U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 2, 2014 ISSN 1223-7027

SHRINKING PROJECTION METHOD FOR A SEQUENCE OF RELATIVELY QUASI-NONEXPANSIVE MULTIVALUED MAPPINGS AND EQUILIBRIUM PROBLEM IN BANACH SPACES

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Strong convergence of a new iterative process based on the Shrinking projection method to a common element of the set of common fixed points of an infinite family of relatively quasi-nonexpansive multivalued mappings and the solution set of an equilibrium problem in a Banach space is established. Our results improved and extend the corresponding results announced by many others.
Key words: Shrinking projection method , equilibrium problem, common fixed point, relatively quasi-nonexpansive multivalued mappings.
2000 Mathematics Subject Classification: 47H10,47H09.

1. Introduction

A nonempty subset C of a Banach space E is called proximinal if for each $x \in E$, there exists an element $y \in C$ such that

$$\| x - y \| = dist(x, C) = inf\{\| x - z \| : z \in C\}.$$

We denote by $N(C), CB(C)$ and $P(C)$ the collection of all nonempty subsets, nonempty closed bounded subsets and nonempty proximinal bounded subsets of C, respectively. The Pompeiu-Hausdorff metric H on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A)\},$$

for all $A, B \in CB(C)$.
Let $T : E \longrightarrow N(E)$ be a multivalued mapping. An element $x \in E$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 1.1. A multivalued mapping $T : E \longrightarrow CB(E)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in E.$$

(ii) quasi-nonexpansive if

$$F(T) \neq \emptyset \qquad and \qquad H(Tx, Tp) \leq \|x - p\|, \quad x \in E, \quad p \in F(T).$$

The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of nonexpansive multivalued mappings is harder than the corresponding theory of nonexpansive single valued mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings (see [1-7]).

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Let E be a real Banach space and let E^* be the dual space of E. Let C be a closed convex subset of E. Let F be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow \mathbb{R}$ is to find $\widehat{x} \in C$ such that

$$F(\widehat{x}, y) \geq 0, \qquad \forall y \in C.$$

The set of solutions is denoted by $EP(F)$. Equilibrium problems, have had a great impact and influence in the development of several branches of pure and applied sciences. Numerous problems in physics, optimization and economics reduce to finding a solution of the equilibrium problem . Some methods have been proposed to solve the equilibrium problem in a Hilbert space. See [8-10].

Let E be a real Banach space with norm $\|.\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle ., . \rangle$ the generalized duality pairing between E and E^*. As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \longrightarrow C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [11] recently introduced a generalized projection operator E^* in a smooth Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \qquad x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y)$ reduces to $\|x - y\|^2$. The generalized projection $\Pi_C : E \longrightarrow C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$ that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x)$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [11, 12, 13]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2 \qquad \forall x, y \in E. \tag{1.1}$$

Remark 1: If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$ (see [13, 14]).

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T be a mapping from C into itself. A mapping T is said to be relatively quasi-nonexpansive ([15, 17]) if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is called closed if $x_n \longrightarrow w$ and $\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$ then $w = T(w)$.

In the recent years, approximation of fixed points of relatively quasi-nonexpansive mappings by iteration has been studied by many authors, see [15-22].

Very recently, Eslamian and Abkar [7] introduce the relatively quasi-nonexpansive multivalued mapping as follows:

Definition 1.2. Let C be a closed convex subset of a smooth Banach space E, and $T : C \longrightarrow N(C)$ be a multivalued mapping. We set

$$\Phi(Tx, Tp) = max\{\sup_{q \in Tp} \inf_{y \in Tx} \phi(y, q), \sup_{y \in Tx} \inf_{q \in Tp} \phi(y, q)\}.$$

We call T is relatively quasi-nonexpansive multivalued mapping if $F(T) \neq \emptyset$ and

$$\Phi(Tx, Tp) \leq \phi(x, p), \qquad \forall p \in F(T), \quad \forall x \in C.$$

We present an example of a multivalued mapping such that P_T is relatively quasi-nonexpansive, but T is not relatively quasi-nonexpansive.

