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FIXED POINT STABILITY FOR α_*-ψ-CONTRACTION MAPPINGS

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Choudhury and Bandyopadhyay [17] discussed the stability of fixed point sets of mappings satisfying the notion of multivalued α-ψ-contraction and raised an open problem: can α-ψ-contractions extended to multivalued case in some other way and in those case whether the stability of fixed point sets still holds? As an answer to this problem, in this paper, we study the stability of fixed point sets of mappings satisfying a new multivalued generalization of α-ψ-contractive mappings.

Keywords: Gauge function; α-ψ-almost contraction; α_*-ψ-δ-almost contraction.

1. Introduction

Banach initiated the study of fixed points through iterative sequences, which appeared as a base for metric fixed point theory. Many authors continue this pattern of finding fixed points, see for eaxmple [1]-[29]. Samet *et al.* [1] introduced the ideas of α-ψ-contractive and α-admissible mappings and got fixed points of the mappings through iterative sequence satisfying these ideas on complete metric space. Some generalizations of these ideas are available in [2-13].

The stability of fixed point sets have a vital role in fixed point theory and other branches of mathematics like in differential equations and integral equations etc. Some basis results about the stability of fixed point sets for multivalued mappings are available in [14, 15]. Lim [16] gave some classical stability results without assuming the restricted conditions like, the domain of the mappings being closed convex bounded subset of a Hilbert space, or, the image of each point under each map being a closed convex subset. Recently Choudhury and Bandyopadhyay [17] introduced the notion of multivalued α-ψ-contraction and discussed fixed points and stability of fixed point sets for such mappings. They also raised an open problem, that is, can α-ψ-contraction extended to

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multivalued case in some other way and in those case whether the stability of fixed point sets still holds? This paper is a positive answer to their open problem.

2. Preliminaries

Here, we recollect some basis definitions and results for completeness. We denote by $N(X)$ the space of all nonempty subsets of X, by $B(X)$ the space of all nonempty bounded subsets of X and by $CL(X)$ the space of all nonempty closed subsets of X. For $A \in N(X)$ and $x \in X$, $d(x, A) = \inf\{d(x,a) : a \in A\}$. For every $A, B \in B(X)$, $\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$. When $A = \{x\}$ we denote $\delta(A,B)$ by $\delta(x,B)$. For every $A, B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

Such a map H is called generalized Hausdorff metric induced by d.

Lim [16] introduced the basic stability result as follows.

Lemma 2.1 *[16] Let (X,d) be a complete metric space, let T_1 and T_2 are contractions from X into $CL(X)$ with same contraction constant λ. Then*

$$H(Fix(T_1), Fix(T_2)) \le \frac{\lambda}{1-\lambda} \sup_{x\in X} H(T_1x, T_2x),$$

where $\lambda \in [0,1)$.

Theorem 2.2 *[16] Let (X,d) be a complete metric space, let $T_i : X \to CL(X)$ be a sequence of contractions, $i \in \mathbb{N} \cup \{0\}$. If $\lim_{i\to\infty} H(T_ix, T_0x) = 0$ uniformly for each $x \in X$, then we have $\lim_{i\to\infty} H(Fix(T_i), Fix(T_0)) = 0$.*

Throughout this paper $J = [0,\infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + \ldots + t^{n-1}$ for each $n \in \mathbb{N}$ and $S_0 = 0$. We use the abbreviation ψ^n for the nth iterate of a function $\psi : J \to J$.

Definition 2.3 *[19] A nondecreasing function $\psi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [19] on J if*

$$\sum_{n=0}^{\infty}\psi^n(t) < \infty, \ for all\, t \in J. \qquad (2.1)$$

Definition 2.4 *[18] A function $\psi : J \to J$ is said to be a gauge function of order $r \geq 1$ on J if it satisfies the following conditions:*

(i) $\psi(\lambda t) \leq \lambda^r \psi(t)$ for all $\lambda \in (0,1)$ and $t \in J$;

(ii) $\psi(t) < t$ for all $t \in J - \{0\}$.

