A CLASS OF HARMONIC FUNCTIONS ASSOCIATED WITH A q-SĂLĂGEAN OPERATOR

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In this paper, a class $S_H^0(n,q,A,B)$ of harmonic functions $f \in \mathcal{H}^0$, associated with a q-Sălăgean operator is defined. A necessary and sufficient convolution condition for the functions $f \in \mathcal{H}^0$ to be in this class is proved. A sufficient coefficient condition for the functions $f \in \mathcal{H}^0$ to be sense preserving and univalent and in the same class is obtained. It is proved that this coefficient condition is necessary for the functions in its sub class $\mathcal{T}S_H^0(n,q,A,B)$. Using this necessary and sufficient coefficient condition, results based on the convexity and compactness of the class $\mathcal{T}S_H^0(n,q,A,B)$, and results on the radii of q-starlikeness and q-convexity of order α, extreme points for the functions in the class $\mathcal{T}S_H^0(n,q,A,B)$ are obtained.

Keywords: q-Sălăgean operator; univalent functions; harmonic functions; subordination

1. Introduction

The theory of q-calculus has motivated the researchers due to its applications in the field of physical sciences, specially in quantum physics. Jackson [11, 12] was the first to give some applications of q-calculus by introducing the q-analogues of derivative and integral. Jackson's q-derivative operator ∂_q on a function h analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is defined for $0 < q < 1$, by

$$\partial_q h(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z}, & z \neq 0 \\ h'(0), & z = 0. \end{cases}$$

For a power function $h(z) = z^k, \quad k \in \mathbb{N} = \{1, 2, 3, ...\}$,

$$\partial_q h(z) = \partial_q(z^k) = [k]_q z^{k-1},$$

where $[k]_q$ is the q-integer number k defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ...q^{k-1}. \tag{1}$$

For more detailed study see [3]. Clearly, $\lim_{q\to 1^-} [k]_q = k$ and $\lim_{q\to 1^-} \partial_q h(z) = h'(z)$.

Let $\mathcal{A}$ denote the class of functions h that are analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $h(0) = h'(0) - 1 = 0$.

Complex-valued harmonic functions of the form: $f = u + iv$, where u and v are real-valued harmonic functions in $\mathbb{D}$, can also be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. The Jacobian of the function $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the Lewy [17], every harmonic function $f = h + \overline{g}$ is locally univalent and

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sense preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$ which is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$ such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

The function $\omega(z)$ is called the dilatation of f. By requiring harmonic function to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principal, and zeros being isolated (see for detail [4]).

A class of harmonic functions $f = h + \overline{g}$ with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $\mathcal{H}$ and functions therein are of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{n=1}^{\infty} \overline{b_k z^k}. \tag{2}$$

A sub class of functions $f = h + \overline{g} \in \mathcal{H}$ with the additional condition $g'(0) = 0$ is denoted by $\mathcal{H}^0$. The class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}\ (\mathcal{H}^0)$ is denoted by $S_{\mathcal{H}}\ (S^0_{\mathcal{H}})$. Further, if $g(z) \equiv 0$, the class $S_{\mathcal{H}}$ reduces to the class S of univalent functions in $\mathcal{A}$.

The convolution of two analytic functions $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is defined by $(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$. The convolution $\widetilde{*}$ of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by $(f \widetilde{*} F)(z) = (g * G)(z) + \overline{(h * H)(z)}$.

The q-Sălăgean operator D_q^n of order $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for an analytic function h, is defined by ([8])

$$D_q^0 h(z) = h(z),\ D_q^1 h(z) = D_q h(z) = z\partial_q h(z)$$

and for $n \in \mathbb{N}$,

$$D_q^n h(z) = D_q(D_q^{n-1} h(z)). \tag{3}$$

Observe that

$$D_q h(z) = h(z) * D_q \left(\frac{z}{1-z} \right), \tag{4}$$

and

$$\begin{aligned} D_q \left(\frac{z}{1-z} \right) &= z + \sum_{k=2}^{\infty} [k]_q z^k \\ &= \frac{z}{(1-z)(1-qz)}, \end{aligned} \tag{5}$$

where $[k]_q$ is the q-integer number k defined by (1). The operator D_q^n reduces to the well known Sălăgean operator D^n [21] as $q \to 1^-$.