Example 1.1. *Let* $I=[0,1]$, $E=L^p(I)$, $1<p<\infty$ *and* $C=\{f\in E: f(x)\geq 0, \forall x\in I\}$. *Let* $T:C\longrightarrow CB(C)$ *be defined by*

$$T(f)=\{g\in C: f(x)\leq g(x)\leq 2\,f(x)\}.$$

Then we have $P_T(f)=\{f\}$ *and hence*

$$\Phi(P_T(f_1),P_T(f_2))\leq \phi(f_1,f_2), \qquad \forall f_1,f_2\in C,$$

that is P_T *is relatively quasi-nonexpansive. Now putting* $f_1(x)=0$ *and* $f_2(x)=2$ *we have* $T(f_1)=0$ *and* $T(f_2)=\{g\in C: 2\leq g(x)\leq 4\}$, *hence* $\Phi(T0,T2)=\sup_{g\in T2}\phi(0,g)=\phi(0,4)$. *On the other hand* $\phi(0,2)=\|2\|_p^2=4$ *and* $\phi(0,4)=\|4\|_p^2=16$, *which shows that*

$$\Phi(T0,T2)>\phi(0,2).$$

Hence T *is not relatively quasi-nonexpansive.*

Definition 1.3. A multivalued mapping T is called closed if $x_n\longrightarrow w$ and $\lim_{n\longrightarrow\infty} dist(x_n,Tx_n)=0$, then $w\in T(w)$.

In [7], Eslamian and Abkar proved the following theorem.

Theorem 1.4. *Let* E *be a uniformly smooth and uniformly convex Banach space, and let* C *be a nonempty closed convex subset of* E. *Let* F *be a bifunction from* $C\times C$ *into* $\mathbb{R}$ *satisfying* $(A1)-(A4)$. *Let* $T_i: C\longrightarrow P(C)$, $i=1,2,...,m$, *be a finite family of multivalued mappings such that* P_{T_i} *is closed and relatively quasi-nonexpansive. Assume that* $\mathcal{F}=\bigcap_{i=1}^m F(T_i)\bigcap EP(f)\neq\emptyset$. *For* $x_0\in C$ *and* $C_0=C$, *let* $\{x_n\}$ *be a sequences generated by the following algorithm:*

$$\begin{cases} y_{n,1}=J^{-1}((1-a_{n,1})Jx_n+a_{n,1}Jz_{n,1}),\\ y_{n,2}=J^{-1}((1-a_{n,2})Jx_n+a_{n,2}Jz_{n,2}),\\ \dots\\ y_{n,m}=J^{-1}((1-a_{n,m})Jx_n+a_{n,m}Jz_{n,m}),\\ u_n\in C\, such\, that\, f(u_n,y)+\frac{1}{r_n}\langle y-u_n, Ju_n-Jy_{n,m}\rangle\geq 0, \qquad \forall y\in C,\\ C_{n+1}=\{z\in C_n: \phi(z,u_n)\leq\phi(z,x_n)\},\\ x_{n+1}=\prod_{C_{n+1}}x_0, \qquad \forall n\geq 0\end{cases}$$

where $z_{n,1}\in P_{T_1}x_n$ *and* $z_{n,i}\in P_{T_i}y_{n,i-1}$ *for* $i=2,...,m$ *and* J *is the duality mapping on* E. *Assume that* $\sum_{i=1}^m a_{n,i}=1$, $\{a_{n,i}\}\in[a,b]\subset(0,1)$ *and* $\{r_n\}\subset[c,\infty)$ *for some* $c>0$. *Suppose that* P_{T_i} *is uniformly continuous with respect to the Hausdorff metric for* $i=2,3,...,m$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}}x_0$.

In this paper, we introduce a new shirking projection algorithm for finding a common element of the set of common fixed points of an infinite family of relatively quasi-nonexpansive multivalued mappings and the set of solutions of an equilibrium problem in uniformly smooth and uniformly convex Banach spaces. Strong convergence to common elements of two set is established. Our results improved and extend the corresponding results announced by many others.

2. **Preliminaries**

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\|<1$ for all $x,y\in E$ with $\|x\|=\|y\|=1$ and $x\neq y$. It said to be uniformly convex if $\lim_{n\longrightarrow\infty}\|x_n-y_n\|=0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\|=\|y_n\|=1$ and $\lim_{n\longrightarrow\infty}\|\frac{x_n+y_n}{2}\|=1$. Let $U=\{x\in E:\|x\|=1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\longrightarrow 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Lemma 2.1. *([12]) Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If $\phi(x_n, y_n) \longrightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \longrightarrow 0$.*

Lemma 2.2. *([11]) Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \qquad \forall y \in C.$$