Remark 2.5 *[18] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.*

Lemma 2.6 *[18] Let ψ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\psi(t) = t\phi(t) \quad for\, all \ \ t \in J, \qquad (2.2)$$

then it has the following properties:

(i) $0 \leq \phi(t) < 1$ for all $t \in J$;

(ii) $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ for all $\lambda \in (0,1)$ and $t \in J$.

Moreover, for each $n \geq 0$ we have

(iii) $\psi^n(t) \leq t\phi(t)^{S_n(r)}$ for all $t \in J$,

(iv) $\phi(\psi^n(t)) \leq \phi(t)^{r^n}$ for all $t \in J$.

Samet *et al.* [1] defined α-ψ-contractive and α-admissible mappings in the following way:

Definition 2.7 *[1] Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is called α-admissible if for each $x, y \in X$ with $\alpha(x,y) \geq 1$, we have $\alpha(Tx,Ty) \geq 1$. A mapping $T : X \to X$ is called α-ψ-contractive if for each $x, y \in X$ we have $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$, where ψ is a Bianchini Grandolfi gauge function.*

Asl *et al.* [3] extended these notions to multivalued mappings as follow:

Definition 2.8 *[3] Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a fucntion. A mapping $T : X \to CL(X)$ is called α_*-admissible if for each $x, y \in X$ with $\alpha(x,y) \geq 1$, we have $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\} \geq 1$.*

Definition 2.9 *[3] Let* (X,d) *be a metric space and* $\alpha : X \times X \to [0,\infty)$ *be a function. A mapping* $T : X \to CL(X)$ *is called* α_**-*ψ*-contractive if for each* $x, y \in X$ *we have*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le \psi(d(x,y))$$

where ψ *is a Bianchini Grandolfi gauge function.*

Choudhury and Bandyopadhyay [17] introduced the notion of multivalued α-ψ-contraction and proved a stability result for the fixed point sets of a sequence of mappings satisfying multivalued α-ψ-contraction.

Definition 2.10 *[17] Let* (X,d) *be a metric space,* $\alpha : X \times X \to [0,\infty)$ *be a mapping and* $T : X \to 2^X$ *be a closed valued multifunction. Let* ψ_* *be a nondecreasing and continuous function with* $\sum \psi_*^i(t) < \infty$ *and* $\psi_*(t) < t$ *for each* $t > 0$*. We say that* T *is a multivalued* α-ψ*-contraction if*

$$\alpha(x,y)H(Tx,Ty) \le \psi_*(d(x,y)) \;\; \forall\, x,y \in X.$$

Theorem 2.11 *[17] Let* (X,d) *be a complete metric space, let* $\{T_i\}$ *is a sequence of multivalued* α-ψ*-contractions which are also* α*-admissible with same* α*. Further, for all* $i \in \mathbb{N}$ *for any* $x \in F(T_i)$*, we have* $\alpha(x,y) \ge 1$ *whenever* $y \in Tx$ *and for any* $x \in F(T)$*, we have* $\alpha(x,y) \ge 1$ *whenever* $y \in T_i x$*. Then* $\lim_{i\to\infty} H(Fix(T_i), Fix(T)) = 0$*, that is, the fixed point sets of* T_i *are stable.*

3. Main results

We begin this section with the following definition.

Definition 3.1 *Let* (X,d) *be a metric space and* $\alpha : X \times X \to [0,\infty)$ *be a function. A mapping* $T : X \to CL(X)$ *is called* α-ψ*-almost contraction if for each* $x, y \in X$ *we have*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le \psi(d(x,y)) + Ld(y,Tx) \tag{3.1}$$

with strict inequality holds when $x \ne y$. Where $L \ge 0$ and ψ *is a gauge function of order* $r \ge 1$ *on an interval* J *.*

Theorem 3.2 *Let* (X,d) *be a complete metric space,* ψ *is a gauge function of order* $r \geq 1$ *on an interval* J*, and let* T *be an* α-ψ*-almost contraction satisfying the following conditions:*

(i) T is α_*-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;

(iii) (a) T is continuous;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis, we have $\alpha_*(x_0, Tx_0) \geq 1$. Suppose $x_1 \in Tx_0$ and $x_1 \neq x_0$, otherwise x_0 is a fixed point of T. Thus, $\alpha(x_0, x_1) \geq 1$, by α_*-admissibility of T, we have $\alpha_*(Tx_0, Tx_1) \geq 1$. From (3.1) we have

$$H(Tx_0, Tx_1) \leq \alpha_*(Tx_0, Tx_1) H(Tx_0, Tx_1) < \psi(d(x_0, x_1)) + Ld(x_1, Tx_0).$$

