# DYNAMICAL BEHAVIOR OF RANDOM FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION VIA HILFER FRACTIONAL DERIVATIVE

Sumbel Begum[1], Akbar Zada[1], Shahid Saifullah[1] and Ioan-Lucian Popa [2,3]

*This article deals with the study of the random Hilfer fractional integro-differential equation with integral boundary condition. Using Banach and Schauder's fixed point theorems we show that for the aforesaid model the solution exists, is unique and is at least one. Also, Pachpatte's inequality is used in order to provide Hyers–Ulam and Hyers–Ulam–Rassias stability results for the mentioned equation. Finally, an example is provided to verify our results.*

**Keywords:** Random fractional, Integro-differential equation, Existence theory, Hilfer fractional derivative, Hyers–Ulam Stability, Banacah fixed point theorem

**MSC2020:** 34B10; 34A12; 26A33; 45M10.

## 1. Introduction

Fractional derivatives ($\mathbb{FD}$) are the generalized forms of integer order derivatives. The idea about $\mathbb{FD}$ was introduced at the end of sixteenth century (1695), when Leibniz used the notation $\frac{d^n}{dx^n}$ for $n^{th}$ order derivative. By writing a letter to him, L'Hospital asked what we can say about $n = \frac{1}{2}$? Leibniz answered in such words, "An apparent Paradox, a day will come to get benefits of this notion" and this question becomes the foundation of fractional calculus ($\mathbb{FC}$). In that time many mathematicians like Fourier and Laplace contributed to the development of $\mathbb{FC}$. After that when Riemann and Liouville introduced Riemann-Liouville ($\mathcal{R} - \mathcal{L}$) derivative which is a fundamental concept in $\mathbb{FC}$, then $\mathbb{FC}$ became an important area for researchers. In 1891 it was introduced Hadamard $\mathbb{FD}$s and in 1997 Caputo presented a new $\mathbb{FD}$s. $\mathbb{FD}$ is a global operator, which is used as a tool for modelling different processes

[1]Department of Mathematics, University of Peshawar, Peshawar, Pakistan, e-mail: `sumbel.776@gmail.com`, e-mail: `akbarzada@uop.edu.pk`, e-mail: `shahidsaif78@gmail.com`

[2]Department of Computing, Mathematics and Electronics, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, 510009, Romania, e-mail: `lucian.popa@uab.ro`

[3]Faculty of Mathematics and Computer Science, Transilvania University of Braşov, Iuliu Maniu Street 50, 500091, Braşov, Romania

and physical phenomena like mathematical biology [19], electro-chemistry [11], control theory [18], dynamical process [15], image and signal processing [14] etc. For more applications of fractional differential equations ($\mathbb{FDE}$)s, we refer the reader to [1, 2, 6, 9, 13, 21, 22, 23, 24, 25, 26].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for $\mathbb{FDE}$s with different $\mathbb{FD}$s. Several mathematicians have been working on different random fractional differential equations ($\mathbb{RFDE}$)s with different random effects. In [4], El- Sayed et al. introduced the existing theory and stability for the $\mathbb{RFDE}$ using Caputo $\mathbb{FD}$ with non-local boundary condition. Vu et al. in [17] for the $\mathbb{RFDE}$ with impulses proved existence and uniqueness ($\mathbb{EU}$) of the solution by using Banach and Schauder fixed point theorem. In [5] Harikrishnan et al. investigated stability and dynamical behavior of $\mathbb{RFDE}$s involving Ψ- Hilfer $\mathbb{FD}$. For the class of implicit $\mathbb{RFDE}$s with non-local and impulsive conditions involving Hilfer $\mathbb{FD}$ Jarad et al. in [7] studied $\mathbb{EU}$ of the solution and stability. Dong et al. [3] showed the $\mathbb{EU}$ and Ulam stability for the following random fractional integro-differential equation ($\mathbb{RFIDE}$) by using mean square sense Caputo $\mathbb{FD}$

$$\mathcal{D}_0^\alpha \mathbb{X}(t) = \mathbb{F}(t, \mathbb{X}(t)) + \int_0^t \mathbb{G}(t, r, \mathbb{X}(r))dr, \quad t, r \in \mathbb{J}.$$

From the literature, it has been observed that in most of the time to prove the exact solution of nonlinear differential equations is a tough job. To overcome this difficulty different approximation techniques were introduced. The difference between exact and approximate solutions is nowadays dealing with the help of Ulam–Hyers ($\mathbb{UH}$) stability, which was first initiated in 1940 by Ulam [16] and then extended by Hyers in the next year, in the context of Banach spaces. Many researchers investigated both $\mathbb{UH}$ and Ulam–Hyers–Rassias ($\mathbb{UHR}$) stabilities for different problems with different approaches (see [10, 20, 27, 28, 29, 30].)

Based on the motivation stated in the works of Dong et al. [3] in this paper we study the existence and uniqueness of the solution for $\mathbb{RFIDE}$ via Hilfer $\mathbb{FD}$, using Banach and Schauder fixed point technique. $\mathbb{UHR}$ stability and $\mathbb{UH}$ stability for $\mathbb{RFIDE}$ are also proved.

