ANALYSIS OF ACOUSTIC WAVE PROPAGATION IN AN INFINITE COAXIAL DUCT WITH A PARTIAL ABSORBING INTERNAL SURFACE

Hülya Öztürk[1]

The propagation of acoustic waves along a rigid infinite coaxial duct with a partial lining on the outer wall is developed by using the mode-matching technique. This technique is based on deriving the solution in the way of an infinite sum of orthogonal functions and then matching them across the boundaries between the regions. In addition to the analytical derivations, the affects of the parameters such as waveguide radii, surface impedance and its length on the propagation phenomenon are presented. The problem is also compared numerically with the Wiener-Hopf approach which is more difficult to implement and very good agreement is obtained. This model can be used as a dissipative silencer which makes a significant contribution for reducing unwanted noise.

Keywords: Mode-matching method, coaxial duct, acoustics, absorbent lining.

1. Introduction

Discontinuities in circular or coaxial waveguides have received wide attention in the literature since these structures contribute significantly to the noise pollution. Various analytical methods have been used by researchers in order to reduce unwanted noise. One of the most applied of these methods is the Wiener-Hopf technique. This technique has been mostly used in studies [1-8]. It was first studied by Levine and Schwinger [9] where they analyzed a semi-infinite rigid circular unflanged duct. Later, in 1978, Rawlins [10] investigated the semi-infinite tube with an acoustically lined interior surface. Çınar et al. [11] studied the infinite coaxial waveguide with a finite gap on the inner wall as a reactive silincer. Finally, Öztürk [12] analyzed the wave scattering problem by a coaxial waveguide with an impedance-coated groove. The mode-matching method is also capable of providing very accurate results for sound reduction and has advantages since it has much less unknown variables. It has been successfully applied to a variety of structures [13-17]. Mahmood-ul-Hassan et al. [18] presented solutions to both triple and pentafurcated spaced ducts by applying mode-matching and the Wiener-Hopf methods. Then, in 2017, the coupled wave scattering characteristics of a two dimensional waveguide structure was studied by Shafique et al [19].

The aim of this work is to analyze the acoustic wave propagation along a coaxial duct with a partial lining on the outer wall by using the mode-matching method. This method is applied by dividing the waveguide into three regions. The field terms are obtained by matching the eigenmodes at the interfaces of the sections where the structure is discontinuous. The solution leads to infinite set of linear equations and these equations are solved by means of numerical procedures. We also give numerical results for the comparision of the mode-matching and Wiener-Hopf methods and illustrating the effects of some parameters.

[1] Asst. Prof. Dr., Department of Mathematics, Gebze Technical University, Kocaeli, Turkey, e-mail: h.ozturk@gtu.edu.tr

It is obtained that these results can be used in engineering applications, in particular, noise reduction devices.

2. Problem Statement

Consider a partially lined coaxial duct as shown in Figure 1. The walls of duct are located at $S = \{a \leq \rho \leq b, -\infty < z < \infty\}$. The inner surface part $0 < z < l$ of the outer wall is assumed to be lined with specific admittance which is characterized by $\eta = \rho_0 c/Z$ where ρ_0, c and Z stand for the density of undisturbed medium, speed of sound and the liner impedance, respectively.

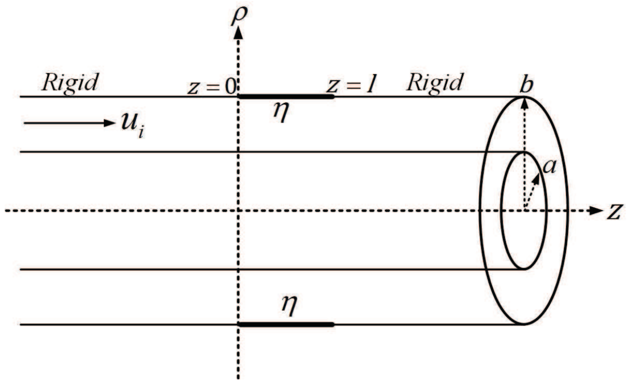

Fig. 1. Geometry of the coaxial waveguide.

Assuming a time dependence of $e^{-i\omega t}$ where ω is the angular frequency, the incident sound field is given as

$$u_i(z) = e^{ikz} \tag{1}$$

where $k = \frac{\omega}{c}$.