Lemma 2.3. *([11]) Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C.$$

Lemma 2.4. *([23]) Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \| x \| \leq r\}$, for $r > 0$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} a_n = 1$ there exists a continuous, strictly increasing and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that that for any positive integers i, j with $i < j$,*

$$\| \sum_{n=1}^{\infty} a_n x_n \|^2 \leq \sum_{n=1}^{\infty} a_n \|x_n\|^2 - a_i a_j g(\|x_i - x_j\|.$$

For solving the equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$,
(A3) F is upper-hemicontinuous, i.e. for each $x, y, z \in C$,

$$\limsup_{t \longrightarrow 0^+} F(tz + (1 - t)x, y) \leq F(x, y)$$

(A4) $F(x, .)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma was proved in [8].

Lemma 2.5. *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \qquad \forall y \in C.$$

The following lemma was given in [21].

Lemma 2.6. *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in E$. define a mapping $T_r : E \longrightarrow C$ as follows:*

$$S_r x = \{z \in C : F(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C\}.$$

Then, the following hold:

(i) *S_r is single valued;*
(ii) *S_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle;$$

(iii) *$F(S_r) = EP(F)$;*
(iv) *$EP(F)$ is closed and convex.*

Lemma 2.7. *([21]) Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, and let $r > 0$. Then for all $x \in E$ and $q \in F(S_r)$,*

$$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x).$$

Lemma 2.8. [7] *Let C be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Suppose $T : C \longrightarrow P(C)$ is a multivalued mapping such that P_T is a relatively quasi-nonexpansive multivalued mapping where*

$$P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}.$$

If $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

3. **Main Result**

In this section, we prove strong convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of an infinite family of relatively quasi- nonexpansive multivalued mappings in a Banach space.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $T_i : C \longrightarrow N(C)$, be a sequence of multivalued mappings such that for each $i \in \mathbb{N}$, P_{T_i} is closed relatively quasi- nonexpansive multivalued mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \neq \emptyset$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_n = J^{-1}(a_{n,0} Jx_n + \sum_{i=1}^{\infty} a_{n,i} Jz_{n,i}), \\ u_n \in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \qquad \forall n \geq 0, \end{cases}$$

where $\sum_{i=0}^{\infty} a_{n,i} = 1$ and $z_{n,i} \in P_{T_i} x_n$. Assume further that $\liminf_n a_{n,0}\, a_{n,i} > 0$, $\{r_n\} \subset (0, \infty)$ and $\liminf_n r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the projection of E onto $\mathcal{F}$.

Proof. First, we show by induction that $\mathcal{F} = (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap EP(F) \subset C_n$ for all $n \geq 0$. From $C_0 = C$, we have $\mathcal{F} \subset C_0$. We suppose that $\mathcal{F} \subset C_n$ for some $n \geq 0$. Let $u \in \mathcal{F}$, then we have $P_{T_i} u = \{u\}, (i \in \mathbb{N})$. Since S_{r_n} and T_i are relatively quasi- nonexpansive, we have

$$\begin{aligned} \phi(u, u_n) = \phi(u, S_{r_n} y_n) \leq \phi(u, y_n) &= \phi(u, J^{-1}(a_{n,0} Jx_n + \sum_{i=1}^{\infty} a_{n,i} Jz_{n,i})) \\ &= \|u\|^2 - 2\langle u, a_{n,0} Jx_n + \sum_{i=1}^{\infty} a_{n,i} Jz_{n,i}\rangle + \|a_{n,0} Jx_n + \sum_{i=1}^{\infty} a_{n,i} Jz_{n,i}\|^2 \\ &\leq \|u\|^2 - 2a_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} a_{n,i}\langle u, Jz_{n,i}\rangle + a_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty} a_{n,i}\|z_{n,i}\|^2 \\ &= a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\phi(u, z_{n,i}) \\ &= a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i} \inf_{u \in P_{T_i} u} \phi(u, z_{n,i}) \\ &\leq a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\Phi(P_{T_i} u, P_{T_i} x_n) \\ &\leq a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\phi(u, x_n) = \phi(u, x_n), \end{aligned}$$

which implies that $u \in C_{n+1}$. Hence

$$\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \subset C_n, \qquad \forall n \geq 0.$$

We observe that C_n is closed and convex (see [20, 21]). From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \qquad \forall z \in C_n. \tag{3.1}$$

Since $\mathcal{F} \subset C_n$ for all $n \geq 0$, we obtain that

$$\langle x_n - u, Jx_0 - Jx_n \rangle \geq 0 \qquad \forall u \in \mathcal{F}.$$