The rest of manuscript is organized as follows. Section 2 contains some weighted and non weighted spaces, important definitions. In Section 3 by using Banach fixed point theorems we derive the existence of at least one solution of $\mathbb{RFIDE}$ (1), which is unique. In Section 4, using Pachpatte's inequality we study $\mathbb{UH}$ and $\mathbb{UHR}$ stability of $\mathbb{RFIDE}$ (1). Finally, in Section 5 we provide an example to verify our results.

## 2. **Notation and auxiliary results**

Here, we define some spaces, definitions, which will be used throughout this paper. These definitions and results are taken from [8, 12].

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probabilistic space. Let $\eta(u,\omega) := \{\eta(u), u \in \mathcal{J} = [0,T] \text{ and } \omega \in \Omega\}, T > 0$ be a second-order stochastic process, i.e., $E\big(\eta^2(u)\big) < \infty$ and $L_2(\Omega)$ represents the Banach space of random variables $\eta : \Omega \to \mathbb{R}$. Consider the following model:

$$\begin{aligned} \mathcal{D}^{p,q}_{0^+}\eta(u) &= \mathbb{F}(u, \eta(u)) + \int_0^u \mathbb{G}(u, r, \eta(r))dr, \quad u, r \in \mathcal{J} \\ \mathcal{I}^{1-\upsilon}\eta(u,\omega)\big|_{u=0} &= \mu, \end{aligned} \tag{1}$$

where $\mathcal{D}^{p,q}_{0^+}$ is Hilfer $\mathbb{FD}$ of order $p \in (0,1)$ and type $q \in [0,1]$, $\mathcal{I}^{1-\upsilon}$ is the $\mathcal{R}-\mathcal{L}$ fractional integral of order $1-\upsilon$ where $\upsilon = p+q-pq$. In addition, $\mu : J \to L_2(\Omega)$ is the random variable with $E(\mu^2) < \infty$. Let $\mathbb{F}, \mathbb{G}$ be two m.s. continuous functions such that $\mathbb{F} : \mathcal{J} \times L_2(\Omega) \to L_2(\Omega)$ and $\mathbb{G} : \mathbb{J} \times L_2(\Omega) \to L_2(\Omega)$, where $\mathbb{J} = \{(u,r) \in \mathcal{J} \times \mathcal{J} \text{ such that } r \leq u\}$.

Let $\mathcal{C}(\mathcal{J}, L_2(\Omega))$ be the Banach space of all continuous functions from $\mathcal{J} \times \Omega$ into $\mathbb{R}$ with norm

$$||\eta||_{\mathcal{C}} = max_{u\in\mathcal{J}}||\eta(u)||_2, \text{ where } ||\eta(u)||_2 = \big(E(\eta^2(u))\big)^{\frac{1}{2}}.$$

**Definition 2.1.** [8] *The stochastic m.s. fractional integral of order* $p > 0$ *is defined as*

$$\mathcal{I}^p\eta(s) = \frac{1}{\Gamma(p)}\int_0^s (t-s)^{p-1}\eta(s)ds.$$

**Definition 2.2.** [8] *Let* $p \in (0,1)$, $q \in [0,1]$ *and* $\omega \in \Omega$. *The Hilfer fractional derivative of order* $p$ *and type* $q$ *is defined as;*

$$\big(\mathcal{D}^{p,q}_0\eta\big)(s) = \Big(\mathcal{I}^{q(1-p)}_0 \frac{d}{dt}\mathcal{I}^{(1-p)(1-q)}_0\eta\Big)(s)$$

*Properties*: Let $p \in (0,1)$, $q \in [0,1]$ and $\upsilon = p+q-pq$.

(1) The operator $\big(\mathcal{D}^{p,q}_0\eta\big)(s)$ can be written as;

$$\big(\mathcal{D}^{p,q}_0\eta\big)(s) = \Big(\mathcal{I}^{q(1-p)}_0\frac{d}{dt}\mathcal{I}^{(1-\upsilon)}_0\eta\Big)(s) = \Big(\mathcal{I}^{q(1-p)}_0\mathcal{D}^{\upsilon}_0\eta\Big)(s). \tag{2}$$

(2) The generalization of (2) for $q=0$, coincides with the $\mathcal{R}-\mathcal{L}$ derivative $\mathcal{D}^{p,0}_0 = \mathcal{D}^p_0$and $q=1$ with Caputo fractional derivative $\mathcal{D}^{p,1}_0 =^c \mathcal{D}^p_0$.

