration range. If the integration range increases to **B = H = 10**, the "two-steps" final results obtained using one cell are given in first 4 rows of Table 4.

Table 4

CF	η_0	Values for (B=10;H=10) obtained with **1 element** by binding two CFs			
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual
$[Cz]_{binded}^{\{3/4\}}$	1.066	9.62529	0.68019	-0.11285	0.00743
$[Cz]_{binded}^{\{4/5\}}$	**0.714**	**9.62529**	**0.68026**	**-0.12738**	**-0.00613**
$[Cz]_{binded}^{\{5/6\}}$	0.233	9.62433	0.68059	-0.12799	-0.254
$[Cz]_{binded}^{\{6/7\}}$	-0.160	9.64339	0.67455	-0.11797	4.98
Values corresponding to the computation with **80×80=6400 elements**					
CF3	*	**9.62524**	**0.68026**	**-0.11273**	**-0.00613**

Taking into account the absolute value of the Target residual, the results corresponding to $[Cz]_{binded}^{\{4/5\}}$, are considered the best. The variation of the function z(x,y) with a single cell is given in Fig.5.1. Because the dimensions of the domain are so great, it is necessary to verify the results with the ordinary procedure, namely to use a mesh. The results obtained with **80×80=6400 elements** are also given in the last row of Table 4. Comparing these values to those corresponding to $[Cz]_{binded}^{\{4/5\}}$ it results a surprisingly matching. The error of the Target value is[4]

Error z_T=(9.62529-9.62524)/ 9.62524=5.19×10^{-6} namely 5.19×10^{-4} %.

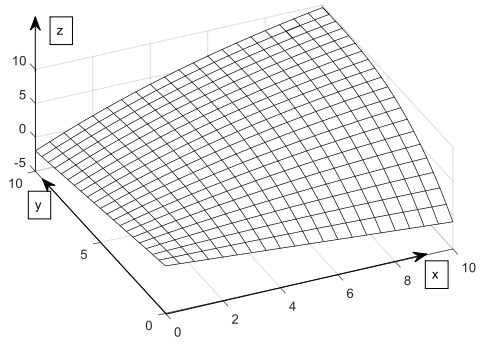

Fig.5.1 Solution with a single cell

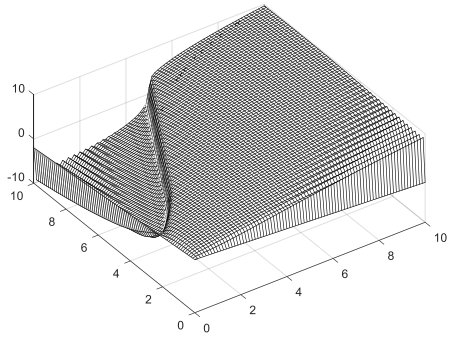

Fig.5.2 Solution with 80×80 elements

The graph of z(x,y) obtained with 6400 elements represented in Fig.5.2 constitutes an unpleasant surprise: except the Target values, which – as it was seen above - coincide, there are great differences between this graph and that given in Fig.5.1. The most important is the *perturbation* (discontinuity) that appears in the North-West quasi-triangular area of Fig.5.2 (x<5,y>2.5), **which is totally ignored** in Fig.5.1.

[4] The very good results obtained for B=H =10 (and also beyond this value) can be explained by the fact that the nonlinear term enters *Step 1* in the range of the degrees testing (3,4,5,6,7) used for the Concordant function .

3.2 Finding the existence of a perturbation

A large part of Chapter 3 from [1] was dedicated to the *"perturbations"* in the graph of the function, of its derivatives or of the residuals, produced by *"the boundary conditions imposed by the user on a (possible) "calm and gentle" solution"*. The perturbations can be produced also by other causes, such as unexpected discontinuities of the function or its derivatives. As a result of the presence of the perturbations, an increase of the target errors was observed. These errors were eliminated in [1] by removing the perturbations from the target area, which can be achieved **by modifying the ratio between the numbers of elements used along the x-axis, respectively y**.

It is obvious that this procedure - which requires many elements - can not be used if *the calculation is done with a single cell*. Instead, becomes compulsory to inform the user about this circumstance and of its consequences.