Further, the q-Sălăgean operator $\mathcal{D}_q^n$ of order $n \in \mathbb{N}_0$ for the harmonic function $f = h + \overline{g}$ is defined by ([13])

$$\mathcal{D}_q^n f(z) = D_q^n h(z) + (-1)^n \overline{D_q^n g(z)}. \tag{6}$$

As $q \to 1^-$, the operator $\mathcal{D}_q^n$ reduces to the operator $\mathcal{D}^n$ which is the modified Sălăgean operator for a harmonic function $f = h + \overline{g}$ ([16]).

We say that a function $h : \mathbb{D} \to \mathbb{C}$ is subordinate to a function $g : \mathbb{D} \to \mathbb{C}$ and write $h(z) \prec g(z),\ \ z \in \mathbb{D}$, if there exists a complex-valued function w which map $\mathbb{D}$ into itself such that $w(0) = 0$ and $h(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$h(z) \prec g(z),\ z \in \mathbb{D} \iff h(0) = g(0) \text{ and } h(\mathbb{D}) \subset g(\mathbb{D}).$$

The above definition of subordination " $\prec$ " was earlier used by Dziok in [5] (see also [2, 6, 7, 15] and in the recent work [1]).

Associated with the q-Sălăgean operator $\mathcal{D}_q^n$, we define a subclass $S_H^0(n, q, A, B)$ of harmonic functions $f \in \mathcal{H}^0$ that satisfy the subordination condition

$$\frac{\mathcal{D}_q^{n+1} f(z)}{\mathcal{D}_q^n f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-B \leq A < B \leq 1; z \in \mathbb{D}) \tag{7}$$

which is equivalent to the condition

$$\left| \frac{\mathcal{D}_q^{n+1} f(z) - \mathcal{D}_q^n f(z)}{B\mathcal{D}_q^{n+1} f(z) - A\mathcal{D}_q^n f(z)} \right| < 1, \ z \in \mathbb{D}. \tag{8}$$

We denote by $\mathcal{T}S_H^0(n, q, A, B)$ a subclass of harmonic functions $f = h + \overline{g} \in S_H^0(n, q, A, B)$, where for this n, functions h and g are of the form:

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \quad g(z) = (-1)^n \sum_{k=2}^{\infty} |b_k| z^k \quad (z \in \mathbb{D}). \tag{9}$$

Clearly, the functions $f \in S_H^0(n, q, A, B)$ satisfy the condition

$$\left| \frac{\mathcal{D}_q^{n+1} f(z)}{\mathcal{D}_q^n f(z)} - \frac{1 - AB}{1 - B^2} \right| < \frac{B - A}{1 - B^2}, \text{ if } B \neq 1,$$

and

$$\Re e \left(\frac{\mathcal{D}_q^{n+1} f(z)}{\mathcal{D}_q^n f(z)} \right) > \frac{1 + A}{2}, \text{ if } B = 1.$$

In particular, if we take $B = q \ (0 < q < 1)$, then for the same q, the class $S_H^0(n, q, A, q)$ may equivalently be defined by

$$\left| \frac{\mathcal{D}_q^{n+1} f(z)}{\mathcal{D}_q^n f(z)} - \frac{1 - Aq}{1 - q^2} \right| < \frac{q - A}{1 - q^2} \ (-q \leq A < q; z \in \mathbb{D}).$$

Observe that as $q \to 1^-$, the class $S_H^0(\lambda, q, A, B) = H^\lambda(A, B)$ was studied by Dziok *et al.* [7] and the class $H^\lambda(A, B)$ for $\lambda = 0, 1$ was studied in [5]. Certain generalized classes of the class $H^\lambda(A, B)$ were studied in [2, 15]. We denote the class $S_H^0(n, q, (1 + q)\alpha - 1, q) \ (0 \leq \alpha < 1)$ by $\mathcal{H}_q^n(\alpha)$ and hence, the classes $\mathcal{H}_q^0(\alpha)$ and $\mathcal{H}_q^1(\alpha)$ are the q-analogue of harmonic starlike and harmonic convex functions of order α, respectively. Further, as $q \to 1^-$, the classes $\mathcal{H}_q^0(\alpha) =: S_H^*(\alpha)$ and $\mathcal{H}_q^1(\alpha) =: S_H^c(\alpha)$ are the well known classes of the functions $f \in S_H^0$ which are starlike and convex functions of order α, respectively, in $\mathbb{D}$ and are investigated by Jahangiri [14].