(3) If $\big(\mathcal{D}^{q(1-p)}\eta\big)$ exists, then

$$\big(\mathcal{D}^{p,q}_0\mathcal{I}^p_0\eta\big)(s) = \Big(\mathcal{I}^{q(1-p)}_0\mathcal{D}^{q(1-p)}_0\eta\Big)(s).$$

(4) If $\big(\mathcal{D}^{\upsilon}\eta\big)$ exists, then

$$\Big(\mathcal{I}^p_0\mathcal{D}^{p,q}_0\eta\Big)(s) = \Big(\mathcal{I}^{\upsilon}_0\mathcal{D}^{\upsilon}_0\eta\Big)(s) = \eta(s) - \frac{\mathcal{I}^{1-\upsilon}(0^+)}{\Gamma(\upsilon)}s^{\upsilon-1};$$

## 3. Main results

In this section, the existence of the solutions to Eq. (1) is presented. In the sequel, we need the following hypotheses:

($H_1$) There exist positive constants $\mathcal{A}$ and $\mathcal{B}$, such that

$$\big|\big|\mathbb{F}(u,\eta)-\mathbb{F}(u,\theta)\big|\big|_2\leq \mathcal{A}\big|\big|\eta-\theta\big|\big|_2$$

and

$$\big|\big|\mathbb{G}(u,r,\eta)-\mathbb{G}(u,r,\theta)\big|\big|_2\leq \mathcal{B}\big|\big|\eta-\theta\big|\big|_2$$

($H_2$) There exist positive constant $\mathcal{D}$ such that

$$\max\{||\mathbb{F}(u,0)||_2,||\mathbb{G}(u,r,0)||_2\}\leq \mathcal{D}.$$

($H_3$) For functions $\mathbb{F}$ and $\mathbb{G}$, we have that

$$||\mathbb{F}(u,\eta)||_2\leq \sup\{\mathrm{f}(u)\phi_1(||\eta||_{\mathcal{C}})\}$$

and

$$||\mathbb{G}(u,s,\eta)||_2\leq \sup\{\mathrm{g}(u,s)\phi_2(||\eta||_{\mathcal{C}})\},$$

where $\mathrm{f},\mathrm{g},\phi_1$ and $\phi_2$ lies in $\mathcal{C}_{\mathcal{C}}(\mathcal{J},L_2(\Omega))$ are non-decreasing on $\mathcal{J}$.

**Lemma 3.1.** *A function $\eta(u)$ is the solution of (1), if and only if $\eta(u)$ satisfied the random integral equation*

$$\eta(u)=\frac{\mu}{\Gamma(\upsilon)}u^{\upsilon-1}+\frac{1}{\Gamma(p)}\int_0^u(u-s)^{p-1}\mathbb{F}(s,\eta(s))ds$$
$$+\frac{1}{\Gamma(p)}\int_0^u(u-s)^{p-1}\int_0^s\mathbb{G}(s,r,\eta(r))drds. \tag{3}$$

**Theorem 3.1.** *Assume that hypothesis* ($H_1$) *and* ($H_2$) *are satisfied. If*

$$\mathcal{A}\leq\frac{\Gamma(1+p)}{6T^p}\quad \textit{and}\quad \mathcal{B}\leq\frac{(p+1)\Gamma(p)}{6T^{p+1}}, \tag{4}$$

*then* (1) *has a unique solution.*

*Proof.* We divide the proof of this theorem in two steps.
*Step:1* Define the operator $\mathcal{Q}:\mathcal{C}(\mathcal{J},L_2(\Omega))\to\mathcal{C}(\mathcal{J},L_2(\Omega))$. Hence, $\eta(u)$ is the solution of (1), where the equivalent integral equation (3) can be written in the operator form

$$(\mathcal{Q}\eta)(u)=\frac{\mu}{\Gamma(\upsilon)}u^{\upsilon-1}+\frac{1}{\Gamma(p)}\int_0^u(u-s)^{p-1}\mathbb{F}(s,\eta(s))ds$$
$$+\frac{1}{\Gamma(p)}\int_0^u(u-s)^{p-1}\int_0^s\mathbb{G}(s,r,\eta(r))drds. \tag{5}$$

Set $\mathcal{B}_\alpha = \{\eta \in L_2(\Omega) : ||\eta||_{\mathcal{C}} \leq \alpha\}$. Now, we will prove that $\mathcal{Q}\mathcal{B}_\alpha \subset \mathcal{B}_\alpha$, for any $\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$. We have that

$$\begin{aligned}
&||(\mathcal{Q}\eta)(u)||_2 \leq \Big|\Big|\frac{\mu}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\eta(s))ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\eta(r))drds\Big|\Big|_2 \\
\leq& \frac{||\mu||_2}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\big|\big|\mathbb{F}(s,\eta(s)) - \mathbb{F}(s,0)\big|\big|_2 + \big|\big|\mathbb{F}(s,0)\big|\big|_2\Big)ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \big|\big|\mathbb{G}(s,r,\eta(r)) - \mathbb{G}(s,r,0)\big|\big|_2 + ||\mathbb{G}(s,r,0)\big|\big|_2 drds \\
\leq& \frac{||\mu||_2}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}(\mathcal{A}||\eta||_2) + ||\mathbb{F}(s,0)||_2 ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\int_0^s (\mathcal{B}||\eta||_2) + ||\mathbb{G}(s,r,0)||_2 dr\Big)ds \\
\leq& \frac{||\mu||_2}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}(\mathcal{A}||\eta||_2 + \mathcal{D})ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s (\mathcal{B}||\eta||_2 + \mathcal{D})drds \\
\leq& \frac{||\mu||_2}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{(\mathcal{A}\alpha + \mathcal{D})u^p}{(1+p)\Gamma(p)} + \frac{(\mathcal{B}\alpha + \mathcal{D})u^{p+1}}{(1+p)\Gamma(p)}.
\end{aligned}$$