The aim of this paragraph is to find a fast methodology to detect the existence of a perturbation. In the above example the perturbation was observed only after solving the PDE with 6400 elements (Fig.5.2), because the perturbation was totally ignored by the "single-cell method" (Fig.5.1). This omission can be easily explained analyzing the information that is furnished to the "single cell" procedure and the conditions imposed by the system of equations:

1. The information furnished as input represent the boundary conditions, namely the values of the function - imposed by the user - along the axes x and y that remain unmodified till the end of the computation. The perturbation near the axis y appears in Fig.5.2 (when y>2.5) as a sudden discontinuity of the function, without any other outside intervention.

2. The system of equations imposes a lot of conditions (See *Appendix B*), but they are all connected to the target point. Consequently, the target results fit with those obtained using a large number of elements, but the "binded CF" can not ensure a rigorous representation of the z (x, y) path in the areas away from the target.

The method analyzed here tries to find a single binded CF to describe a complicated phenomenon that develops on a quite large domain. In some cases the graphic agreement between "single-cell " and "many-elements" solutions may be possible as it happened in the examples examined until now[5]; in other cases, when some perturbations appear far from the Target, one may obtain reliable results only on a limited part of the domain or not at all.

[5] See Figs.3.4 and 3.5 or Figs.3.8 and 3.9

Table 5

CF	η_0	Values for (**B=4.30;H=4.30**) obtained with **1 element** by binding two CFs				
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual	Maximum Residual Test
$[Cz]^{\{3/4\}}_{binded}$	1.937	6.03032	0.78865	-0.08751	0.373	114.56
$[Cz]^{\{4/5\}}_{binded}$	0.346	6.02277	0.79931	-0.07716	-0.385	119.82
$[Cz]^{\{5/6\}}_{binded}$	0.7418	6.02401	0.79867	-0.07723	-0.254	111.58
$[Cz]^{\{6/7\}}_{binded}$	-0.243	6.03015	0.79472	-0.07680	0.388	111.13

Table 6

CF	η_0	Values for (**B=4.29;H=4.29**) obtained with **1 element** by binding two CFs				
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual	Maximum Residual Test
$[Cz]^{\{3/4\}}_{binded}$	1.564	0.38645	0.33947	100.88	-0.121	6.3×10^{5}
$[Cz]^{\{4/5\}}_{binded}$	-3×10^{-20}	8.22×10^{-4}	-6.67916	107.41	3×10^{-13}	1.4×10^{5}
$[Cz]^{\{5/6\}}_{binded}$	1	1.47×10^{5}	-2×10^{8}	2×10^{8}	3×10^{15}	2×10^{53}
$[Cz]^{\{6/7\}}_{binded}$	4×10^{-29}	2.39×10^{5}	-2×10^{8}	2×10^{8}	3×10^{16}	3×10^{53}

There are many possibilities to choose a "check parameter" that indicate the existence of a perturbation. The author has chosen one that is easy to use but is not always very effective: *the maximum (absolute) value of the Residual* across the entire integration domain. For start, based on fig.5.3 a test is made for a domain limited to **B=H=4.30**. The results do not indicate any perturbation and seem to be credible (see Table 5, last column). Instead, when the target is a little changed to **B = H = 4.29**, the results become suddenly incoherent and, obviously, can not be taken into account (Table 6). Even without looking at the other values that are also alarming, the Maximum Residual test, which increases more than 1,000 times, seems appropriate to lead to the decision to quit the computation and inform the user accordingly.

Example 3. The PDE

$$4\frac{\partial\phi}{\partial x}+2\frac{\partial\phi}{\partial y}+3\phi-9.9\cos 2\phi-2.8-3.7x-4.6y-3.8x^2-1.2xy-4.5y^2=0 \quad (5.1)$$

must be integrated on a two-dimensional square domain [(B=1)×(H=1)], with the boundary conditions

-along x axis (y=0) : $\Psi(x)=\phi(x,y=0)=2+3x-3x^2$ (5.2)

-along y axis (x=0) : $\Omega(y)=\phi(x=0,y)=2+4y-4y^2$ (5.3)

The second step of the computation leads to the results given in Table 7. The Concordant Function $[Cz]_{binded}^{\{4/5\}}$ is given in *Appendix A*. Based on it, the variation of z(x,y) is represented in Fig.5.1. The conclusion based on all the results obtained is: *"the computation is valid"*.