3. Mode-Matching Analysis

The coaxial duct is separated to three regions and the eigenfunction expansions are obtained in these regions. Let the total acoustic pressure $u_T(\rho, z)$ be

$$u_T(\rho, z) = \begin{cases} u_i(z) + u_1(\rho, z) & , \ z \in (-\infty, 0), \ \rho \in (a, b) \\ u_2(\rho, z) & , \ z \in (0, l), \ \rho \in (a, b) \\ u_3(\rho, z) & , \ z \in (l, \infty), \ \rho \in (a, b) \end{cases} \tag{2}$$

Above, $u_j(\rho, z)$, $j = 1 - 3$ are the unknown pressures in their relevant regions and they satisfy the following Helmholtz equation :

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right] u_j(\rho, z) = 0, \qquad j = 1 - 3 \tag{3}$$

The inner and outer duct, with the exception of finite part, walls are rigid, so that

$$\frac{\partial u_T(\rho, z)}{\partial \rho} \big|_{\rho=a} = 0, \ z \in (-\infty, \infty) \tag{4}$$

$$\frac{\partial u_1(\rho, z)}{\partial \rho} \big|_{\rho=b} = 0, \ z \in (-\infty, 0) \tag{5}$$

$$\frac{\partial u_3(\rho, z)}{\partial \rho} \big|_{\rho=b} = 0, \ z \in (l, \infty) \tag{6}$$

By introducing the scalar potential u which defines the acoustic pressure p and velocity $\mathbf{v}$ by $p = i\omega\rho_0 u$ and $\mathbf{v} = \text{grad } u$, respectively, the boundary condition on the absorbing lining reads

$$\frac{\partial u}{\partial n} + ik\eta u = 0 \tag{7}$$

where n the normal pointing outward the lining. Therefore, the lined impedance wall can be written as

$$-\frac{\partial u_2(\rho, z)}{\partial \rho} \mid_{\rho=b} + ik\eta u_2(\rho, z) \mid_{\rho=b} = 0, \quad z \in (0, l) \tag{8}$$

At $z = 0, l$, the pressure and the velocity are continuous, that is

$$ik + \frac{\partial u_1(\rho, 0)}{\partial z} = \frac{\partial u_2(\rho, 0)}{\partial z} \tag{9}$$

$$1 + u_1(\rho, 0) = u_2(\rho, 0) \tag{10}$$

$$\frac{\partial u_2(\rho, l)}{\partial z} = \frac{\partial u_3(\rho, l)}{\partial z} \tag{11}$$

$$u_2(\rho, l) = u_3(\rho, l) \tag{12}$$

3.1. Region I $\{-\infty < z < 0\}$

The solution in this region is given as

$$u_1(\rho, z) = \sum_{n=1}^{\infty} a_n e^{-i\alpha_n z} \varphi_n(\rho) \tag{13}$$

where $\varphi_n(\rho) = \left[J_0(K_n\rho) - \frac{J_1(K_n b)}{Y_1(K_n b)} Y_0(K_n\rho)\right]$ are the eigenfunctions of the Helmholtz equation (3), a_n is the magnitude of the reflected wave and $\alpha_n' s$ being the zeros of $K\left[J_1(Ka) - \frac{J_1(Kb)}{Y_1(Kb)} Y_1(Ka)\right]$ satisfying

$$K_n\left[J_1(K_n a) - \frac{J_1(K_n b)}{Y_1(K_n b)} Y_1(K_n a)\right] = 0, \; n = 1, 2, .., \tag{14}$$

with

$$\alpha_n = \sqrt{k^2 - K_n^2}, \; n = 1, 2, .., \tag{15}$$

Above, J_n and Y_n $(n = 0, 1)$ are the Bessel and Neumann functions. K_n are the eigenvalues and α_n are the associated eigenvalues. The physical meaning of K_n and α_n is radial wave numbers and axial wave numbers, respectively.