Research work in connection with function theory and q-calculus was first introduced by Ismail *et al.* [10]. Recently, q-calculus is involved in the theory of analytic functions in the work [8, 9, 18, 20] etc.. But research on q-calculus in connection with harmonic functions is fairly new and not much published (one may find papers [13], [19] and most recently [1]).

In this paper, a class $S_H^0(n, q, A, B)$ of harmonic functions $f \in \mathcal{H}^0$, associated with q-Sălăgean operator is defined as above (7). A necessary and sufficient convolution condition for the functions $f \in \mathcal{H}^0$ to be in this class is proved as Theorem 2.1 below. A sufficient coefficient condition for the functions $f \in \mathcal{H}^0$ to be sense preserving and univalent and in the same class is obtained as Theorem 2.2. It is proved that this coefficient condition is necessary for the functions in its sub class $\mathcal{T}S_H^0(n, q, A, B)$ as Theorem 2.3. Using this necessary and sufficient coefficient condition, in the subsequent work, results on convexity and compactness; results based on the radii of q-starlikeness and q-convexity of order α, and extreme points for the functions in the class $\mathcal{T}S_H^0(n, q, A, B)$ are obtained. This research work will motivate future research to work in the area of q-calculus operators together with harmonic functions.

2. MAIN RESULTS

Theorem 2.1. *Let $f \in \mathcal{H}^0$. Then the function $f \in S_H^0(n,q,A,B)$ if and only if*

$$\mathcal{D}_q^n f(z) \widetilde{*} \Phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where

$$\Phi(z;\zeta) = \frac{(B-A)\zeta z + (1+A\zeta)\, qz^2}{(1-z)(1-qz)} - \overline{\left(\frac{2z + (A+B)\,\overline{\zeta} z - \left(1+A\overline{\zeta}\right) qz^2}{(1-z)(1-qz)}\right)}. \tag{10}$$

Proof. Let $f = h + \overline{g} \in \mathcal{H}^0$ be of the form (2). Then $f \in S_H^0(n,q,A,B)$ if and only if (7) holds or equivalently

$$\frac{\mathcal{D}_q^{n+1} f(z)}{\mathcal{D}_q^n f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\})$$

which by (6) is given by

$$\begin{aligned} & (1+B\zeta)\left[D_q^n\left(D_q h(z)\right) + (-1)^{n+1}\,\overline{D_q^n\left(D_q g(z)\right)}\right] \\ & -(1+A\zeta)\left[D_q^n h(z) + (-1)^n\,\overline{D_q^n g(z)}\right] \\ \neq\ & 0. \end{aligned} \tag{11}$$

On using (4) and (5), the condition (11) may also be given by

$$\begin{aligned} & D_q^n h(z) * \left[(1+B\zeta)\frac{z}{(1-z)(1-qz)} - (1+A\zeta)\,\frac{z}{1-z}\right] \\ & -(-1)^n\overline{D_q^n g(z)} * \left[(1+B\zeta)\frac{\bar{z}}{(1-\bar{z})(1-q\bar{z})} + (1+A\zeta)\,\frac{\bar{z}}{1-\bar{z}}\right] \neq 0 \end{aligned}$$

which on using the convolution $\widetilde{*}$ between two harmonic functions, we get

$$\mathcal{D}_q^n f(z) \widetilde{*} \Phi(z;\zeta) \neq 0,$$

where the harmonic function $\Phi(z;\zeta)$ is given by (10). □

If we consider $q \to 1^-$ in Theorem 2.1, we get following result involving the Sălăgean operator $\mathcal{D}^n$:

Corollary 2.1. *Let $f \in \mathcal{H}^0$. Then the function $f \in S_H^0(n,A,B)$ if and only if*

$$\mathcal{D}^n f(z) \widetilde{*} \phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where

$$\phi(z;\zeta) = \frac{(B-A)\zeta z + (1+A\zeta)\, z^2}{(1-z)^2} - \overline{\left(\frac{2z + (A+B)\,\overline{\zeta} z - \left(1+A\overline{\zeta}\right) z^2}{(1-z)^2}\right)}. \tag{12}$$

Remark 2.1. The result of Corollary 2.1 with $\phi(z;\zeta)$ given by (12) improves the results of Dziok *et al.* [7, Theorem 1, p.3].