From the estimation, we have

$$||(\mathcal{Q}\eta)(u)||_2 \leq \frac{||\mu||_{\mathcal{C}}}{\Gamma(\upsilon)}u^{\upsilon-1} + \frac{(\mathcal{A}\alpha + \mathcal{D})T^p}{(1+p)\Gamma(p)} + \frac{(\mathcal{B}\alpha + \mathcal{D})T^{p+1}}{(1+p)\Gamma(p)} = \alpha, \ \ \forall u \in \mathcal{J}.$$

This proves that $\mathcal{Q}$ transform the ball $\mathcal{B}_\alpha = \{\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega)) : ||\eta||_{\mathcal{C}}(\mathcal{J}, L_2(\Omega)) \leq \alpha\}$ into itself, that is $\mathcal{Q}(\mathcal{B}_\alpha) \subset \mathcal{B}_\alpha$.
*Step:2* In this step, we are going to show that $\mathcal{Q}$ is contractive. For any $\eta, \theta \in \mathcal{B}_\alpha$, we get

$$\begin{aligned}
&||(\mathcal{Q}\eta)(u) - (\mathcal{Q}\theta)(u)||_2 = \\
&= \Big|\Big|\frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\eta(s))ds + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\eta(r))drds \\
&- \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\theta(s))ds - \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\theta(r))drds\Big|\Big|_2 \\
&\leq \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\big|\big|\mathbb{F}(s,\eta(s)) - \mathbb{F}(s,\theta(s))\big|\big|_2 ds \\
&+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \big|\big|\mathbb{G}(s,r,\eta(r)) - \mathbb{G}(s,r,\theta(r))\big|\big|_2 drds
\end{aligned}$$

$$\leq \frac{\mathcal{A}u^p}{\Gamma(1+p)}||\eta-\theta||_{\mathcal{C}} + \frac{\mathcal{B}}{\Gamma(p)}\Big(\frac{u^{p+1}}{p} - \frac{u^{p+1}}{p+1}\Big)||\eta-\theta||_{\mathcal{C}}$$
$$\leq \Big(\frac{\mathcal{A}T^p}{\Gamma(1+p)} + \frac{\mathcal{B}T^{p+1}}{(p+1)\Gamma(p)}\Big)||\eta-\theta||_{\mathcal{C}}.$$

Therefore, by assumption (4), we imply that $\frac{\mathcal{A}T^p}{\Gamma(1+p)} + \frac{\mathcal{B}T^{p+1}}{(p+1)\Gamma(p)} < 1$. Therefore, we deduced that $\mathcal{Q}$ is a contraction. Finally, applying Banach contraction theorem we deduce that there exists a unique solution of the problem (1). The proof is completed. □

**Theorem 3.2.** *Assume that the hypothesis from* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *are satisfied. Then the problem (1) has at least one solution.*

*Proof.* By hypothesis $(\mathrm{H}_3)$, the functions $\mathbb{F}$ and $\mathbb{G}$ are continuous. So, we can find constants $L_1$ and $L_2$, such that

$$||\mathbb{F}(u,\eta)||_2 \leq \sup\{\mathrm{f}(u)\phi_1(||\eta||_{\mathcal{C}})\} := L_1$$

and

$$||\mathbb{G}(u,s,\eta)||_2 \leq \sup\{\mathrm{g}(u,s)\phi_2(||\eta||_{\mathcal{C}})\} := L_2.$$

Consider the operator, $\mathcal{P} : \mathcal{B}_\beta \to \mathcal{B}_\beta$ given by

$$(\mathcal{P}\eta)(u) = \frac{\mu}{\Gamma(v)}u^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\mathbb{F}(s,\eta(s))ds + \int_0^s \mathbb{G}(s,r,\eta(r))dr\Big)ds,$$

where

$$\mathcal{B}_\beta := \{\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega)) : ||\eta - \mu||_2 \leq \beta\},$$

such that $\beta \geq \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)}$.

Firstly, we see that the operator $\mathcal{P}$ maps into itself. For this we take any $u \in [0,T]$ and $\eta \in \mathcal{B}_\beta$, we get

$$||(\mathcal{P}\eta)(u)||_2 \leq \frac{||\mu||_2}{\Gamma(v)}u^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}||\mathbb{F}(s,\eta(s))||_2 ds$$
$$+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s ||\mathbb{G}(s,r,\eta(r))||_2 dr ds$$
$$\leq \frac{||\mu||_2}{\Gamma(v)}u^{v-1} + \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)}$$
$$\leq ||\mu||_2 + \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)}.$$

Thus, we have

$$||(\mathcal{P}\eta)(u) - \mu||_2 \leq \frac{L_1 u^p}{\Gamma(1+p)} + \frac{L_2 u^{p+1}}{(1+p)\Gamma(p)} \leq \beta.$$

That is, $\mathcal{P}(\mathcal{B}_\beta)$ is uniformly bounded. This proves that $\mathcal{P}(\mathcal{B}_\beta) \subset \mathcal{B}_\beta$.