Table 7

CF	η_0	Values for (**B=1;H=1**) obtained with **1 element** by binding two CFs			
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Target Residual
$[Cz]_{binded}^{\{3/4\}}$	-0.117	4.20267	0.69466	-0.00336	-0.0338
$[Cz]_{binded}^{\{4/5\}}$	**0.746**	**4.17016**	**0.57847**	**0.64248**	**0.1365**
$[Cz]_{binded}^{\{5/6\}}$	-1.354	4.21902	0.41850	0.53138	0.2529
$[Cz]_{binded}^{\{6/7\}}$	1.568	4.38891	0.02702	0.12674	0.1267
Values corresponding to the computation with **10×20=200 elements**					
CF3	*	**4.17381**	**0.52380**	**0.63825**	**-0.0158**

In order to have a confirmation, the verification of the results obtained with of 10×20=200 elements, leads to the values given in the last row of Table 7. The error of the Target value that corresponds is

Error z_T=(5.17016-4.17381)/4.17381= -8.74×10^{-4} namely -0.0874 %.
This value confirms the validity of the results.

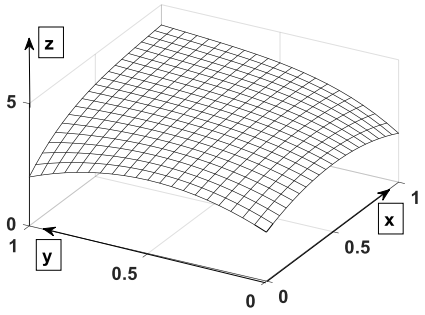

Fig.5.3 Solution with a single cell

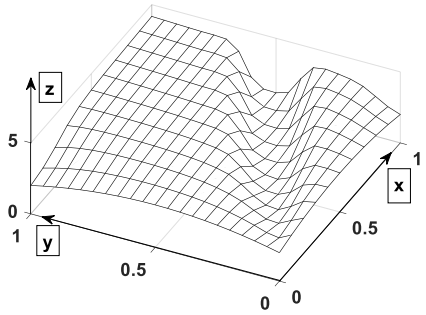

Fig.5.4 Solution with 10×20 elements

However, the comparison between the graphs of z (x, y) obtained with a single cell (Fig.5.3) and with 200 elements (Fig.5.4) indicates the existence of a perturbation. This perturbation is not signaled - as is now expected according the previous thorough analysis - by the "single-cell" method. But the perturbation is not close to the target, so it does not influence the results obtained above or the conclusion about the validity of the calculation.

Table 8

CF	η_0	Values for (**B=1;H=0.5703**) obtained with **1 element** by binding two CFs			
		Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Maximum Residual (Test)
$[Cz]_{binded}^{\{3/4\}}$	-2.041	4.06318	-0.11772	0.36924	29.67
$[Cz]_{binded}^{\{4/5\}}$	-2.399	3.97391	0.38731	0.45446	29.67
$[Cz]_{binded}^{\{5/6\}}$	-3.704	4.05911	0.27957	0.05332	29.67
$[Cz]_{binded}^{\{6/7\}}$	344.8	5.49205	4.19714	-20.9683	3980.4
Values corresponding to the computation with **10×20=200 elements**					
CF3	*	1.23889	-1.99379	6.13487	*

Table 9

CF	Values for (**B=1;H=0.5702**) obtained with **1 element** by binding two CFs			
	Target Function z_T	Target Deriv. $(\partial z/\partial x)_T$	Target Deriv. $(\partial z/\partial y)_T$	Maximum Residual (Test)
$[Cz]_{binded}^{\{3/4\}}$	6.93750	-6.1×10^{12}	1.2×10^{13}	1.6×10^{14}
$[Cz]_{binded}^{\{4/5\}}$	3.5×10^{14}	-2×10^{14}	2.8×10^{15}	1.8×10^{27}
$[Cz]_{binded}^{\{5/6\}}$	-5.1×10^{25}	9.2×10^{25}	-5.4×10^{26}	2.7×10^{38}
$[Cz]_{binded}^{\{6/7\}}$	7.5×10^{49}	-2.9×10^{58}	5.9×10^{58}	2.2×10^{61}

What happens if the target is changed in the proximity of the perturbation, namely if the target point becomes (B=1, H=0.5703) ? The values for this test given in Table 8 are quite normal. The proximity of the perturbation "feels" in the target values, the error of the function z(B=1,H=0.5703) being more than 200%. If tried - based on the experience of paragraph 3.2 – a calculation for a very close target point (B=1, H=0.5702) the result is similar: discontinuities of results that clearly indicate the occurrence of a perturbation (see Table 9). Some numerical experiments made by the author have confirmed this behavior.