Thus there exists an $\varepsilon_1 > 0$ such that

$$H(Tx_0, Tx_1) + \varepsilon_1 \leq \psi(d(x_0, x_1)).$$

Then we have $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d(x_1, Tx_1) + \varepsilon_1 \leq H(Tx_0, Tx_1) + \varepsilon_1 \leq \psi(d(x_0, x_1)).$$

By applying ψ in the above inequality we have

$$\psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)).$$

As T is α_*-admissible, we have $\alpha(x_1, x_2) \geq 1$ implies $\alpha_*(Tx_1, Tx_2) \geq 1$. By continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \textit{ for each } n \in \mathbb{N}.$$

For each $m > n$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(d(x_0, x_1)).$$

Since ψ is a gauge function of order $r \geq 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. There exists $x^* \in X$ such that $x_n \to x^*$. If T is continuous then $Tx_n \to Tx^*$. Thus we have $x^* \in Tx^*$. We assume (iii)-(b) holds. By using the triangular inequality and α_*-admissibility of T, we have

$$d(x^*,Tx^*)\le d(x^*,x_{n+1})+H(Tx_n,Tx^*)\le d(x^*,x_{n+1})+\alpha_*(Tx_n,Tx^*)H(Tx_n,Tx^*)$$
$$\le d(x^*,x_{n+1})+\psi(d(x_n,x^*))+Ld(x^*,x_{n+1}).$$

Letting $n\to\infty$ in the above inequality, we have $d(x^*,Tx^*)=0$. Thus, x^* is a fixed point T.

We use following lemma in our next result.

Lemma 3.3 *Let (X,d) be a complete metric space, ψ is a gauge function of order $r\ge 1$ on an interval J, $\phi: J\to J$ is a nondecreasing function defined by (2.2), and let T_1 and T_2 be α-ψ-almost contractions satisfying following conditions:*

(i) T_2 is α_*-admissible;

(ii) there exists $x_0\in Fix(T_1)$ such that $\alpha_*(x_0,T_2x_0)\ge 1$;

(iii) (a) T_2 is continuous;

or

(b) for any sequence $\{x_n\}$ in X with $x_n\to x$ as $n\to\infty$ and $\alpha(x_{n-1},x_n)\ge 1$ for each $n\in\mathbb{N}$, then $\alpha(x_n,x)\ge 1$ for each $n\in\mathbb{N}$.

Then we have

$$H(Fix(T_1),Fix(T_2))\le \frac{\lambda}{1-\lambda}\sup_{x\in X}H(T_1x,T_2x),$$

where $\lambda=\phi(d(x_0,x_1))$ with $x_0\in Fix(T_1)$ and for any $x_1\in T_2x_0$.

Proof. Suppose that $\eta=\sup_{x\in X}H(T_1x,T_2x)$. By hypothesis, we have $x_0\in Fix(T_1)$ such that $\alpha_*(x_0,T_2x_0)\ge 1$. As $H(T_1x_0,T_2x_0)\le\eta$, then for some fixed $\varepsilon>0$ we have $x_1\in T_2x_0$ and $x_1\ne x_0$, otherwise x_0 is a fixed point of T_2 such that

$$d(x_0,x_1)\le d(x_0,T_2x_0)+\varepsilon\le H(T_1x_0,T_2x_0)+\varepsilon\le\eta+\varepsilon.$$

As $\alpha_*(x_0,T_2x_0)\ge 1$, then $\alpha(x_0,x_1)\ge 1$, by α_*-admissibility of T_2, we have $\alpha_*(T_2x_0,T_2x_1)\ge 1$. From (3.1) we have

$$H(T_2x_0,T_2x_1)\le\alpha_*(T_2x_0,T_2x_1)H(T_2x_0,T_2x_1)<\psi(d(x_0,x_1))+Ld(x_1,T_2x_0).$$

