CLOSED FORM EVALUATION OF RESTRICTED SUMS CONTAINING SQUARES OF FIBONOMIAL COEFFICIENTS

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We give a systematic approach to compute certain sums of squares Fibonomial coefficients with powers of generalized Fibonacci and Lucas numbers as coefficients; the range of the summation is not the natural one but about half of it. The technique is to rewrite everything in terms of a variable q, and then to use generating functions and Rothe's identity from classical q-calculus.

Keywords: Gaussian q-binomial coefficients, Fibonomial coefficients, q-analysis, sums identities

11B39, 05A30

1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.$$

These recurrence relations can also be extended in the backward direction. Thus

$$U_{-n} = U_{-n+2} - pU_{-n+1} = (-1)^{n+1} U_n,$$
$$V_{-n} = V_{-n+2} - pV_{-n+1} = (-1)^{n} V_n.$$

For $n \geq k \geq 1$ and an integer m, define the generalized Fibonomial coefficient with indices in an arithmetic progression by

$$\left\{ {n \atop k} \right\}_{U;m} := \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ {n \atop 0} \right\}_{U;m} = \left\{ {n \atop n} \right\}_{U;m} = 1$ and 0 otherwise. When $p = m = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_F$. When $m = 1$, we obtain the generalized Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_{U;1}$. We will frequently denote $\left\{ {n \atop k} \right\}_{U;1}$ by $\left\{ {n \atop k} \right\}_U$.

In this paper, we are interested in sums including the square of Fibonomial coefficients of the form $\left\{ {2n \atop n+k} \right\}_U^2$. An additional challenge is here that the range of summation is not the full range $-n \leq k \leq n$ but only about half of it, namely $0 \leq k \leq n$. We mainly present three sets of identities which are expressed in the notion of $\left\{ {2n \atop k} \right\}_{U;m}$ with $m = 1, 2$. More importantly, we describe a general methodology how to evaluate these sums, which will be applicable to many others as well.

Our approach is as follows. For an integer n, we use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

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with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} (1 + xq^k),$$

and *Rothe's* formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k).$$

All the identities we will derive hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

Recently, the authors of [2, 4] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if n and m are both nonnegative integers, then

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix}_U U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U U_{2mk} = P_{n,m} \sum_{k=0}^{m} \begin{Bmatrix} 2m \\ 2k \end{Bmatrix}_U U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U V_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix}_U V_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U V_{2mk} = P_{n,m} \sum_{k=0}^{m} \begin{Bmatrix} 2m \\ 2k \end{Bmatrix}_U V_{(2n+1)2k},$$

as well as their alternating analogues were also presented, where

$$P_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m. \end{cases}$$

More recently, Kilic and Prodinger [3] give a systematic approach for computing the sums of the form:

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U^2 U_{\lambda_1 k + r_1} \ldots U_{\lambda_s k + r_s}$$

in closed form, where r_i and $\lambda_i \geq 1$ are integers.

In this paper we investigate sums containing the square of Fibonomial coefficients with the powers of generalized Fibonacci and Lucas numbers, over half the natural summation range. The approach works for Fibonacci and Lucas (-type) numbers as factors likewise.

We discuss both, the Fibonacci and Lucas instances, where the range of summations is over all non-negative integers (i. e., about half of the possible number of terms). For instances with the full range of summations, we refine ourselves to present a few representative formulæ without presenting their derivations.

2. A systematic approach

We are now interested to evaluate the following two kinds of sums

$$i)\ \sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{km_1} V_{km_3}^{m_2} \quad \text{and} \quad ii) \sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{km_1} U_{km_3}^{m_2}$$

as well as

$$iii)\ \sum_{k=-n}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{km_1} V_{km_3}^{m_2} \quad \text{and} \quad iv)\ \sum_{k=-n}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{km_1} U_{km_3}^{m_2}$$

in closed form where m_i are integers.