From Lemma 2.3 we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \leq \phi(u, x_0)$$

for all $u \in \mathcal{F} \subset C_n$. Then the sequence $\phi(x_n, x_0)$ is bounded. Thus $\{x_n\}$ is bounded. From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$ we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \qquad \forall n \geq 0.$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. So the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n for any positive integer $m \geq n$ we have

$$x_m = \Pi_{C_m} x_0 \in C_m \subset C_n.$$

It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned}$$

Letting $m, n \longrightarrow \infty$ we have

$$\lim_{n \longrightarrow \infty} \phi(x_m, x_n) = 0. \tag{3.2}$$

It follows from Lemma 2.1 that $x_m - x_n \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since C is closed and convex subset of Banach space E, we can assume that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Next we show $z \in \bigcap_{i=1}^{\infty} F(T_i)$. By taking $m = n+1$ in (3.2) we get

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.3}$$

It follows from Lemma 2.1 that

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

From $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \qquad n \geq 0,$$

it follows from (3.3) that

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

By Lemma 2.1 we have

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0. \tag{3.5}$$

Combining (3.4) with (3.5) one see

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| \leq \lim_{n \longrightarrow \infty} (\|x_{n+1} - x_n\| + \|x_{n+1} - u_n\|) = 0. \tag{3.6}$$

It follows from $x_n \longrightarrow z$ as $n \longrightarrow \infty$ that $u_n \longrightarrow z$ as $n \longrightarrow \infty$. Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0. \tag{3.7}$$

We show that $\{z_{n,i}\}$ is bounded for $i \in \mathbb{N}$. Indeed, for $u \in \mathcal{F}$ we have

$$(\|z_{n,i}\| - \|u\|)^2 \leq \phi(z_{n,i}, u) \leq \phi(x_n, u) \leq (\|x_n\| + \|u\|)^2.$$

Since $\{x_n\}$ is bounded, we obtain $\{z_{n,i}\}$ is bounded for $i \in \mathbb{N}$. Let

$$r = sup_{n\geq 0}\{\|x_n\|, \|z_{n,i}\| : i \in \mathbb{N}\}.$$

Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore from Lemma 2.4 there exists a continuous strictly increasing, and convex function g with $g(0) = 0$ such that

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) \\
&= \phi(u, J^{-1}(a_{n,0}Jx_n + \sum_{i=1}^{\infty} a_{n,i}Jz_{n,i})) \\
&= \|u\|^2 - 2\langle u, a_{n,0}Jx_n + \sum_{i=1}^{\infty} a_{n,i}Jz_{n,i}\rangle + \|a_{n,0}Jx_n + \sum_{i=1}^{\infty} a_{n,i}Jz_{n,i}\|^2 \\
&\leq \|u\|^2 - 2a_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} a_{n,i}\langle u, Jz_{n,i}\rangle + a_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty} a_{n,i}\|z_{n,i}\|^2 - a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|) \\
&= a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\phi(u, z_{n,i}) - a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|) \\
&\leq a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\Phi(P_{T_i}u, P_{T_i}x_n) - a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|) \\
&\leq a_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} a_{n,i}\phi(u, x_n) - a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|) \\
&\leq \phi(u, x_n) - a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|). \qquad (3.8)
\end{aligned}$$

It follows that

$$a_{n,0}a_{n,i}g(\|Jx_n - Jz_{n,i}\|) \leq \phi(u, x_n) - \phi(u, u_n) \qquad n \geq 0. \qquad (3.9)$$

On the other hand

$$\begin{aligned}
\phi(u, x_n) - \phi(u, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\leq |\, \|x_n\|^2 - \|u_n\|^2 \,| + 2|\langle u, Jx_n - Ju_n\rangle\,| \\
&\leq |\, \|x_n\| - \|u_n\| \,|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\| \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.
\end{aligned}$$

It follows from (3.6) and (3.7) that

$$\lim_{n\longrightarrow\infty}(\phi(u, x_n) - \phi(u, u_n)) = 0. \qquad (3.10)$$

Using (3.9) and by assumption that $\liminf a_{n,0}a_{n,i} > 0$ we have that

$$\lim_{n\longrightarrow\infty} g(\|Jx_n - Jz_{n,i}\|) = 0, \quad (i \in \mathbb{N}).$$