Thus there exists an $\varepsilon_1>0$ such that

$$H(T_2x_0,T_2x_1)+\varepsilon_1\le\psi(d(x_0,x_1)).$$

Then we have $x_2\in T_2x_1$ such that

$$d(x_1,x_2)\le d(x_1,T_2x_1)+\varepsilon_1\le H(T_2x_0,T_2x_1)+\varepsilon_1\le\psi(d(x_0,x_1)).$$

By applying ψ in the above inequality we have

$$\psi(d(x_1,x_2)) \le \psi^2(d(x_0,x_1)).$$

As T_2 is α_*-admissible, we have $\alpha(x_1,x_2)\ge 1$ implies $\alpha_*(T_2x_1,T_2x_2)\ge 1$. By continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_n \in T_2x_{n-1}$, $\alpha(x_{n-1},x_n)\ge 1$ and

$$d(x_n,x_{n+1}) \le \psi^n(d(x_0,x_1)) \ \textit{for each}\, n\in \mathbb{N}.$$

For each $m>n$, we have

$$d(x_n,x_m) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{m-1}\psi^i(d(x_0,x_1)).$$

Since ψ is a gauge function of order $r\ge 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. There exists $x^*\in X$ such that $x_n \to x^*$. If T_2 is continuous then $T_2x_n \to T_2x^*$. Thus we have $x^*\in T_2x^*$. We assume (iii)-(b) holds. By the triangular inequality and α_*-admissibility of T_2, we have

$$\begin{aligned} d(x^*,T_2x^*) &\le d(x^*,x_{n+1}) + H(T_2x_n,T_2x^*) \le d(x^*,x_{n+1}) + \alpha_*(T_2x_n,T_2x^*)H(T_2x_n,T_2x^*) \\ &\le d(x^*,x_{n+1}) + \psi(d(x_n,x^*)) + Ld(x^*,x_{n+1}). \end{aligned}$$

Letting $n\to\infty$ in the above inequality, we have $d(x^*,T_2x^*)=0$. Moreover,

$$\begin{aligned} d(x_0,x^*) &\le \sum_{i=0}^{\infty}(d(x_i,x_{i+1})) \le \sum_{i=0}^{\infty}\psi^i(d(x_0,x_1)) \le d(x_0,x_1)[1+\lambda^{S_1(r)}+\lambda^{S_2(r)}+\cdots] \\ &= d(x_0,x_1)\sum_{i=0}^{\infty}\lambda^{S_i(r)} \le d(x_0,x_1)\sum_{i=0}^{\infty}\lambda^i = \frac{\lambda}{1-\lambda}d(x_0,x_1), \end{aligned}$$

since $S_i(r)\ge i$. Hence, we have

$$d(x_0,x^*) \le \frac{\lambda}{1-\lambda}d(x_0,x_1) \le \frac{\lambda}{1-\lambda}(\eta+\varepsilon).$$

Thus,

$$H(Fix(T_1),Fix(T_2)) \le \frac{\lambda}{1-\lambda}\sup_{x\in X} H(T_1x,T_2x).$$

Lemma 3.4 *Let (X,d) be a complete metric space, ψ is a gauge function of order $r\ge 1$ on an interval J, $\phi: J\to J$ is a nondecreasing function defined by (2.2), and let T_1 and T_2 be α-ψ-almost contractions satisfying following conditions:*

(i) T_1 is α_*-admissible;

(ii) there exists $x_0\in Fix(T_2)$ such that $\alpha_*(x_0,T_1x_0)\ge 1$;

(iii) (a) T_1 is continuous;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then we have

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1-\lambda} \sup_{x \in X} H(T_1 x, T_2 x),$$

where $\lambda = \phi(d(x_0, x_1))$ with $x_0 \in Fix(T_1)$ and for any $x_1 \in T_2 x_0$.

Theorem 3.5 *Let (X,d) be a complete metric space, ψ is a gauge function of order $r \geq 1$ on an interval J, $\phi: J \to J$ is a nondecreasing function defined by (2.2), and let $\{T_n\}_{n=0}^{\infty}$ be a sequence of α-ψ-almost contractions under same α and ψ, satisfying the following conditions:*

(i) T_n is α_*-admissible for each $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} H(T_n x, T_0 x) = 0$ uniformly for each $x \in X$;

(iii) there exists $x_0 \in Fix(T_0)$ such that $\alpha_*(x_0, T_n x_0) \geq 1$ for each $n \in \mathbb{N}$;

(iv) (a) T_n is continuous for each $n \in \mathbb{N}$;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then we have $\lim_{n\to\infty} H(Fix(T_n), Fix(T_0)) = 0$.