The sums of the type (i) will be translated into q-notation:

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 q^{k^2-n^2} (-1)^{km_1+n+k} \mathbf{i}^{km_3m_2} q^{-\frac{1}{2}km_3m_2} \left(1+q^{km_3}\right)^{m_2}.$$

For our method to work, firstly the factor $(-1)^k$ must appear and secondly the term for "k" must coincide with the term for "$-k$". That means that we have two possibilities for the first condition:

- m_2 is even,
- m_3 is even

such that $2m_1 = m_2m_3$.

Then we are able to evaluate the sums

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{krs} V_{kr}^{2s} \quad \text{and} \quad \sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n-k \end{matrix} \right\}_U^2 (-1)^{krs} V_{2kr}^{s}$$

in closed form. Their q-forms are

$$(-1)^n q^{-n^2} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{kr}\right)^{2s}$$

and

$$(-1)^n q^{-n^2} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{2kr}\right)^{s}, \tag{1}$$

respectively.

Now we will examine whether the second condition is satisfied. We consider the term for $-k$ and ignore constant factors,

$$\begin{aligned} &\begin{bmatrix} 2n \\ n-(-k) \end{bmatrix}_q^2 (-1)^{(-k)} q^{(-k)^2-(-k)rs} \left(1+q^{(-k)r}\right)^{2s} \\ &= \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2+krs} \left(1+q^{-kr}\right)^{2s} = \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(q^{kr}+1\right)^{2s}, \end{aligned}$$

which is the term for k. Thus we have that

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{kr}\right)^{2s} \\ &\qquad = \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 2^{2s-1} + \frac{1}{2} \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{kr}\right)^{2s}. \end{aligned}$$

Now we compute the sum over the full range:

$$\begin{aligned}
&\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{kr}\right)^{2s} \\
&= \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^{k-n} q^{(k-n)^2-(k-n)rs} \left(1+q^{(k-n)r}\right)^{2s} \\
&= (-1)^n q^{n^2+nrs} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-krs} \sum_{t=0}^{2s} \binom{2s}{t} q^{(k-n)rt} \\
&= (-1)^n q^{n^2+nrs} \sum_{t=0}^{2s} \binom{2s}{t} q^{-nrt} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-krs+krt} \\
&= (-1)^n q^{n^2+nrs} \sum_{t=0}^{2s} \binom{2s}{t} q^{-nrt} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-k(2n+r(s-t))}.
\end{aligned}$$

Thus, concentrating on the inner sums, we have to evaluate a *finite number* of terms of the form

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-k(2n+\mu)},$$

where $\mu = r(s-t)$ is an integer. Now we will explain how this can be done. Consider

$$\begin{aligned}
&\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-k(2n+\mu)} \\
&= q^{-2n^2+n} \left[z^{2n}\right] \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{\binom{k}{2}} z^k \right) \cdot \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ 2n-k \end{bmatrix}_q (-1)^k q^{\binom{2n-k}{2}-\mu k} z^k \right) \\
&= q^{-2n^2+n} \left[z^{2n}\right] \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{\binom{k}{2}} z^k \right) \cdot \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-\mu k} z^k \right) \\
&= q^{-2n^2+n} \left[z^{2n}\right] (-z;q)_{2n} \left(zq^{-\mu};q\right)_{2n}.
\end{aligned}$$

Summarizing, the sum of interest is evaluated as

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{kr}\right)^{2s} \\
&= 2^{2s-1} \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 + (-1)^n \frac{1}{2} q^{-n^2+n(rs+1)} \sum_{t=0}^{2s} \binom{2s}{t} q^{-nrt} \left[z^{2n}\right] (-z;q)_{2n} \left(zq^{-\mu};q\right)_{2n},
\end{aligned}$$

where μ is defined as before.