Now, we shall show that the operator $\mathcal{P}$ satisfies all the conditions of Schauder's theorem. The following are the steps of Schauder's theorem.
*Step 1:* $\mathcal{P}$ is continuous.
Let $\eta_n$ be a sequence such that $\eta_n \to \eta$ in $\mathcal{B}_\beta$.

$$||((\mathcal{P}\eta_n)(u) - (\mathcal{P}\eta)(u))||_2 \leq \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} ||\mathbb{F}(s,\eta_n(s)) - \mathbb{F}(s,\eta(s))||_2 ds$$
$$+ \frac{1}{\Gamma(p)} \int_0^u (u-s)^{p-1} \int_0^s ||\mathbb{G}(s,r,\eta_n(r)) - \mathbb{G}(s,r,\eta(r))||_2 drds.$$

Since, $\mathbb{F}, \mathbb{G}$ are continuous functions, by the the Lebesgue dominated convergence theorem, we get

$$\|(\mathcal{P}\eta_n)(u) - (\mathcal{P}\eta)(u)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\mathcal{P}$ is continuous in $\mathcal{B}_\beta$.
*Step 2:* $\mathcal{P}(\mathcal{B}_\beta)$ is uniformly bounded. This is clear since $\mathcal{P}(\mathcal{B}_\beta) \subset \mathcal{B}_\beta$ is bounded.
*Step 3:* For any $u_1, u_2 \in [0,T]$ $u_1 < u_2$, we have

$$\begin{aligned} &||(\mathcal{P}\eta)(u_2) - (\mathcal{P}\eta)(u_1)||_2 = \\ &\Big|\Big|\frac{1}{\Gamma(p)} \int_0^{u_2} (u-s)^{p-1} \Big(\mathbb{F}(s,\eta(s))ds + \int_0^s \mathbb{G}(s,r,\eta(r))dr\Big)ds \\ &- \Big(\frac{1}{\Gamma(p)} \int_0^{u_1} (u-s)^{p-1} \mathbb{F}(s,\eta(s))ds + \frac{1}{\Gamma(p)} \int_0^{u_1} (u-s)^{p-1} \\ &\int_0^s \mathbb{G}(s,r,\eta(r))drds\Big)\Big|\Big|_2 \leq \frac{L_1({u_2}^p - {u_1}^p)}{\Gamma(1+p)} + \frac{L_2({u_2}^{p+1} - {u_1}^{p+1})}{(1+p)\Gamma(p)} \end{aligned} \tag{6}$$

As $u_2 \to u_1$, the right hand side of of Eq (6) tends to zero, i.e.,

$$||(\mathcal{P}\eta)(u_2) - (\mathcal{P}\eta)(u_1)|| \to 0.$$

This means that $(\mathcal{P}\eta)(u)$ is equi-continuous on $[0,U]$ and completely continuous. So, by Schauder's theorem together with the steps $1-3$ we obtain that the operator $\mathcal{P}$ has at least one fixed point in $\mathcal{B}_\beta$. This completes the proof. □

## 4. **Ulam- Hyers stability**

In this section of the manuscript, we will present the stability results for the problem (1);

**Definition 4.1.** *A system is* $\mathbb{UH}$ *stable, if there exists a real number* $\mathbb{C}_1$ *such that for each* $\epsilon$ *and for each solution* $\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$ *of the following inequality*

$$||\mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr||_2 \leq \epsilon, \tag{7}$$

*there exists a solution* $\theta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$, *with* $||\eta - \theta||_2 \leq \mathbb{C}_1\epsilon$.

**Definition 4.2.** *The problem (1) is* $\mathbb{UHR}$ *stable, if there exists a real number* $\mathbb{C}_2$ *such that for each* $\epsilon$ *and for each solution* $\eta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$ *of the following inequality*

$$||\mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr||_2 \leq \epsilon\psi(u), \tag{8}$$

*there exists a solution* $\theta \in \mathcal{C}(\mathcal{J}, L_2(\Omega))$*, with* $||\eta - \theta||_2 \leq \mathbb{C}_2\epsilon\psi(u)$.

**Theorem 4.1.** *Under the hypotheses* $(\mathrm{H}_1)$*,* $(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *we have that* $\mathbb{RFIDE}$ (1) *is* $\mathbb{UH}$ *stable*

*Proof.* Let $\eta(u)$ be a solution of the inequality

$$||\mathcal{D}_0^{p,q}\eta(u) - \mathbb{F}(u,\eta(u)) - \int_0^u \mathbb{G}(u,r,\eta(r))dr||_2 \leq \epsilon. \tag{9}$$

Let $\theta(u)$ be the solution of the following equation

$$\mathcal{D}_0^{p,q}\theta(u) = \mathbb{F}(u,\theta(u)) + \int_0^u \mathbb{G}(u,r,\theta(r))dr \tag{10}$$

and

$$\mathcal{I}_0^{1-v}\eta(u)\Big|_{u=0} = \mu \quad \text{and} \quad \mathcal{I}_0^{1-v}\theta(u)\Big|_{u=0} = \lambda.$$