Proof. By hypothesis (ii), for any $\frac{1-\lambda}{\lambda}\varepsilon_* > 0$, where $\lambda = \phi(d(x_0, \rho))$, with $x_0 \in Fix(T_0)$ and $\rho \in T_n x_0$ for some n, such that

$$d(x_0, \rho) = \max\{d(x_0, x_j) : x_0 \in Fix(T_0) \text{ and } x_j \in T_n x_0 \text{ for any } n\},$$

we can find N such that $\sup_{x\in X} H(T_n x, T_0 x) < \frac{1-\lambda}{\lambda}\varepsilon_*$ for each $n \geq N$. Hence by using Lemma 3.4 we have $H(Fix(T_n), Fix(T_0)) < \varepsilon_*$ for each $n \geq N$.

Example 3.6 *Let $X = [0,\infty)$ be endowed with the usual metric d. Define $\{T_n : X \to CL(X)\}_{n=1}^{\infty}$ such that*

$$T_n x = \begin{cases} [0, \frac{x+3}{4n}] & \text{if } x \leq 1 \\ [x^2, e^x] & \text{otherwise} \end{cases}$$

and $T_0 : X \to CL(X)$ by

$$T_0 x = \begin{cases} \{0\} & if\ x \le 1 \\ [x^2, e^x] & otherwise \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x, y \in [0,1] \\ 0 & otherwise. \end{cases}$$

First we show that each T_n is α-ψ-almost contraction with $\psi(t) = \frac{t}{2}$ for each $t \ge 0$ and $L = 0$. If $x \ne y \in [0,1]$, then

$$\alpha_*(T_n x, T_n y) H(T_n x, T_n y) = \frac{1}{4n} | x - y | < \frac{1}{2} | x - y | = \psi(d(x, y)),$$

otherwise

$$\alpha_*(T_n x, T_n y) H(T_n x, T_n y) \le \psi(d(x, y)).$$

It is easy to see that T_0 is also an α-ψ-almost contraction with $\psi(t) = \frac{t}{2}$ for each $t \ge 0$ and $L = 0$. Further, for each $n \in \mathbb{N}$, T_n is α_*-admissible and for $x_0 = 0 \in Fix(T_0)$ we have $\alpha_*(x_0, T_n x_0) = 1$. Also, $T_n \to T_0$ as $n \to \infty$. For any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) = 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) = 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 3.5, we have $H(Fix(T_n), Fix(T_0)) \to 0$ as $n \to \infty$.

Definition 3.7 *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to B(X)$ is called α_*-ψ-δ-almost contraction if for each $x, y \in X$ we have*

$$\alpha_*(Tx, Ty)\delta(Tx, Ty) \le \psi(d(x, y)) + Ld(y, Tx) \tag{3.2}$$

where $L \ge 0$ and ψ *is a gauge function of order $r \ge 1$ on an interval J.*

Theorem 3.8 *Let (X, d) be a complete metric space, ψ is a gauge function of order $r \ge 1$ on an interval J, and let T be an α_*-ψ-δ-almost contraction satisfying following conditions:*

(i) T is α_*-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \ge 1$;

(iii) (a) T is continuous;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.
Then T has an end point.

Proof. By hypothesis, we have $\alpha_*(x_0, Tx_0) \geq 1$, suppose $x_1 \in Tx_0$, then $\alpha(x_0, x_1) \geq 1$, by α_*-admissibility of T, $\alpha_*(Tx_0, Tx_1) \geq 1$. From (3.2) we have

$$\delta(Tx_0, Tx_1) \leq \alpha_*(Tx_0, Tx_1)\delta(Tx_0, Tx_1) \leq \psi(d(x_0, x_1)) + Ld(x_1, Tx_0).$$