In order to evaluate $\left[z^{2n}\right] (-z;q)_{2n} (zq^{-\mu};q)_{2n}$, we observe that there are factors $(1-zq^i)$ and $(1+zq^i)$ that can be combined to $(1-z^2q^{2i})$. (That is the reason that we need the factor $(-1)^k$ in our sums, as mentioned before.) In fact, there are $2n-|\mu|$ such pairs, and only $2|\mu|$ separate factors. They mess up the final result, but since μ is a constant (not depending on n), we still get a closed from evaluation. We have to evaluate a *finite number* of terms of the form

$$[z^{2n}] z^a q^b (z^2q^c;q^2)_{2n-|\mu|} = q^b [z^{2n-a}] (z^2q^c;q^2)_{2n-|\mu|}.$$

This is either 0 for $2n-a$ odd or

$$q^{b+\frac{c(2n-a)}{2}} \begin{bmatrix} 2n-|\mu| \\ \frac{2n-a}{2} \end{bmatrix}_{q^2} (-1)^{n-\frac{a}{2}} q^{(n-a/2)(n-a/2-1)}$$

otherwise.

Eventually we end up with a (finite) linear combination of terms of the form

$$\begin{bmatrix} 2n-|\mu| \\ n-a/2 \end{bmatrix}_{q^2}$$

for some integers μ and a. The final step is to translate such a result back to expressions in terms of $\left[\begin{smallmatrix} 2n-|\mu| \\ n-a/2 \end{smallmatrix}\right]_{U;2}$ and simplify according to the Binet formula related to the recursion of second order for U_n.

For (1), the second part of (i), we can check in the same way that our two conditions are satisfied. Thus we may write

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{2kr}\right)^s$$
$$= \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 2^{s-1} + \frac{1}{2} \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{2kr}\right)^s .$$

Now, similar to the previous case, we write

$$\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 (-1)^k q^{k(k-rs)} \left(1+q^{2kr}\right)^s$$
$$= q^{n^2+nrs} (-1)^n \sum_{t=0}^{s} \binom{s}{t} q^{-2nrt} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-k(2n+r(s-2t))},$$

which, by using our previous result, equals

$$= q^{-n^2+n(rs+1)} (-1)^n \sum_{t=0}^{s} \binom{s}{t} q^{-2nrt} \left[z^{2n}\right] (-z;q)_{2n} \left(zq^{-\mu};q\right)_{2n},$$

where $\mu = r(s-2t)$.

Therefore we have

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k^2-krs} \left(1+q^{2kr}\right)^s$$
$$= \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 2^{s-1} + \frac{1}{2} q^{-n^2+n(rs+1)} (-1)^n \sum_{t=0}^{s} \binom{s}{t} q^{-2nrt} \left[z^{2n}\right] (-z;q)_{2n} \left(zq^{-\mu};q\right)_{2n},$$

where $\mu = r(s-2t)$.

Now we move to sums of type (ii) and translate them into q-notation:

$$(1-q)^{-m_2} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 q^{k^2-n^2} (-1)^{km_1+n-k} \mathbf{i}^{m_2(km_3-1)} q^{-\frac{1}{2}m_2(km_3-1)} \left(1-q^{km_3}\right)^{m_2}.$$

For our method to work, we require that m_2 is even such that $2m_1 = m_2 m_3$.

Then we are able to evaluate the sums

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n-k \end{Bmatrix}_U^2 (-1)^{krs} U_{kr}^{2s}$$

in closed form. In q-notation, we have to evaluate

$$(1-q)^{-2s} (-1)^{n-s} q^{s-n^2} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 (-1)^k q^{k(k-rs)} \left(1-q^{kr}\right)^{2s}.$$

Since the term for $k=0$ evaluates to 0 and we have again symmetry between $-k$ and k we can write

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2(-1)^k q^{k(k-rs)}\left(1-q^{kr}\right)^{2s} = \frac{1}{2}\sum_{k=-n}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2(-1)^k q^{k(k-rs)}\left(1-q^{kr}\right)^{2s}.$$

Now we deal with the full range summation

$$\begin{aligned}\sum_{k=-n}^{n}&\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2(-1)^k q^{k(k-rs)}\left(1-q^{kr}\right)^{2s}\\ &=\sum_{k=-n}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2(-1)^k q^{k(k-rs)}\sum_{t=0}^{2s}\binom{2s}{t}(-1)^t q^{krt}\\ &=\sum_{t=0}^{2s}\binom{2s}{t}(-1)^t\sum_{k=-n}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2(-1)^k q^{k(k-r(s-t))}\end{aligned}$$

which, by taking $k-n$ instead of k, equals

$$=(-1)^n q^{n^2}\sum_{t=0}^{2s}\binom{2s}{t}(-1)^t q^{nr(s-t)}\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^k q^{k^2-k(2n+r(s-t))}.$$

Thus, we again encounter terms of the form

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\k\end{bmatrix}_q^2(-1)^k q^{k^2-k(2n+\mu)},$$

where $\mu=r(s-t)$ is an integer. The treatment of these terms is covered by the previous discussion.