Using Lemma 3.1 we obtain

$$\begin{aligned}\theta(u) =& \frac{\mu}{\Gamma(v)}(u)^{v-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\theta(s))ds \\ &+ \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\theta(r))drds.\end{aligned}$$

By integration of (10), we have

$$\Big|\Big|\mathcal{I}_0^p\mathcal{D}_0^{p,q}\eta(u) - \mathcal{I}_0^p\Big(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(s,r,\eta(r))dr\Big)\Big|\Big|_2 \leq \mathcal{I}_0^p\epsilon = \frac{\epsilon u^p}{\Gamma(1+p)}.$$

On the other hand we obtain

$$\begin{aligned}&||\eta(u) - \theta(u)||_2 \\ &\leq \Big|\Big|\eta(u) - \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)}\int_0^p (u-s)^{p-1}\Big(\mathbb{F}(u,\theta(u)) + \int_0^s \mathbb{G}(s,r,\theta(r))dr\Big)du\Big|\Big|_2 \\ \leq&\Big|\Big|\eta(u) - \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(u,r,\eta(r))dr\Big)\Big|\Big|_2 \\ &+ \Big|\Big|\frac{\mu u^{1-v}}{\Gamma(v)} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\mathbb{F}(u,\eta(u)) + \int_0^s \mathbb{G}(u,r,\eta(r))dr\Big)\Big|\Big|_2 \\ &- \frac{\mu u^{1-v}}{\Gamma(v)} - \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\Big(\mathbb{F}(u,\theta(u)) + \int_0^s \mathbb{G}(u,r,\theta(r))dr\Big)\Big|\Big|_2,\end{aligned}$$

which implies that

$$\begin{aligned}||\eta(u)-\theta(u)||_2 \leq & \frac{\epsilon u^p}{\Gamma(1+p)}+\frac{\mathcal{A}}{\Gamma(p)}\int_0^u (u-s)^{p-1}||\eta(u)-\theta(u)||ds \\ & +\int_0^u (u-s)^{p-1}\frac{\mathcal{A}}{\Gamma(p)}\int_0^u \frac{\mathcal{B}}{\mathcal{A}}||\eta(r)-\theta(r)||drdu. \end{aligned} \tag{11}$$

Now, using Pachpatte's inequality (see [12]), we get

$$\begin{aligned} & ||\eta(u)-\theta(u)|| \\ & \leq \frac{\epsilon u^p}{\Gamma(1+p)}\Big[1+\int_0^u \frac{\mathcal{A}}{\Gamma(p)}(u-s)^{p-1}\Big(\int_o^u \big(\frac{\mathcal{A}}{\Gamma(p)}(u-s)^{p-1}+\frac{\mathcal{B}}{\mathcal{A}}\big)ds\Big)du\Big] \\ & \leq \frac{u^p}{\Gamma(1+p)}\Big[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\Big(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}\Big)\Big]\epsilon \\ & \leq \mathbb{C}_1\epsilon, \end{aligned}$$

This provides that

$$\mathbb{C}_1=\frac{u^p}{\Gamma(1+p)}\Big[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\Big(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}\Big)\Big].$$

Finally, this implies that

$$||\eta(u)-\theta(u)|| \leq \mathbb{C}_1\epsilon.$$

Thus, we may conclude that (1) is $\mathbb{UH}$ stable. □

**Theorem 4.2.** *Under the hypotheses* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ *and* $(\mathrm{H}_3)$ *there exists a constant* $\sigma_\psi > 0$ *such that*

$$\int_0^u \frac{(u-s)^{p-1}\psi(u)}{\Gamma(p)}du \leq \sigma_\psi\psi(u),$$

*where* $\psi$ *is non decreasing. Then* (1) *is* $\mathbb{UHR}$ *stable.*

*Proof.* Let $\eta(u)$ be a solution for

$$||\mathcal{D}_0^{p;q}\eta(u)-\mathbb{F}(u,\eta(u))-\int_0^u \mathbb{G}(u,r,\eta(r))dr||_2 \leq \epsilon\psi(u). \tag{12}$$

Let $\theta(u)$ be a unique solution of the equation

$$\mathcal{D}_0^{p;q}\theta(u)=\mathbb{F}(u,\theta(u))+\int_0^u \mathbb{G}(u,r,\theta(r))dr. \tag{13}$$

and

$$\mathcal{I}_0^{1-v}\eta(u)\big|_{u=0}=\mu \;\text{ and }\; \mathcal{I}_0^{1-v}\theta(u)\big|_{u=0}=\lambda$$

So, by using Lemma 3.1, we obtain

$$\theta(u) = \frac{\mu}{\Gamma(\upsilon)}(u)^{\upsilon-1} + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\mathbb{F}(s,\theta(s))ds + \frac{1}{\Gamma(p)}\int_0^u (u-s)^{p-1}\int_0^s \mathbb{G}(s,r,\theta(r))drds.$$