3.2. Region II $\{0 < z < l\}$

$u_2(\rho, z)$ in region II is given as

$$u_2(\rho, z) = \sum_{n=1}^{\infty} \left[b_n e^{i\chi_n z} + c_n e^{-i\chi_n z}\right] \psi_n(\rho) \tag{16}$$

where $\psi_n(\rho) = \left[J_0(\xi_n\rho) - \frac{[ik\eta J_0(\xi_n b) + \xi_n J_1(\xi_n b)]}{[ik\eta Y_0(\xi_n b) + \xi_n Y_1(\xi_n b)]} Y_0(\xi_n\rho)\right]$ and χ_n is the root of the equation

$$\xi_n[J_1(\xi_n a) - \frac{[ik\eta J_0(\xi_n b) + \xi_n J_1(\xi_n b)]}{[ik\eta Y_0(\xi_n b) + \xi_n Y_1(\xi_n b)]} Y_1(\xi_n a)] = 0, \; n = 1, 2, .. \tag{17}$$

with

$$\varkappa_n = \sqrt{k^2 - \xi_n^2}, \; n = 1, 2, .., \tag{18}$$

3.3. Region III $\{l < z < \infty\}$

The transmitted field $u_3(\rho, z)$ in region III is given as

$$u_3(\rho, z) = \sum_{n=1}^{\infty} d_n e^{i\alpha_n z} \varphi_n(\rho) \tag{19}$$

where d_n is the magnitude of the transmitted duct mode.

By taking into account the matching conditions (9) and (10), one gets

$$-\sum_{m=1}^{\infty} a_m \alpha_m \varphi_m(\rho) + k = \sum_{n=1}^{\infty} \chi_n [b_n - c_n] \psi_n(\rho) \tag{20}$$

$$\sum_{m=1}^{\infty} a_m \varphi_m(\rho) + 1 = \sum_{n=1}^{\infty} [b_n + c_n] \psi_n(\rho) \tag{21}$$

Multiplying (20) and (21) with $\rho\psi_n(\rho)$ and integration along $\rho \in [a, b]$ give

$$-\frac{2}{\pi^2 b} S(b, k, \eta, \xi_n) \sum_{m=1}^{\infty} \frac{a_m \alpha_m}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} + \frac{k}{\pi \xi_n^2} S(b, k, \eta, \xi_n) = \varkappa_n P_n [b_n - c_n] \tag{22}$$

$$\frac{2S(b, k, \eta, \xi_n)}{\pi^2 b} \sum_{m=1}^{\infty} \frac{a_m}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} + \frac{S(b, k, \eta, \xi_n)}{\pi \xi_n^2} = \varkappa_n P_n [b_n + c_n] \tag{23}$$

where

$$S(b, k, \eta, \xi_n) = \frac{2ik\eta}{[ik\eta Y_0(\xi_n b) + \xi_n Y_1(\xi_n b)]}\ , \quad P_n = \int_a^b \psi_n^2(\rho)\rho d\rho \tag{24}$$

b_n and c_n can be obtained from (22) and (23) as

$$b_n = \frac{S(b, k, \eta, \xi_n)}{P_n \chi_n \pi^2 b} \sum_{m=1}^{\infty} \frac{a_m(\chi_n - \alpha_m)}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} + \frac{(\chi_n + k)}{2\pi \xi_n^2 P_n \chi_n} S(b, k, \eta, \xi_n) \tag{25}$$

$$c_n = \frac{S(b, k, \eta, \xi_n)}{P_n \chi_n \pi^2 b} \sum_{m=1}^{\infty} \frac{a_m(\chi_n + \alpha_m)}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} + \frac{(\chi_n - k)}{2\pi \xi_n^2 P_n \chi_n} S(b, k, \eta, \xi_n) \tag{26}$$

Substituting (16) and (19) and their derivatives into (11) and (12), we get

$$\sum_{n=1}^{\infty} \chi_n \left[b_n e^{i\chi_n l} - c_n e^{-i\chi_n l}\right] \psi_n(\rho) = \sum_{m=1}^{\infty} d_m \alpha_m e^{i\alpha_m l} \varphi_m(\rho) \tag{27}$$

$$\sum_{n=1}^{\infty} \left[b_n e^{i\chi_n l} + c_n e^{-i\chi_n l}\right] \psi_n(\rho) = \sum_{m=1}^{\infty} d_m e^{i\alpha_m l} \varphi_m(\rho) \tag{28}$$