Summarizing, our evaluation takes the form

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 q^{k^2-krs}(-1)^k\left(1-q^{kr}\right)^{2s} = \frac{1}{2}(-1)^n q^{-n^2+n}\sum_{t=0}^{2s}\binom{2s}{t}(-1)^t q^{n\mu}\left[z^{2n}\right](-z;q)_{2n}\left(zq^{-\mu};q\right)_{2n},$$

where μ is defined as before.

In the remaining sections, this general program will be illustrated in more detail on four examples. Further, we will list several attractive formulæ that were obtained using the procedure just described. Finally we present results on the sums with full summation range.

3. Illustrative Examples

Now we work out four examples that fall into the general scheme mentioned above in more detail. Also we will present some additional examples without proof.

Theorem 3.1. *For $n>1$,*

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_{U;1}^2 V_{2k}^2 = 2\begin{Bmatrix}2n\\n\end{Bmatrix}_{U;1}^2+2\begin{Bmatrix}2n-2\\n\end{Bmatrix}_{U;2} + \begin{Bmatrix}2n\\n\end{Bmatrix}_{U;2}+\left(\Delta U_{2n}^2+V_2\right)\begin{Bmatrix}2n-2\\n-1\end{Bmatrix}_{U;2},$$

where Δ *is defined as before.*

Proof. First we convert the left-hand side of the claim in q-notation:

$$\begin{aligned}\sum_{k=0}^{n}\left\{\begin{matrix}2n\\n-k\end{matrix}\right\}_U^2 V_{2k}^2 &= \sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 \alpha^{2(n^2-k^2)}\alpha^{4k}(1+q^{2k})^2\\ &= \alpha^{2n^2}\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 \alpha^{-2k^2+4k}(1+q^{2k})^2\\ &= \alpha^{2n^2}\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k\, q^{k(k-2)}(1+q^{2k})^2.\end{aligned}$$

Second we convert the right-hand side of the claim in q-notation, skipping details:

$$\begin{aligned}&2\left\{\begin{matrix}2n\\n\end{matrix}\right\}_{U;1}^2 + 2\left\{\begin{matrix}2n-2\\n\end{matrix}\right\}_{U;2} + \left\{\begin{matrix}2n\\n\end{matrix}\right\}_{U;2} + \left(\Delta^2 U_{2n}^2 + V_2\right)\left\{\begin{matrix}2n-2\\n-1\end{matrix}\right\}_{U;2}\\ &\quad= \alpha^{2n^2}\Bigg(2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + 2q^{2n}\begin{bmatrix}2n-2\\n\end{bmatrix}_{q^2} + \begin{bmatrix}2n\\n\end{bmatrix}_{q^2}\\ &\qquad\qquad - q^{-1}\left(\left(1-q^{2n}\right)^2 - q^{2n-1}\left(1+q^2\right)\right)\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2}\Bigg).\end{aligned}$$

So we need to prove that

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k\, q^{k^2-2k}\left(1+q^{2k}\right)^2 &= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + 2q^{2n}\begin{bmatrix}2n-2\\n\end{bmatrix}_{q^2}\\ &+ \begin{bmatrix}2n\\n\end{bmatrix}_{q^2} - q^{-1}\left(\left(1-q^{2n}\right)^2 - q^{2n-1}\left(1+q^2\right)\right)\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2}.\end{aligned}$$