Thus, for each $x_2 \in Tx_1$ we have

$$d(x_1, x_2) \leq \delta(Tx_0, Tx_1) \leq \psi(d(x_0, x_1)).$$

By applying ψ in the above inequality we have

$$\psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)).$$

As T is α_*-admissible, we have $\alpha(x_1, x_2) \geq 1$ implies $\alpha_*(Tx_1, Tx_2) \geq 1$. By continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \text{ for each } n \in \mathbb{N}.$$

For each $m > n$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(d(x_0, x_1)).$$

Since ψ is a gauge function of order $r \geq 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. There exists $x^* \in X$ such that $x_n \to x^*$. If T is continuous then $Tx_n \to Tx^*$. Thus we have $x^* \in Tx^*$. We assume (iii)-(b) holds. By using the triangular inequality and α_*-admissibility of T, we have

$$\delta(x^*, Tx^*) \leq d(x^*, x_{n+1}) + \delta(Tx_n, Tx^*) \leq d(x^*, x_{n+1}) + \alpha_*(Tx_n, Tx^*)\delta(Tx_n, Tx^*)$$
$$\leq d(x^*, x_{n+1}) + \psi(d(x_n, x^*)) + Ld(x^*, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we have $\delta(x^*, Tx^*) = 0$. Thus $\{x^*\} = Tx^*$.

Lemma 3.9 *Let (X, d) be a complete metric space, ψ is a gauge function of order $r \geq 1$ on an interval J, $\phi : J \to \mathbb{R}^+$ is a nondecreasing function defined by (2.2), and let T_1 and T_2 be α_*-ψ-δ-almost contractions satisfying the following conditions:*

(i) T_2 is α_*-admissible;

(ii) there exists $x_0 \in End(T_1)$ such that $\alpha_*(x_0, T_2x_0) \geq 1$;

(iii) (a) T_2 is continuous;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then we have

$$\delta(End(T_1), End(T_2)) \leq \frac{\lambda}{1-\lambda} \sup_{x \in X} \delta(T_1 x, T_2 x),$$

where $\lambda = \phi(d(x_0, x_1))$ with $x_0 \in End(T_1)$ and for any $x_1 \in T_2 x_0$.

Proof. Suppose that $\eta = \sup_{x \in X} \delta(T_1 x, T_2 x)$. By hypothesis, we have $x_0 \in End(T_1)$ such that $\alpha_*(x_0, T_2 x_0) \geq 1$. As $\delta(T_1 x_0, T_2 x_0) \leq \eta$, then for each $x_1 \in T_2 x_0$ we have

$$d(x_0, x_1) \leq \delta(T_1 x_0, T_2 x_0) \leq \eta.$$

As $\alpha_*(x_0, T_2 x_0) \geq 1$, then $\alpha(x_0, x_1) \geq 1$, by α_*-admissibility of T_2, $\alpha_*(T_2 x_0, T_2 x_1) \geq 1$. From (3.2) we have

$$\delta(T_2 x_0, T_2 x_1) \leq \alpha_*(T_2 x_0, T_2 x_1)\delta(T_2 x_0, T_2 x_1) \leq \psi(d(x_0, x_1)) + Ld(x_1, Tx_0).$$

Thus for each $x_2 \in T_2 x_1$ we have

$$d(x_1, x_2) \leq \delta(T_2 x_0, T_2 x_1) \leq \psi(d(x_0, x_1)).$$

By applying ψ in the above inequality we have

$$\psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)).$$

As T_2 is α_*-admissible, we have $\alpha(x_1, x_2) \geq 1$ implies $\alpha_*(T_2 x_1, T_2 x_2) \geq 1$. By continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_n \in T_2 x_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \textit{ for each } n \in \mathbb{N}.$$

For each $m > n$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(d(x_0, x_1)).$$