Theorem 2.2. *Let $f = h + \overline{g} \in \mathcal{H}^0$ be of the form (2) and let $-B \leq A < B \leq 1$. If*

$$\sum_{k=2}^{\infty} \left(C_k|a_k| + D_k|b_k|\right) \leq B - A, \tag{13}$$

where

$$C_k = ([k]_q)^n\{[k]_q(1+B) - (1+A)\}, \tag{14}$$

$$D_k = ([k]_q)^n\{[k]_q(1+B) + (1+A)\}, \tag{15}$$

and $[k]_q$ is the q-integer number k defined by (1), then

(i) *the function f is locally univalent and sense-preserving as $q \to 1^-$ and univalent in $\mathbb{D}$,*
(ii) *the function $f \in S_H^0(n, q, A, B)$.*

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f = h + \overline{g}$, where h and g of the form (2) and assume that there exist $k \in \{2, 3, \dots\}$ such that $a_k \neq 0$ or $b_k \neq 0$. Since, from (1), $[k]_q > 1$, we observe from (14) and (15) that $D_k \geq C_k > [k]_q(B - A) \quad (k = 2, 3, \dots)$, by which the condition (13) implies the condition

$$\sum_{k=2}^{\infty} [k]_q \left(|a_k| + |b_k|\right) < 1 \tag{16}$$

and hence, we get for any $q \ (0 < q < 1)$,

$$\begin{aligned} |\partial_q h(z)| - |\partial_q g(z)| &\geq 1 - \sum_{k=2}^{\infty} [k]_q |a_k||z^{k-1}| - \sum_{k=2}^{\infty} [k]_q |b_k||z^{k-1}| \\ &> 1 - |z| \sum_{k=2}^{\infty} [k]_q \left(|a_k| + |b_k|\right) > 1 - |z| > 0 \end{aligned}$$

in $\mathbb{D}$ which implies as $q \to 1^-$ that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ that is the function f is locally univalent and sense-preserving in $\mathbb{D}$. Moreover, if $z_1, z_2 \in \mathbb{D}$ and for some $q \,(0 < q < 1)\,, z_1 \neq qz_2$. Then for that q,

$$\left| \frac{z_1^k - (qz_2)^k}{z_1 - (qz_2)} \right| = \left| \sum_{l=1}^{k} z_1^{l-1} (qz_2)^{k-l} \right| \leq \sum_{l=1}^{k} |z_1|^{l-1} q^{k-l} |z_2|^{k-l} < [k]_q \quad (k = 2, 3, \dots).$$

Hence, for that value of q, from (16), we have

$$\begin{aligned} |f(z_1) - f(qz_2)| &\geq |h(z_1) - h(qz_2)| - |g(z_1) - g(qz_2)| \\ &\geq \left| z_1 - qz_2 - \sum_{k=2}^{\infty} a_k (z_1^k - (qz_2)^k) \right| - \left| \sum_{k=2}^{\infty} \overline{b_k (z_1^k - (qz_2)^k)} \right| \\ &\geq |z_1 - qz_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left| \frac{z_1^k - (qz_2)^k}{z_1 - qz_2} \right| - \sum_{k=2}^{\infty} |b_k| \left| \frac{z_1^k - (qz_2)^k}{z_1 - qz_2} \right| \right) \\ &> |z_1 - qz_2| \left(1 - \sum_{k=2}^{\infty} [k]_q |a_k| - \sum_{k=2}^{\infty} [k]_q |b_k| \right) > 0 \end{aligned}$$

which proves that f is univalent in $\mathbb{D}$. This proves the result (i). To prove result (ii), it needs to show that the function f satisfy the condition (8). Consider for $f = h + \overline{g}$, where h and g of the form (2) and for $|z| = r \ (0 < r < 1)$,