Thus, according to our approach, we write

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k\, q^{k^2-2k}\left(1+q^{2k}\right)^2\\ &= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + (-1)^n\frac{1}{2}q^{-n^2+3n}\sum_{t=0}^{2}\binom{2}{t}q^{-2nt}\left[z^{2n}\right](-z;q)_{2n}\left(zq^{-2(1-t)};q\right)_{2n}\\ &= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + (-1)^n\frac{1}{2}q^{-n^2+3n}\Big(\left[z^{2n}\right](-z;q)_{2n}\left(zq^{-2};q\right)_{2n}\\ &+ 2q^{-2n}\left[z^{2n}\right](-z;q)_{2n}(z;q)_{2n} + q^{-4n}\left[z^{2n}\right](-z;q)_{2n}\left(zq^2;q\right)_{2n}\Big)\\ &= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + (-1)^n\frac{1}{2}q^{-n^2+3n}\Big(\left[z^{2n}\right]\left(z^2;q^2\right)_{2n-2}\left(1+zq^{2n-1}\right)\left(1+zq^{2n-2}\right)\\ &\times\left(1-z/q^2\right)(1-z/q) + 2q^{-2n}\left[z^{2n}\right]\left(z^2;q^2\right)_{2n} + q^{-4n}\left[z^{2n}\right](1+z)(1+zq)\\ &\times\left(1-zq^{2n+1}\right)\left(1-zq^{2n}\right)\left(-zq^2;q\right)_{2n-2}\left(zq^2;q\right)_{2n-2}\Big)\\ &= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + (-1)^n\frac{1}{2}q^{-n^2+3n}\Big(2q^{-2n}\left[z^{2n}\right]\left(z^2;q^2\right)_{2n} + \left[z^{2n}\right]\left(z^2;q^2\right)_{2n-2}\\ &\times(1+q^{-3}\left(1-q^{2n-1}-2q^{2n}-q^{2n+1}+q^{4n}\right)z^2 + q^{4n-6}z^4) + q^{-4n}\left[z^{2n}\right]\end{aligned}$$

which, by Rothe's formula and after some rearrangements, equals

$$
\begin{aligned}
&= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + (-1)^n \frac{1}{2} q^{-n^2+3n}\bigg(2q^{-2n}\begin{bmatrix}2n\\n\end{bmatrix}_{q^2}(-1)^n q^{2\binom{n}{2}}\\
&\times\bigg((-1)^n q^{2\binom{n}{2}}\begin{bmatrix}2n-2\\n\end{bmatrix}_{q^2} + q^{4n-6}\begin{bmatrix}2n-2\\n-2\end{bmatrix}_{q^2}(-1)^n q^{2\binom{n-2}{2}}\\
&- q^{-3}\left(1-q^{2n-1}-2q^{2n}-q^{2n+1}+q^{4n}\right)\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2}(-1)^n q^{2\binom{n-1}{2}}\\
&+ q^{-4n}(-1)^n q^{2\binom{n}{2}+4n}\begin{bmatrix}2n-2\\n\end{bmatrix}_{q^2} + q^{4n+2}(-1)^n q^{2\binom{n-2}{2}+4n-8}\begin{bmatrix}2n-2\\n-2\end{bmatrix}_{q^2}\\
&- q\left(1-q^{2n-1}-2q^{2n}-q^{2n+1}+q^{4n}\right)(-1)^n q^{2\binom{n-1}{2}+4n-4}\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2}\bigg)\bigg)\\
&= 2\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + 2q^{2n}\begin{bmatrix}2n-2\\n\end{bmatrix}_{q^2} + \begin{bmatrix}2n\\n\end{bmatrix}_{q^2}\\
&- q^{-1}\Big(\left(1-q^{2n}\right)^2 - q^{2n-1}\left(1+q^2\right)\Big)\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2},
\end{aligned}
$$

as claimed. □

Theorem 3.2. *For $n > 1$,*

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_{U;1}^2 (-1)^k V_{2k} = \begin{Bmatrix}2n\\n\end{Bmatrix}_{U;1}^2 + 2(-1)^n\begin{Bmatrix}2n-1\\n\end{Bmatrix}_{U;2}.$$

Proof. First we convert the left-hand side of the claim in q-notation:

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_{U;1}^2 (-1)^k V_{2k} = (-1)^n q^{-n^2}\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k q^{k^2-k}\left(1+q^{2k}\right).$$