By integration of (9), we have

$$\Big|\Big|\mathcal{I}_0^p\mathcal{D}_0^{p,q}\eta(u) - \mathcal{I}_0^p\Big(\mathbb{F}(u,\eta(u)) + \int_0^u \mathbb{G}(u,r,\eta(r))dr\Big)\Big|\Big| \leq \mathcal{I}_0^p\epsilon\psi(u) \leq \epsilon\int_0^u \frac{(u-s)^{p-1}\psi(u)}{\Gamma(p)}du \leq \epsilon\sigma_\psi\psi(u)$$

Following the same procedure used in Theorem 4.1, we have

$$||\eta(u)-\theta(u)|| \leq \epsilon\sigma_\psi\psi(u) + \frac{\mathcal{A}}{\Gamma(p)}\int_0^u (u-s)^{p-1}||\eta(u,\omega)-\theta(u,\omega)||ds + \int_0^u (u-s)^{p-1}\frac{\mathcal{A}}{\Gamma(p)}\int_0^u \frac{\mathcal{B}}{\mathcal{A}}||\eta(r)-\theta(r)||drdu.$$

Now, using Pachpatte's inequality (see [12]), we obtain

$$\begin{aligned}||\eta(u)-\theta(u)|| &\leq \epsilon\sigma_\psi\psi(u)\Big[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\Big(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}\Big)\Big]\epsilon \\ &\leq \sigma_\psi\Big[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\Big(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}\Big)\Big]\epsilon\psi(u) \\ &\leq \mathbb{C}_2\epsilon\psi(u),\end{aligned}$$

where,

$$\mathbb{C}_2 = \sigma_\psi\Big[1+\frac{\mathcal{A}u^p}{\Gamma^2(p)}\Big(\frac{\mathcal{A}u^p}{\Gamma(1+p)}+\frac{\mathcal{B}u}{\mathcal{A}}\Big)\Big].$$

Therefore, the equation (1) is $\mathbb{UHR}$ stable. □

## 5. **Example**

In this section, we provide an illustrative example to show the consistency and validity of our results.

**Example 5.1.** *In accordance with Equation* (1)*, we design a* $\mathbb{RFIDE}$ *in the following form*

$$\mathcal{D}_{0^+}^{\frac{2}{3},\frac{1}{2}}\eta(s) = \frac{\tan(s^4)\nu^4e^{-25-s}}{\nu^4+\sin(s)+1}\eta(s) + \int_0^s \frac{e^{\sin(s)}(1+\nu^2)}{1+e^{25}}\eta(r)dr,$$

$$\mathcal{I}^{1-\frac{7}{6}}\eta(s,\omega)\Big|_{s=0} = \eta(\omega). \tag{14}$$

*From* $\mathbb{RFIDE}$ (14)*, we see that* $p=\frac{2}{3}$*,* $q=\frac{1}{2}$ *and* $\upsilon=\frac{7}{6}$*. Also, for* $s\in[0,1]$ *and* $\eta\in L_2(\Omega)$*, we can easily find* $\mathcal{A}=\mathcal{B}=\mathcal{C}_1=\mathcal{C}_2=\frac{1}{e^{25}}$*. From Theorem 3.1, we see that inequalities* $\mathcal{A}\leq\Gamma(\frac{5}{3})$ *and* $\mathcal{B}\leq\frac{5}{3}\Gamma(\frac{2}{3})$ *are satisfied. Hence*

(14) *has a unique solution. Model* (14) *also satisfy the conditions of Theorem 3.2, so* (14) *has at least one solution.*

**Conclusion**

In this manuscript, we focus on the solution of $\mathbb{RFIDE}$ (1) and using Banach contraction theorem we prove that it has a unique solution. With the use of Schauder's theorem we derived that at least one solution of (1) exists. Next, Pachpatte's inequality is used for the results of $\mathbb{UH}$ types stabilities of the proposed model. We also verify our results through an example and proved $\mathbb{UH}$ stability.

## REFERENCES


[1] *N. Ahmad, Z. Ali, K. Shah, A. Zada, G. Rahman*, Analysis of implicit type nonlinear dynamical problem of impulsive fractional differentail equations, Complexity, **2018**, (2018) 1–15.

[2] *Z. Ali, A. Zada, K. Shah*, Ulam stability results for the solutions of nonlinear implicit fractional order differential equations, Hacet. J. Math. Stat., **48**(4), (2019), 1092–1109.

[3] *L. S. Dong, H. Van Ngo, H. Vu*, Existence and Ulam stabiity for random fractional integro-differential equation, Afrika Matematika, **31**, (2020) 1283–1294.

[4] *A. M. A. El- Sayed, E. E. Elasddad, H. F. A. Madkour*, On Cauchy problem of a delay stochastic differential equation of arbitrary (fractional) orders, Fract. Differ. Calculus, **5**(2), (2015) 163–170.