$$\begin{aligned} & \left|\mathcal{D}_q^{n+1} f(z) - \mathcal{D}_q^n f(z)\right| - |B\mathcal{D}_q^{n+1} f(z) - A\mathcal{D}_q^n f(z)| \\ = \; & \left| \sum_{k=2}^{\infty} ([k]_q)^n ([k]_q - 1) a_k z^k - (-1)^n \sum_{k=2}^{\infty} ([k]_q)^n ([k]_q + 1) \overline{b_k z^k} \right| \\ & - \left| (B - A)z + \sum_{k=2}^{\infty} ([k]_q)^n (B[k]_q - A) a_k z^k \right. \\ & \qquad \left. - (-1)^n \sum_{k=2}^{\infty} ([k]_q)^n (B[k]_q + A) \overline{b_k z^k} \right| \end{aligned}$$

$$\le \sum_{k=2}^{\infty}([k]_q)^n([k]_q-1)|a_k|r^k+\sum_{k=2}^{\infty}([k]_q)^n([k]_q+1)|b_k|r^k$$
$$-(B-A)r+\sum_{k=2}^{\infty}([k]_q)^n(B([k]_q-A)|a_k|r^k$$
$$+\sum_{k=2}^{\infty}([k]_q)^n(B[k]_q+A)|b_k|r^k$$
$$<\sum_{k=2}^{\infty}(C_k|a_k|+D_k|b_k|)r^{k-1}-(B-A)$$
$$\le \sum_{k=2}^{\infty}(C_k|a_k|+D_k|b_k|)r^{k-1}-(B-A)\le 0$$

if the condition (13) holds, where C_k and D_k are given, respectively, by (14) and (15). This proves the condition (8). This completes the proof of Theorem 2.2. □

Remark 2.2. Equality in (13) occurs for the function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{B-A}{C_k}\alpha_k z^k+\sum_{n=2}^{\infty}\frac{B-A}{D_k}\overline{\beta_k}\overline{z}^k,$$

where C_k and D_k are defined, respectively, by (14) and (15), $-B\le A<B\le 1, z\in\mathbb{D}$ and $\sum_{k=2}^{\infty}(|\alpha_k|+|\beta_k|)=1$.

Theorem 2.3. *Let $f=h+\overline{g}\in\mathcal{H}^0$, where h and g are given by (9). Then $f\in\mathcal{T}S_H^0(n,q,A,B)$, if and only if the condition (13) holds.*

Proof. If part is proved in Theorem 2.2. To prove only if part let $f=h+\overline{g}\in\mathcal{T}S_H^0(n,q,A,B)$, where h and g are given by (9). Then by the class condition (7) we have from (8) that for any $z\in\mathbb{D}$,

$$\left|\frac{\sum_{k=2}^{\infty}([k]_q)^n([k]_q-1)|a_k|z^k+\sum_{k=2}^{\infty}([k]_q)^n([k]_q+1)|b_k|\overline{z}^k}{(B-A)z-\sum_{k=2}^{\infty}([k]_q)^n(B([k]_q-A)|a_k|z^k-\sum_{k=2}^{\infty}([k]_q)^n(B([k]_q+A)|b_k|\overline{z}^k}\right|<1,$$

where for $z=r$ $(0\le r<1)$, we obtain

$$\frac{\sum_{k=2}^{\infty}([k]_q)^n([k]_q-1)|a_k|r^{k-1}+\sum_{k=2}^{\infty}([k]_q)^n([k]_q+1)|b_k|r^{k-1}}{(B-A)-\sum_{k=2}^{\infty}([k]_q)^n(B([k]_q-A)|a_k|r^{k-1}-\sum_{k=2}^{\infty}([k]_q)^n(B([k]_q+A)|b_k|r^{k-1}}<1$$

which proves for C_k and D_k defined, respectively, by (14) and (15), that

$$\sum_{k=2}^{\infty}(C_k|a_k|+D_k|b_k|)\,r^{k-1}<B-A\quad(0\le r<1). \tag{17}$$

Let σ_k be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty}(C_k|a_k|+D_k|b_k|)\,.$$

Then σ_k is a non decreasing sequence and by (17) it is bounded above. Thus, as $r\to 1^-$, it is convergent and

$$\sum_{k=2}^{\infty}[C_k|a_k|+D_k|b_k|]=\lim_{k\to\infty}\sigma_k\le B-A.$$

This gives the condition (13).