Second we convert the right-hand side of the claim in q-notation:

$$\begin{Bmatrix}2n\\n\end{Bmatrix}_{U;1}^2 + 2(-1)^n\begin{Bmatrix}2n-1\\n\end{Bmatrix}_{U;2} = (-1)^n q^{-n^2}\left(\begin{bmatrix}2n\\n\end{bmatrix}_q^2 + 2q^n\begin{bmatrix}2n-1\\n\end{bmatrix}_{q^2}\right).$$

Thus we need to prove that

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k q^{k^2-k}\left(1+q^{2k}\right) = \begin{bmatrix}2n\\n\end{bmatrix}_q^2 + 2q^n\begin{bmatrix}2n-1\\n\end{bmatrix}_{q^2}.$$

Thus, according to our approach, we write

$$
\begin{aligned}
&\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q^2 (-1)^k q^{k^2-k}\left(1+q^{2k}\right)\\
&= \begin{bmatrix}2n\\n\end{bmatrix}_q^2 + \frac{1}{2}q^{-n^2+2n}(-1)^n\\
&\times\Big(\left[z^{2n}\right](-z;q)_{2n}\left(zq^{-1};q\right)_{2n} + q^{-2n}\left[z^{2n}\right](-z;q)_{2n}(zq;q)_{2n}\Big)\\
&= \begin{bmatrix}2n\\n\end{bmatrix}_q^2 + \frac{1}{2}q^{-n^2+2n}(-1)^n\Big(\left[z^{2n}\right]\left(z^2;q^2\right)_{2n-1}(1-z/q)\left(1+zq^{2n-1}\right)\\
&+ q^{-2n}\left[z^{2n}\right](-zq;q)_{2n-1}(zq;q)_{2n-1}(1+z)\left(1-zq^{2n}\right)\Big)
\end{aligned}
$$

$$= \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 + \frac{1}{2} q^n \left(2 \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2} + 2 \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_{q^2} \right)$$
$$= \begin{bmatrix} 2n \\ n \end{bmatrix}_q^2 + 2q^n \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2},$$

as desired; we skipped a few simple intermediate steps for brevity. □

Theorem 3.3. *For nonnegative n,*

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^2 U_k^2 (-1)^k = -U_n^2 \begin{Bmatrix} 2n-1 \\ n-1 \end{Bmatrix}_{U;2}.$$

Proof. Following the program outlined before, we need to prove that

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 (-1)^k q^{k(k-1)} (1-q^k)^2 = - \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_{q^2} (1-q^n)^2.$$

Thus, according to our approach, we evaluate the sum as follows, only giving some key steps:

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 q^{k^2-k} (-1)^k \left(1-q^k\right)^2$$
$$= \frac{1}{2} (-1)^n q^{-n^2+n} \Big(q^n \left[z^{2n}\right] (-z;q)_{2n} (z/q;q)_{2n} - 2 \left[z^{2n}\right] (-z;q)_{2n} (z;q)_{2n}$$
$$+ q^{-n} \left[z^{2n}\right] (-z;q)_{2n} (zq;q)_{2n} \Big)$$
$$= \frac{1}{2} (-1)^n q^{-n^2+n} \Big(q^n \left[z^{2n}\right] (1-z/q) (z;q)_{2n-1} (-z;q)_{2n-1} \left(1+zq^{2n-1}\right)$$
$$- 2 \left[z^{2n}\right] \left(z^2;q^2\right)_{2n} + q^{-n} \left[z^{2n}\right] (1+z) (-zq;q)_{2n-1} (zq;q)_{2n-1} \left(1-zq^{2n}\right) \Big)$$
$$= \frac{1}{2} (-1)^n q^{-n^2+n} \left(2 (-1)^n q^{n(n+1)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2} - 2 (-1)^n q^{n^2} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \right)$$

which, by some simple rearrangements, equals

$$= -\left(1-q^n\right)^2 \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2},$$

as claimed. □

Theorem 3.4. *For nonnegative n,*

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^2 U_{2k}^2 = U_2 U_{2n-1} U_{2n} \begin{Bmatrix} 2n-2 \\ n-1 \end{Bmatrix}_{U;2}.$$