Multiplying (27) and (28) with $\rho\psi_n(\rho)$ and integration along $\rho \in [a, b]$ give

$$\frac{2}{\pi^2 b} S(b, k, \eta, \xi_n) \sum_{m=1}^{\infty} \frac{d_m \alpha_m e^{i\alpha_m l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} = \varkappa_n P_n \left[b_n e^{i\chi_n l} - c_n e^{-i\chi_n l}\right] \tag{29}$$

$$\frac{2}{\pi^2 b} S(b, k, \eta, \xi_n) \sum_{m=1}^{\infty} \frac{d_m e^{i\alpha_m l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} = P_n \left[b_n e^{i\chi_n l} + c_n e^{-i\chi_n l}\right] \tag{30}$$

b_n and c_n can be obtained easily from (29) and (30) as

$$b_n = \frac{S(b, k, \eta, \xi_n)}{P_n \chi_n \pi^2 b} \sum_{m=1}^{\infty} \frac{d_m(\chi_n + \alpha_m) e^{i(\alpha_m - \chi_n)l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} \tag{31}$$

$$c_n = \frac{S(b,k,\eta,\xi_n)}{P_n \chi_n \pi^2 b} \sum_{m=1}^{\infty} \frac{d_m(\chi_n - \alpha_m) e^{i(\alpha_m+\chi_n)l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} \tag{32}$$

From (25, 26) and (31, 32), we have

$$\sum_{m=1}^{\infty} \frac{d_m(\chi_n + \alpha_m) e^{i(\alpha_m-\chi_n)l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} - \sum_{m=1}^{\infty} \frac{a_m(\chi_n - \alpha_m)}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} = \frac{\pi b(\chi_n + k)}{2\xi_n^2} \tag{33}$$

$$\sum_{m=1}^{\infty} \frac{d_m(\chi_n - \alpha_m) e^{i(\alpha_m+\chi_n)l}}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} - \sum_{m=1}^{\infty} \frac{a_m(\chi_n + \alpha_m)}{K_m Y_1(K_m b)(K_m^2 - \xi_n^2)} = \frac{\pi b(\chi_n - k)}{2\xi_n^2} \tag{34}$$

The solution is obtained in terms of unknown constants d_m and a_m. Considering the equations (33) and (34) together, these constants are evaluated numerically.

4. Computational Results

Some numerical results displaying the affect of the different values of the parameters such as absorbent lining, length of the lining and radii of the coaxial duct on the propagation phenomenon are presented. For some parameter values, we have used the previous studies [4, 11, 20]. The transmission loss is determined by using the following formula

$$TL = -20\log_{10}|T_0|$$

Since analysis of the problem involves infinite sets of linear equations, first of all, convergence regarding the truncation number (N) is analyzed. In Figures 2 and 3, the magnitudes of the transmission reflection coefficients of the fundamental mode versus N is examined for various values of ka. It is obtained that solution becomes insensitive when $N > 3$. Since similar results are obtained for other sets of the problem parameters, in our computations we use $N = 10$.

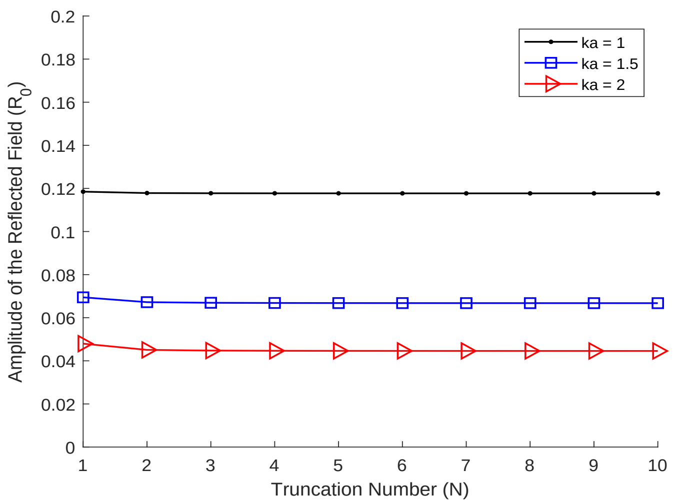

Fig. 2. Truncation number (N) analysis for the reflected field for various values of ka with $b = 2a$, $kl = 10$, $\eta^{-1} = 1 - 3i$.

Figure 4 and 5 examine the effect of surface impedance on the transmission phenomena. We observe that beyond 900 Hz approximately, the transmission loss decreases with increasing values of $|\eta^{-1}|$.