□

Remark 2.3. As $q \to 1^-$, the result of Theorem 2.3 coincides with the result [7, Theorem 2, p.4].

Taking $B = q \quad (0 < q < 1)$ and $A = (1 + q)\alpha - 1 \quad (0 \le \alpha < 1)$ in Theorem 2.3, we get following result:

Corollary 2.2. *Let $f = h + \overline{g} \in \mathcal{H}^0$, where h and g are given by (9). Then $f \in \mathcal{TH}_q^n(\alpha)$, if and only if the condition*

$$\sum_{k=2}^{\infty}([k]_q)^n \left[([k]_q - \alpha)\,|a_k| + ([k]_q + \alpha)\,|b_k|\right] \le 1 - \alpha \tag{18}$$

holds, where $[k]_q$ is the q-integer number k defined by (1).

Remark 2.4. Corollary 2.2 gives a necessary and sufficient condition for the functions $f = h + \overline{g} \in \mathcal{H}^0$, where h and g are given by (9) to be q-starlike and q-convex of order α in $\mathbb{D}$ if we put $n = 0$ and 1, respectively, in (18) and are given by

$$\sum_{k=2}^{\infty}\left[([k]_q - \alpha)\,|a_k| + ([k]_q + \alpha)\,|b_k|\right] \le 1 - \alpha \tag{19}$$

and

$$\sum_{k=2}^{\infty}[k]_q \left[([k]_q - \alpha)\,|a_k| + ([k]_q + \alpha)\,|b_k|\right] \le 1 - \alpha. \tag{20}$$

Theorem 2.4. *The class $\mathcal{T}S_H^0(n, q, A, B)$ is a convex and compact subclass of the class of functions $f = h + \overline{g} \in \mathcal{H}^0$, where h and g are of the form (9).*

Proof. Let for $t = 1, 2$, $f_t \in \mathcal{T}S_H^0(n, q, A, B)$, and let for this n it is of the form

$$f_t(z) = z - \sum_{k=2}^{\infty}|a_{t,k}|z^k + (-1)^n \sum_{k=2}^{\infty}|b_{t,k}|\overline{z}^k \quad (\, z \in \mathbb{D}). \tag{21}$$

Then for $0 \le \rho \le 1$,

$$\begin{aligned} F(z) \quad : \quad &= \rho f_1(z) + (1-\rho) f_2(z) \\ = \quad & z - \sum_{k=2}^{\infty}\left\{\rho|a_{1,k}| + (1-\rho)|a_{2,k}|\right\} z^k + (-1)^n \sum_{k=2}^{\infty}\left\{\rho|b_{1,k}| + (1-\rho)|b_{2,k}|\right\} \overline{z}^k \end{aligned}$$

and by Theorem 2.3, we get for C_k and D_K defined by (14), that

$$\begin{aligned} &\sum_{k=2}^{\infty}\left[C_k \left\{\rho|a_{1,k}| + (1-\rho)|a_{2,k}|\right\} + D_k \left\{\rho|b_{1,k}| + (1-\rho)|b_{2,k}|\right\}\right] \\ &= \rho \sum_{k=2}^{\infty}\{C_k|a_{1,k}| + D_k|b_{1,k}|\} + (1-\rho)\sum_{k=2}^{\infty}\{C_k|a_{2,k}| + D_k|b_{2,k}|\} \\ &\le \rho(B - A) + (1-\rho)(B - A) = B - A \end{aligned}$$

This proves that the function $F \in \mathcal{T}S_H^0(n, q, A, B)$. Hence, the class $\mathcal{T}S_H^0(n, q, A, B)$ is convex. On the other hand, if we consider a sequence of functions $f_t \in \mathcal{T}S_H^0(n, q, A, B)$, $t \in \mathbb{N} = \{1, 2, 3, ...\}$ of the form (21), then by Theorem 2.3, we get for C_k and D_k defined by (14),

$$\sum_{k=2}^{\infty}\{C_k|a_{t,k}| + D_k|b_{t,k}|\} \le B - A. \tag{22}$$

Hence, for $|z| \le r$ $(0 < r < 1)$,

$$\begin{aligned}|f_t(z)| &\le r + \sum_{k=2}^{\infty}\{|a_{t,k}| + |b_{t,k}|\}r^k \\ &\le r + \frac{1}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}\sum_{k=2}^{\infty}\{C_k|a_{t,k}| + D_k|b_{t,k}|\}r^k \\ &< r + \frac{B-A}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}r^2.\end{aligned}$$