[5] *S. Harikrishnan, K. Shah, D. Baleanu, K. Kanagarajan*, Note on the solution of random differential equations via $\phi$- Hilfer fractional derivative. Adv. Diff. equ., **224**; 2018;

[6] *M. Iswarya, R. Raja, G. Rajchakit, J. Cao, J. Alzabut, C. Huang*, Existence, uniqueness and exponential stability of periodic solution for discrete-time delayed BAM neural networks based on coincidence degree theory and graph theoretic method, Mathematics **2019**(7), (2019) 1–18.

[7] *F. Jarad, S. Harikrishnan,K. Shah, K. Kanagarajan* Existence and stability results to a class of fractional random implicit differential equations involving a generalized hilfer fractional derivative, Discret. Continuous Dyn. Syst, **13**(3), (2020) 723–739.

[8] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*, Theory and Applications of Fractional Differential Equations, North–Holland Mathematics Studies, **204**, Elsevier, Amsterdam, (2006).

[9] *A. Khan, M. I. Syam, A. Zada et al.* Stability analysis of nonlinear fractional differential equations with Caputo and Riemann-Liouville derivatives, Eur. Phys. J. Plus., **133**(264), (2018) 1–9.

[10] *M. Obloza*, Hyers stability of the linear differential equation, Rocznik NaukDydakt, Prace Mat., **13** (1993), 259–270.

[11] *K. B. Oldham*, Fractional differential equations in electrochemistry, Advances in Engineering Software., **41** (2010) 9–12.

[12] *B. Pachpatte*, Inequalitis for differential and integral equations, Academic Press, New York, (1997).

[13] *A. Pratap, R. Raja, Jinde Cao, J. Alzabut, C. Huang*, Finite-time synchronization criterion of graph theory perspective fractional order coupled discontinuous neural networks, Adv. Differ. Equ., **97**,(2020), (2020) 1–24.

[14] J. Sabatier, O. P. Agrawal, J. A. T. Machado, Advances in Fractional Calculus, *Dordrecht: Springer.*, (2007).

[15] *V. E. Tarasov*, Fractional Dynamics: Application of Fractional Calculus to Dynamics of particles, Fields and Media, Springer, Heidelberg; Higher Education Press, Beijing., (2010).

[16] *S. M. Ulam*, A Collection of the Mathematical Problems, Interscience, New York., (1960).

[17] *H. Vu, H. Van Ngo*, On initial value problem of random fractional differential equation with impulsive, Hace. J. Math Stat., **49**(1), (2019) 282–293.

[18] *B. M. Vintagre, I. Podlybni, A. Hernandez, V. Feliu*, Some approximations of fractional order operators used in control theory and applications, Fract. Calc. Appl. Anal., **3**(3), (2000) 231–248.

[19] *F. A. Rihan*, Numerical Modeling of Fractional Order Biological Systems, *Abs. Appl. Anal.*, (2013), Article ID 816803.

[20] *U. Riaz, A. Zada, Z. Ali et al.*, Analysis of nonlinear coupled systems of impulsive fractional differential equations with Hadamard derivatives, Math. Probl. Eng., **2020**, (2019) 1–20.

[21] *U. Riaz, A. Zada*, Analysis of $(\alpha, \beta)$-order coupled implicit Caputo fractional differential equations using topological degree method, Int. J. Nonlinear Sci. Numer. Simul., **22**(7–8), 2021, 897–915

[22] *U. Riaz, A. Zada, Z. Ali, Y. Cui, J. Xu*, Analysis of coupled systems of implicit impulsive fractional differential equations involving Hadamard derivatives, Adv. Differ. Equ., **226**(2019), 1–27.

[23] *U. Riaz, A. Zada, Z. Ali, I. L. Popa,, S. Rezapour,, S. Etemad*, On a Riemann-Liouville Type Implicit Coupled System via Generalized Boundary Conditions, Mathematics, **9**(1205), (2021), 1–22.

[24] *J. Wang, A. Zada, H. Waheed*, Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem, Math. Meth. App. Sci., **42**(18), (2019) 6706–6732.

[25] *A. Zada, S. Ali*, Stability Analysis of multi-point boundary value problem for sequential fractional differential equations with noninstantaneous impulses, Int. J. Nonlinear Sci. Numer. Simul., **19**(7), (2018) 763–774.

[26] *A. Zada, S. Ali*, Stability of integral Caputo–type boundary value problem with noninstantaneous impulses, Int. J. Appl. Comput. Math., **5**(55), (2019) 1–18.

[27] *A. Zada, U. Riaz, F. Khan*, Hyers–Ulam stability of impulsive integral equations, Boll. Unione Mat. Ital., **12**(3)(2019), 453–467.

[28] *A. Zada, F. Khan, U. Riaz, T. Li*, Hyers–Ulam stability of linear summation equations, Punjab Uni. J. Math., **49**(1)(2017), 19–24.

[29] *A. Zada, W. Ali, S. Farina*, Hyers–Ulam stability of nonlinear differential equations with fractional integrable impulses, Math. Meth. App. Sci., **40**(15)(2017), 5502–5514.

[30] *A. Zada, S. Ali, Y. Li*, Ulam–type stability for a class of implicit fractional differential equations with non–instantaneous integral impulses and boundary condition, Adv. Differ. Equ., **2017** (2017): 317.