Proof. The identity in q-form is

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 (-1)^k q^{k(k-2)} \left(1-q^{2k}\right)^2$$
$$= -q^{-1} (1+q) \left(1-q^{2n-1}\right) \left(1-q^{2n}\right) \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix}_{q^2}.$$

Thus we write

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 q^{k^2-2k} (-1)^k \left(1-q^{2k}\right)^2$$

$$= \frac{1}{2}(-1)^n q^{-n^2+n}\Big(q^{2n}\left[z^{2n}\right]\left(1-z/q^2\right)(1-z/q)\left(1+zq^{2n-1}\right)$$
$$\times\left(1+zq^{2n-2}\right)(-z;q)_{2n-2}(z;q)_{2n-2} - 2\left[z^{2n}\right]\left(z^2;q^2\right)_{2n} + q^{-2n}\left[z^{2n}\right]$$
$$\times(1+z)(1+zq)\left(1-zq^{2n}\right)\left(1-zq^{2n+1}\right)\left(-zq^2;q\right)_{2n-2}\left(zq^2;q\right)_{2n-2}\Big)$$
$$= \frac{1}{2}(-1)^n q^{-n^2+n}(q^{2n}\left[z^{2n}\right]\left(z^2;q^2\right)_{2n-2}$$
$$\times\left(1+q^{-3}\left(1-q^{2n+1}-q^{2n-1}-2q^{2n}+q^{4n}\right)z^2+q^{4n-6}z^4\right)$$
$$-2\left[z^{2n}\right]\left(z^2;q^2\right)_{2n}+q^{-2n}\left[z^{2n}\right]\left(z^2q^4;q^2\right)_{2n-2}$$

which, by *Rothe's* formula, equals

$$= 2q^{2n}\begin{bmatrix}2n-2\\ n\end{bmatrix}_{q^2} - \begin{bmatrix}2n\\ n\end{bmatrix}_{q^2} - q^{-1}\left(1-q^{2n-1}-2q^{2n}-q^{2n+1}+q^{4n}\right)\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2}$$
$$= -(1+q)\,q^{-1}\left(1-q^{2n-1}\right)\left(1-q^{2n}\right)\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2},$$

as claimed. □

Now we will present a few additional results without explicit proofs; they can be done in exactly the same way as the previous examples:

Theorem 3.5. (1) *For* $n > 1$,

$$\sum_{k=0}^{n}\left\{\begin{matrix}2n\\ n+k\end{matrix}\right\}_U^2 U_{2k}^4 = \frac{2U_{2n-2}+3U_{2n-4}+U_{2n}}{U_{2n-2}} V_1^2 U_{2n-1}U_{2n-3}U_{2n}^2\left\{\begin{matrix}2n-4\\ n-2\end{matrix}\right\}_{U;2}.$$

(2) *For nonnegative* n,

$$\sum_{k=0}^{n}\left\{\begin{matrix}2n\\ n+k\end{matrix}\right\}_U^2 U_k^4 = \frac{U_n^3\left(U_{n+1}+3U_{n-1}\right)}{V_{2n-1}V_{2n}}\left\{\begin{matrix}2n\\ n\end{matrix}\right\}_{U;2}.$$

To finish, we present two examples where the sums are over the full summation range.

Theorem 3.6. (1) *For nonnegative* n,

$$\sum_{k=-n}^{n}\left\{\begin{matrix}2n\\ n+k\end{matrix}\right\}_U^2 V_{2k}(-1)^k = \frac{4(-1)^n}{V_{2n}}\left\{\begin{matrix}2n\\ n\end{matrix}\right\}_{U;2}.$$

(2) *For* $n > 0$,

$$\sum_{k=-n}^{n}\left\{\begin{matrix}2n\\ n+k\end{matrix}\right\}_U^2 V_k^2(-1)^k = \frac{2V_nU_{4n-2}}{U_n}\left\{\begin{matrix}2n-2\\ n-1\end{matrix}\right\}_{U;2}.$$

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