Similarly, we get for $|z| \le r$ $(0 < r < 1)$,

$$|f_t(z)| > r - \frac{B-A}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}r^2.$$

Therefore, class $\mathcal{T}S_H^0(n,q,A,B)$ is locally uniformly bounded. Let $f = h + \overline{g}$, where h and g are given by (9). If we assume that $f_t \to f$, then we conclude that $|a_{t,k}| \to |a_k|$ and $|b_{t,k}| \to |b_k|$ as $t \to \infty$ for any k=2, 3, ... Hence, from (22), we get

$$\sum_{k=2}^{\infty}\{C_k|a_k| + D_k|b_k|\} \le B - A$$

which proves that $f \in \mathcal{T}S_H^0(n,q,A,B)$ and therefore the class $\mathcal{T}S_H^0(n,q,A,B)$ is closed. This proves the compactness of the class $\mathcal{T}S_H^0(n,q,A,B)$. □

Corollary 2.3. *Let $f \in \mathcal{T}S_H^0(n,q,A,B)$. Then for $|z| = r$ $(r < 1)$,*

$$r - \frac{B-A}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}r^2 < |f(z)| < r + \frac{B-A}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}r^2.$$

Furthermore,

$$\left\{w \in \mathbb{C} : |w| < 1 - \frac{B-A}{([2]_q)^n\{[2]_q(1+B)-(1+A)\}}\right\} \subset f(\mathbb{D}).$$

The minimum of all values of the radius $r \in (0,1)$ for functions $f \in \mathcal{T}S_H^0(n,q,A,B)$ such that $\frac{f(rz)}{r} \in \mathcal{H}_q^*(\alpha)$ is called the radius of q-starlikeness of order α and is denoted by $r_{\mathcal{H}_q^*(\alpha)}(\mathcal{T}S_H^0(n,q,A,B))$.

In the following theorem we obtain the radius of q-starlikeness of order α for functions $f \in \mathcal{T}S_H^0(n,q,A,B)$.

Theorem 2.5. *Let $0 \le \alpha < 1$ and C_k, D_k are defined, respectively, by* (14), (15)*. Then*

$$r_{\mathcal{H}_q^*(\alpha)}(\mathcal{T}S_H^0(n,q,A,B)) = \inf_{k\ge 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{C_k}{[k]_q-\alpha}, \frac{D_k}{[k]_q+\alpha}\right\}\right]^{\frac{1}{k-1}}, \tag{23}$$

where $[k]_q$ is the q-integer number k defined by (1).

Proof. Let $f = h + \overline{g} \in \mathcal{T}S_H^0(n,q,A,B)$, where h and g are given by (9). Then by Theorem 2.3, we have

$$\sum_{k=2}^{\infty}\left\{\frac{C_k}{B-A}|a_k| + \frac{D_k}{B-A}|b_k|\right\} \le 1,$$

where C_k and D_k are defined, respectively, by (14) and (15). Let r_0 be the radius of q-starlikeness of order α. Then $\frac{f(r_0 z)}{r_0} \in \mathcal{H}_q^*(\alpha)$ if and only if from (19) that

$$\sum_{k=2}^{\infty}\{([k]_q - \alpha)|a_k| + ([k]_q + \alpha)|b_k|\}r_0^{k-1} \le 1 - \alpha$$

which is true if $\frac{([k]_q-\alpha)}{1-\alpha}r_0^{k-1} \leq \frac{C_k}{B-A}$ $(k=2,3,\dots)$ and $\frac{([k]_q+\alpha)}{1-\alpha}r_0^{k-1} \leq \frac{D_k}{B-A}$ $(k=2,3,\dots)$ or if

$$r_0 \leq \left[\frac{1-\alpha}{B-A}\min\left\{\frac{C_k}{([k]_q-\alpha)}, \frac{D_k}{([k]_q+\alpha)}\right\}\right]^{\frac{1}{k-1}} \quad (k=2,3,\dots).$$