Since ψ is a gauge function of order $r \geq 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. There exists $x^* \in X$ such that $x_n \to x^*$. If T_2 is continuous then $T_2 x_n \to T_2 x^*$. Thus we have $x^* \in T_2 x^*$. We assume (iii)-(b) holds. By the triangular inequality and α_*-admissibility of T_2, we have

$$\delta(x^*, T_2 x^*) \leq d(x^*, x_{n+1}) + \delta(T_2 x_n, T_2 x^*) \leq d(x^*, x_{n+1}) + \alpha_*(T_2 x_n, T_2 x^*)\delta(T_2 x_n, T_2 x^*)$$
$$\leq d(x^*, x_{n+1}) + \psi(d(x_n, x^*)) + Ld(x^*, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we have $\delta(x^*, T_2 x^*) = 0$, that is, $\{x^*\} = T_2 x^*$. By using the triangular inequality, we get

$$d(x_0, x^*) \le \sum_{i=0}^{\infty} d(x_i, x_{i+1}) \le \sum_{i=0}^{\infty} \psi^i(d(x_0, x_1)) \le d(x_0, x_1)[1 + \lambda^{S_1(r)} + \lambda^{S_2(r)} + \cdots]$$

$$= d(x_0, x_1) \sum_{i=0}^{\infty} \lambda^{S_i(r)} \le d(x_0, x_1) \sum_{i=0}^{\infty} \lambda^i = \frac{\lambda}{1-\lambda} d(x_0, x_1),$$

since $S_i(r) \ge i$. Hence, we have

$$d(x_0, x^*) \le \frac{\lambda}{1-\lambda} d(x_0, x_1) \le \frac{\lambda}{1-\lambda}(\eta).$$

Thus,

$$\delta(End(T_1), End(T_2)) \le \frac{\lambda}{1-\lambda} \sup_{x \in X} \delta(T_1 x, T_2 x).$$

Theorem 3.10 *Let (X, d) be a complete metric space, ψ is a gauge function of order $r \ge 1$ on an interval J, $\phi : J \to J$ is a nondecreasing function defined by (2.2), and let $\{T_n\}$ be a sequence of α_*-ψ-δ-almost contractions under same α and ψ. Further, assume that the following conditions:*

(i) T_n is α_*-admissible for each $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \delta(T_n x, T_0 x) = 0$ uniformly for each $x \in X$;

(iii) there exists $x_0 \in End(T_0)$ such that $\alpha_*(x_0, T_n x_0) \ge 1$ for each $n \in \mathbb{N}$;

(iv) (a) T_n is continuous for each $n \in \mathbb{N}$;

or

(b) for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1}, x_n) \ge 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n, x) \ge 1$ for each $n \in \mathbb{N}$.

Then we have $\lim_{n\to\infty} \delta(End(T_n), End(T_0)) = 0$.

Proof. By hypothesis (ii), for any $\frac{1-\lambda}{\lambda}\varepsilon_* > 0$, where $\lambda = \phi(d(x_0, \rho))$ with $x_0 \in End(T_0)$ and $\rho \in T_n x_0$ for some n, such that

$$d(x_0, \rho) = \max\{d(x_0, x_j) : x_0 \in End(T_0) \text{ and } x_j \in T_n x_0 \text{ for any } n\},$$

we can find N such that $\sup_{x \in X} \delta(T_n x, T_0 x) < \frac{1-\lambda}{\lambda}\varepsilon_*$ for each $n \ge N$, Hence by using Lemma 3.9 we have $\delta(Fix(T_n), Fix(T_0)) < \varepsilon_*$ for each $n \ge N$.

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REFERENCES

[1] *B. Samet, C. Vetro, P. Vetro,* Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal., 75 (2012) 2154-2165.

[2] *E. Karapinar, B. Samet,* Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal., 2012 (2012) Article id: 793486.

[3] *J. H. Asl, S. Rezapour, N. Shahzad,* On fixed points of α-ψ-contractive multifunctions, Fixed Point Theory Appl., (2012) 2012:212.

[4] *M. U. Ali, T. Kamran,* On (α^*,ψ)-contractive multi-valued mappings, Fixed Point Theory Appl., 2013, 2013:137.

[5] *B. Mohammadi, S. Rezapour, N Shahzad,* Some results on fixed points of α-ψ-Ciric generalized multifunctions. Fixed Point Theory Appl., (2013) 2013:24.

[6] *P. Amiri, S. Rezapour, N. Shahzad,* Fixed points of generalized α-ψ-contractions, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales Serie A Mate., doi: 10.1007/s13398-013-0123-9.