6. Conclusions on the procedure based on a single cell

The numerical tests made using a "single-cell" - part of which were disclosed above - represent a basis for a first draft of a procedure to implement the method. Let begin with the positive aspects connected to the use of the method:

1. A simple first order non-linear PDE can be numerically integrated over a large domain using only one cell, without resorting to tens or hundreds of elements.

2. The main result is the *Target value* z_T of the unknown function, which may be obtained with acceptable or even good precision[6], **in a very short time.**

[6] The precision may be improved by binding a second time the already binded Concordant Functions.

3. The procedure is based on the use of a *binded* Concordant Function, which may be considered – *if the function z(x,y) has a smooth variation, without perturbations* – as a **quasi-analytic solution**, valid on all the (quite–large) integration domain.

The negative aspects are connected with the possible existence of a perturbation or discontinuity of the function:

1. If a perturbation occurs *quite far from the Target* its presence does not affect the value of the Target function, which can be validated by the actual Residual value; but **the quasi-analytic solution is no longer valid on the whole domain, so must be abandoned.**

2. When a perturbation occurs *near to the Target*, **all the computation based on a single-cell must be abandoned**, and the computation continued with another method.

Taking into account these considerations, it is proposed below a draft for a program monitoring the computation. **The computation starts always using the "single-cell" method.** After finising the "two-steps" procedure the program has to look for a potential perturbation and to decide between some possible variants:

a. If no perturbation is detected, and the target residual is accepted by the user, the computation may be considered succesful on all aspects; besides the Target value, also the quasi-analytical solution is available and may be used.

b. If a perturbation placed far from the Target is detected, only the Target value z_T can be used, obviously if the Target residual is accepted by the user.

c. If a perturbation placed near the Target is detected or the value of the Target residual is rejected by the user, the procedure based on a "single-cell" has to be abandoned and the computation must continue with another method. In this case **the only inconvenience is a very short delay, due to initial use of the "single cell" method.**

Appendix A. Verification of the quasi-analitic solution based on binded Concordant Functions

The reader may solve the any PDE mentioned below using a convenient method, with many elements, which can give a graph z(x,y) of the solution. Then he can compare with the corresponding solution using the relation

$$z(x,y) = \left[X^{(0)}Y^{(0)}\right] * \left[Cz\right]$$

[Cz]↓	PDE (4.1)	PDE (4.16)	PDE (4.19)	PDE (5.1)
-	2	4	2	2
X	0.3	3	3	3
Y	-0.4	2	-4	4
x^2	0	4	-2	-3
xy	-5.311299993772749	6.849264811348015	-7.512708174515619	-5.396256898077837
y^2	0	-2	4	-4
x^3	0	0	0	0
x^2y	5.958301368667835	-7.711717436565404	19.2822934496235	6.754373906579559
xy^2	4.234934649939510	-19.51288306117300	-0.6098420135279419	5.790102630444205
y^3	0	0	0	0

x^4	0	0	0	0
x^3y	-2.874989612660367	8.295844566807396	-12.18390765788577	2.258376389964472
x^2y^2	-2.773949504978351	8.288054534600711	-4.769315094015080	-12.75907144209255
xy^3	-1.792543706767099	17.92080328731461	-2.090475145001517	4.230362666938873
y^4	0	0	0	0
x^5	0	0	0	0
x^4y	0.8780545998646947	-1.757283860876247	3.244524145572669	-1.129690887907409
x^3y^2	0.6079563801315699	-3.551537608065100	4.484752633311077	2.786845209903647
x^2y^3	0.9649828617150603	-8.290407354894782	0.8214107313336749	0.7116417402416660
xy^4	0.3223773992956812	-4.335750045919715	1.827116095134027	-1.076521854388049
y^5	0	0	0	0
x^6	0	0	0	*
x^5y	-0.1520675938622337	-0.3639551578387353	0.1854259380161165	*
x^4y^2	-0.0577995787822082	0.9497147338675938	-1.568590656239227	*
x^3y^3	-0.1650506239501214	-0.9625061382603679	0.9499649695895012	*
x^2y^4	-0.1206652944106619	2.665528346891400	-0.9458399622741006	*
xy^5	-0.0273120825036137	-0.9625061382603679	0.1043893077311010	*
y^6	0	0	0	*
x^7	0	0	*	*
x^6y	0.0112799144284583	0.0717381065909952	*	*
x^5y^2	0.0008.502921848936	0.0602553262973988	*	*
x^4y^3	0.0118014569079755	-0.1111270203496452	*	*
x^3y^4	0.008.7327015760803	-0.2948655774706394	*	*
x^2y^5	0.0062211695348708	-0.0656443948048216	*	*
xy^6	0.0005.913920026163	0.134282666933508	*	*
y^7	0	0	*	*