Therefore from the property of g , we have

$$\lim_{n\longrightarrow\infty} \|Jx_n - Jz_{n,i}\| = 0, \quad (i \in \mathbb{N}).$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded set, we have

$$\lim_{n\longrightarrow\infty} \|x_n - z_{n,i}\| = 0,$$

this implies that

$$\lim_{n\longrightarrow\infty} dist(x_n, P_{T_i}x_n) \leq \lim_{n\longrightarrow\infty} \|x_n - z_{n,i}\| = 0, \quad (i \in \mathbb{N}).$$

Now by closedness of P_{T_i} we obtain that $z \in \bigcap_{i=1}^{\infty} F(T_i)$. By a similar argument as in [20] (see also[21]) we obtain that $z \in EP(F)$. Therefore $z \in \mathcal{F}$. Finally we prove $z = \Pi_{\mathcal{F}} x_0$. By taking limit in (3.1) we have

$$\langle z - u, Jx_0 - Jz\rangle \geq 0, \qquad \forall u \in \mathcal{F}.$$

Hence by Lemma 2.2 we have $z = \Pi_{\mathcal{F}} x_0$, which completes the proof. □

By a similar argument as in the proof of Theorem 3.1, we can prove the following theorem.

Theorem 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $T_i : C \longrightarrow N(C)$, be a sequence of closed relatively quasi-nonexpansive multivalued mappings such that $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \neq \emptyset$ and for all $p \in \mathcal{F}$, $T_i(p) = \{p\}$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_n = J^{-1}(a_{n,0}Jx_n + \sum_{i=1}^{\infty} a_{n,i}Jz_{n,i}), \\ u_n \in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x, \qquad \forall n \geq 0, \end{cases}$$

where $\sum_{i=0}^{\infty} a_{n,i} = 1$ and $z_{n,i} \in T_i x_n$. Assume further that $\liminf_n a_{n,0}a_{n,i} > 0$, $\{r_n\} \subset (0, \infty)$ and $\liminf_n r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the projection of E onto $\mathcal{F}$.

As a result for single valued mappings we obtain the following theorem.

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let Let $T_i : C \longrightarrow C$, be a sequence of closed relatively quasi-nonexpansive mappings such that $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \neq \emptyset$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_n = J^{-1}(a_{n,0}Jx_n + \sum_{i=1}^{\infty} a_{n,i}JT_ix_n), \\ u_n \in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x, \qquad \forall n \geq 0, \end{cases}$$

where $\sum_{i=0}^{\infty} a_{n,i} = 1$. Assume further that $\liminf_n a_{n,0}a_{n,i} > 0$, $\{r_n\} \subset (0, \infty)$ and $\liminf_n r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the projection of E onto $\mathcal{F}$.

Remark : Our main result generalize the result of Eslamian and Abkar [7] of a finite family of multivalued mappings to an infinite family of multivalued mappings. We also remove the uniformly continuity of the mappings.

4. **Application to Hilbert Spaces**

In the Hilbert space setting, we have

$$\phi(x, y) = \|x - y\|^2, \qquad \Phi(Tx, Ty) = H(Tx, Ty)^2 \quad \forall x, y \in H.$$

Therefore

$$\Phi(Tx, Tp) \leq \phi(x, p) \Leftrightarrow H(Tx, Tp) \leq \|x - p\|$$

for every $x \in C$ and $p \in F(T)$. We note that in a Hilbert space H, J is the identity operator.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $T_i : C \longrightarrow P(C)$, $i \in \mathbb{N}$ be a sequence of multivalued mappings such that P_{T_i} is closed quasi- nonexpansive. Assume that*

$\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \neq \emptyset$. *For* $x_0 \in C$ *and* $C_0 = C$, *let* $\{x_n\}$ *be a sequences generated by the following algorithm:*

$$\begin{cases} y_n = a_{n,0}x_n + \sum_{i=1}^{\infty} a_{n,i}z_{n,i}, \\ u_n \in C \, such \, that \, F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0; \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \qquad \forall n \geq 0 \end{cases},$$

where $\sum_{i=0}^{\infty} a_{n,i} = 1$ *and* $z_{n,i} \in P_{T_i}x_n$. *Assume further that* $\liminf_n a_{n,0}a_{n,i} > 0$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathcal{F}}x_0$.

Remark : Theorem 4.1 holds if we assume that T_i is closed quasi-nonexpansive multivalued mapping and $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$.

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