[7] *G. Minak, I. Altun,* Some new generalizations of Mizoguchi-Takahashi type fixed point theorem, J. Inequal. Appl., (2013) 2013:493 doi:10.1186/1029-242X-2013-493.

[8] *P. Salimi, A. Latif, N. Hussain,* Modified α-ψ-contractive mappings with applications, Fixed Point Theory Appl., (2013) 2013:151.

[9] *N. Hussain, P. Salimi, A. Latif,* Fixed point results for single and set-valued α-η-ψ-contractive mappings, Fixed Point Theory Appl., (2013) 2013:212.

[10] *B. Mohammadi, S. Rezapour,* On modified α-φ-contractions, J. Adv. Math. Stud., 6 (2013) 162-166.

[11] *M. U. Ali, T. Kamran, E. Karapinar,* A new approach to (α,ψ)-contractive nonself multivalued mappings, J. Inequal. Appl., (2014) 2014:71.

[12] *M. U. Ali, T. Kamran, E. Karapinar,* (α,ψ,ξ)-contractive multi-valued mappings, Fixed Point Theory Appl., (2014) 2014:7.

[13] *M. U. Ali, T. Kamran, W. Sintunavarat, P. Katchang,* Mizoguchi-Takahashi's fixed point theorem with α,η functions, Abstr. Appl. Anal., 2013 (2013), Article ID 418798.

[14] *J. T. Markin,* A fixed point stability theorem for nonexpansive set valued mappings, J. Math. Anal. Appl., 54 (1976) 441-443.

[15] *S. B. Nadler,* Multivaluedcontraction mappings, Pacific J. Math., 30 (1969) 475-488.

[16] *T. C. Lim,* On fixed point stability for setvalued contractive mappings with application to generalized differential equations, J. Math. Anal. Appl., 110 (1985) 436-441.

[17] *B. S. Choudhury, C.Bandyopadahyay,* A new multivalued contraction and stability of its fixed point sets, J. Egypt. Math. Sco.,(inpress).

[18] *P. D. Proinov,* A generalization of the Banach contraction principle with high order of convergence of successive approximations, Nonlinear Anal., 67 (2007) 2361-2369.

[19] *R. M. Bianchini, M. Grandolfi,* Transformazioni di tipo contracttivo generalizzato in uno spazio metrico, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Math. Natur., 45 (1968), 212-216.

[20] *Q. Kiran, T. Kamran,* Fixed point theorems for generalized contractive multi-valued maps, Comput. Math. Appl., 59 (2010) 3813-3823.
[21] *Q. Kiran, T. Kamran,* Nadler's type principle with high order of convergence, Nonlinear Anal., 69 (2008) 4106-4120.
[22] *R. P. Agarwal, D. O Regan, D. R. Sahu,* Existence Theorems in Metric Spaces, Fixed Point Theory for Lipschitzian-type Mappings with Applications Topological Fixed Point Theory and Its Applications, Springer, Volume 6 (2009) 175-209.
[23] *Sh. Rezapour, R. H. Haghi,* Fixed Point of Multifunctions on Cone Metric Spaces, Numerical Funct. Anal. Optim. 30 (2009) 825-832.
[24] *H.E. Kunze, D. L. Torre, E. R. Vrscay,* Contractive multifunctions, fixed point inclusions and iterated multifunction systems, J. Math. Anal. Appl. 330 (2007) 159-173.
[25] *M. U. Ali, Q. Kiran, and N. Shahzad,* Fixed Point Theorems for Multivalued Mappings Involving α -Function," Abstr. Appl. Anal., (2014), Article ID 409467, 6 pages.
[26] *W. Shatanawi and M. Postolache,* Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, Fixed Point Theory Appl. (2013) 2013:54.
[27] *S. Chandok and M. Postolache,* Fixed point theorem for weakly Chatterjea-type cyclic contractions, Fixed Point Theory Appl., (2013) 2013:28.
[28] *M. A. Miandaragh, M. Postolache and Sh. Rezapour,* Some approximate fixed point results for generalized α -contractive mappings, U.P.B. Sci. Bull., Series A, 75 (2013) 3-10.
[29] *M.A. Miandaragh, M. Postolache and Sh. Rezapour,* Approximate fixed points of generalized convex contractions, Fixed Point Theory Appl.,(2013) 2013:255.