Appendix B. Conditions imposed for obtaining a Concordant Function with a given *D* degree

B.1 The number of imposed limit conditions

The number of limit conditions (boundary and initial) that can be imposed to obtain the coefficients of a Concordant Function depends on its degree noted D. These conditions may be expressed as two polynomials noted $\Psi(x, y=0)$ along x axis and $\Omega(x=0, y)$ along y. Supposing D=3 and using (2.1), the limit conditions can be connected to the CF according to

$$\Psi(x, y=0) = CF3(x, y=0) = C_1 + C_2 x + C_4 x^2 + C_7 x^3 \tag{B.1}$$

$$\Omega(x=0, y) = CF3(x=0, y) = C_1 + C_3 y + C_6 y^2 + C_{10} y^3 \tag{B.2}$$

It results that only 7 conditions can be imposed for a CF of third degree. For a CF of degree D the number of conditions that can be imposed is

$$N_{Lim} = 2D+1 \tag{B.3}$$

In contrast to [1], here only the function values (not also the derivatives) of Ψ and Ω will be used as limit conditions.

B.2 The conditions necessary to obtain a CF of D degree

The total number of conditions for determining unequivocally all the coefficients of a CF is given by [1]

$$N_D = (D+1)\times(D+2)/2 \tag{B.4}$$

If D=3 then $N_3=(3+1)\times(3+2)/2=10$, if D=4 then $N_4=(4+1)\times(4+2)/2=15$ and so on.

For obtaining all the N_D unknown coefficients, it is compulsory to use N_D equations that include the N_{Lim}(A.3) conditions. Therefore there are still necessary – besides the boundary conditions - the following number of equations

$$Ec_D=N_D - N_{Lim}= (D+1)\times(D+2)/2 – (2D+1) = D\times(D-1)/2 \quad (B.5)$$

To obtain a CF of a chosen D degree are therefore necessary: for CF3→Ec_3=3×2/2=3 equations; for CF4→Ec_4=6 eq.; CF5→Ec_5=10 eq.; CF6→Ec_6=15 eq.; CF7→Ec_7=21 eq.

In [1] these equations were selected according to Chapters 8 and 9. Here the choice is a little changed. From these equations *it is maintained the integral of PDE*, to which are added the derivatives of PDE. Consequently, the number of equations still necessary is:

-for CF3 → 2 first order derivatives $\left(\partial(PDE)/dx;\partial(PDE)/dy\right)$, $Total_3=2;$

-for CF4 → 2 previous derivatives + 3 new second order derivatives

$$\left(\partial^2(PDE)/dx^2;\partial^2(PDE)/dy\,dy;\partial^2(PDE)/dy^2\right),\ Total_4=5;$$

-for CF5 → 5 previous derivatives + 4 new second order derivatives, $Total_5=9;$

-similarly,for CF6→ $Total_6$=9+5 (new)=*14* ; for CF7 → $Total_7$=14+6 (new)=20.

If to these *"Totals"* is added the 1 equation (representing the *integral of PDE*) it result the numbers of equations Ec_3; Ec_4; Ec_5 and so on. This is the basis for establishing the necessary number of equations valid also for *any degree of CF greater than D=7* (maximum CF degree used here).

The only special problem is represented by the partial derivatives of the **nonlinear source term** $F[\phi(x,y)]$ included in (1.1). The basis for calculating these derivatives was started in [1], relations (9.2.4)... (9.2.7).

R E F E R E N C E S

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