It follows that the radius $r_{\mathcal{H}_q^*(\alpha)}(\mathcal{T}S_H^0(n,q,A,B))$ is given by (23). □

Similarly, we may find the radius of q-convexity of order α for functions $f \in \mathcal{T}S_H^0(n,q,A,B)$ which is as below:

Theorem 2.6. *Let* $0 \leq \alpha < 1$ *and* C_k *and* D_k *are defined, respectively, by* (14) *and* (15). *Then*

$$r_{\mathcal{H}_q^c(\alpha)}(\mathcal{T}S_H^0(n,q,A,B)) = \inf_{k\geq 2}\left[\frac{1-\alpha}{(B-A)\,[k]_q}\min\left\{\frac{C_k}{[k]_q-\alpha}, \frac{D_k}{[k]_q+\alpha}\right\}\right]^{\frac{1}{k-1}}, \tag{24}$$

where $[k]_q$ *is the q-integer number* k *defined by (1).*

Theorem 2.7. *Let* $f \doteq h+\overline{g}$ *be of the form* (9). *Then* $f \in \mathcal{T}S_H^0(n,q,A,B)$ *if and only if*

$$f(z) = \sum_{k=1}^{\infty}\left[x_k h_k(z) + y_k g_k(z)\right], \tag{25}$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{B-A}{C_k}z^k, g_1(z) = z, g_k(z) = z + (-1)^n\frac{B-A}{D_k}\bar{z}^k, \quad k=2,3,...$$

$$x_k, y_k \geq 0, \quad x_1 = 1 - \sum_{k=2}^{\infty}x_k - \sum_{k=1}^{\infty}y_k. \tag{26}$$

In particular the points $\{h_k\}$ *and* $\{g_k\}$ *are called the extreme points of the closed convex hull of the class* $\mathcal{T}S_H^0(n,q,A,B)$ *denoted by clco*$\mathcal{T}S_H^0(n,q,A,B)$.

Proof. Let f be given by (25). Then from (26), it is of the form

$$f(z) = z - \sum_{k=2}^{\infty}x_k\frac{B-A}{C_k}z^k + (-1)^n\sum_{k=2}^{\infty}y_k\frac{B-A}{D_k}\overline{z}^k$$

which by Theorem 2.3 proves that $f \in \mathcal{T}S_H^0(n,q,A,B)$, since for this function

$$\begin{aligned}&\sum_{k=2}^{\infty}\left(C_kx_k\frac{B-A}{C_k} + D_ky_k\frac{B-A}{D_k}\right) = (B-A)\sum_{k=2}^{\infty}(x_k+y_k)\\ =\ &(B-A)(1-\ x_1-\ y_1) \leq B-A.\end{aligned}$$

Conversely, let $f = h+\overline{g} \in \mathcal{T}S_H^0(n,q,A,B)$ be of the form (9). Set $x_k = \frac{C_k}{B-A}|a_k|$, $y_k = \frac{D_k}{B-A}|b_k|$.

Then on using (26), we obtain

$$\begin{aligned}f(z) &= z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=2}^{\infty}|b_k|\overline{z}^k = z - \sum_{k=2}^{\infty}x_k\frac{B-A}{C_k}z^k + (-1)^n\sum_{k=2}^{\infty}y_k\frac{B-A}{D_k}\overline{z}^k\\ &= z - \sum_{k=2}^{\infty}x_k\{z-h_k(z)\} + \sum_{k=2}^{\infty}y_k\{g_k(z)-z\}\\ &= \left[1-\sum_{k=2}^{\infty}(x_k+y_k)\right]z + \sum_{k=2}^{\infty}\{x_kh_k(z)+y_kg_k(z)\}\end{aligned}$$

which is of the form (25). This proofs Theorem 2.7. □

Corollary 2.4. *Let* $f \in \mathcal{T}S_H^0(n,q,A,B)$ *be of the form* (9)*. Then*

$$|a_k| \leq \frac{B-A}{C_k} \text{ and } |b_k| \leq \frac{B-A}{D_k}, \quad k=2,3,4,..., \tag{27}$$

where C_k *and* D_k *are defined, respectively, by* (14) *and* (15)*. Equality in the inequalities (27) occurs for the extremal functions* $h_k(z)$ *and* $g_k(z)$ *given in (26) for* $k=2,3,4,...$

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