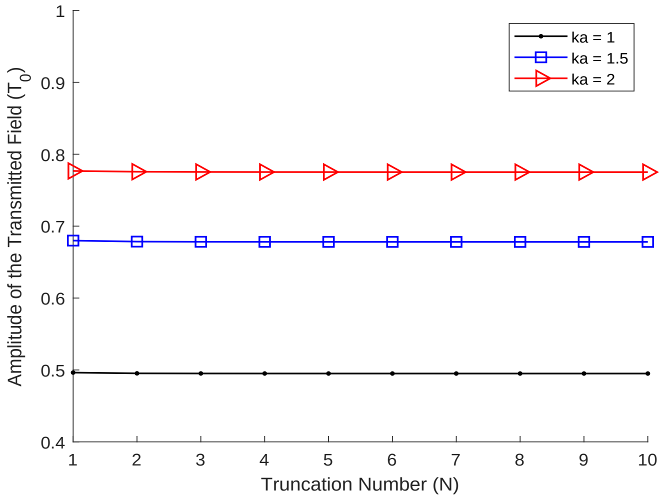

Fig. 3. Truncation number (N) analysis for the transmitted field for various values of ka with $b = 2a$, $kl = 10$, $\eta^{-1} = 1 - 3i$.

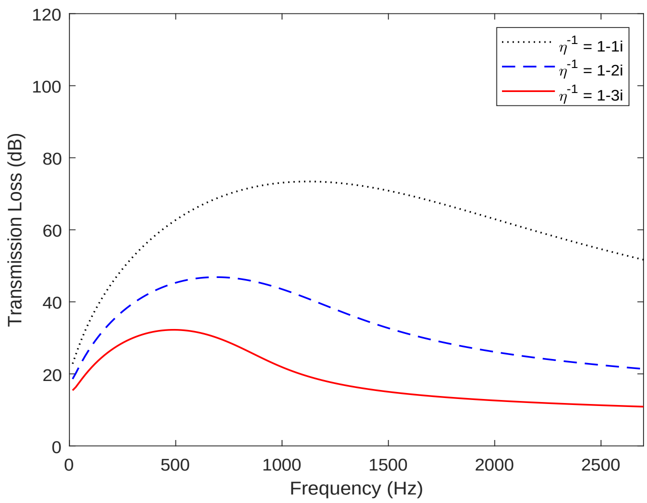

Fig. 4. Transmission loss analysis for different values of Im η^{-1} while $a = 0.025$ m, $b = 2a$, $l = 0.5$ m.

Figures 6 and 7, depict the transmission loss for different waveguide radii a and b. It is obtained that the transmission loss diminishes with the affect of increasing radii. Lower radii improves the performance of dissipative silencer at low to medium frequencies.

Figures 8 displays the affect of the lining length on the transmission. It is seen that higher lining length cause higher transmission loss. The magnitude of the tranmission loss peak is decreased when the length of the lining decreases.

Finally, we compared the two approach in Figures 9 and 10. Variation of the transmission loss is investigated for various values of η^{-1} and l. It is evident that the results of the two approaches agree well.

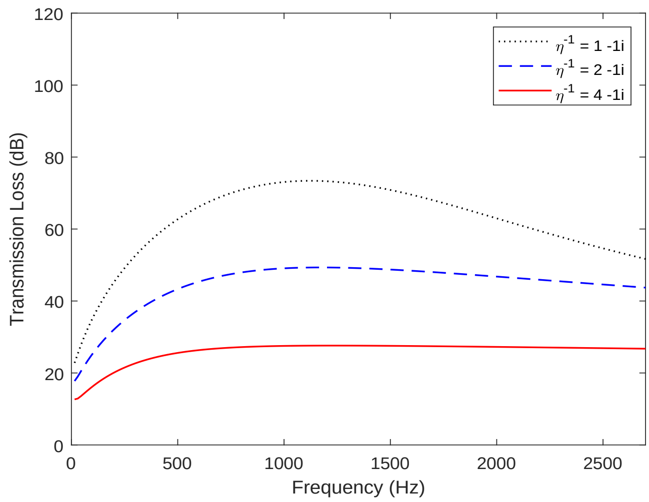

Fig. 5. Transmission loss analysis for different values of Re η^{-1} while $a =$ 0.025 m, $b = 2a$, $l = 0.5$ m.

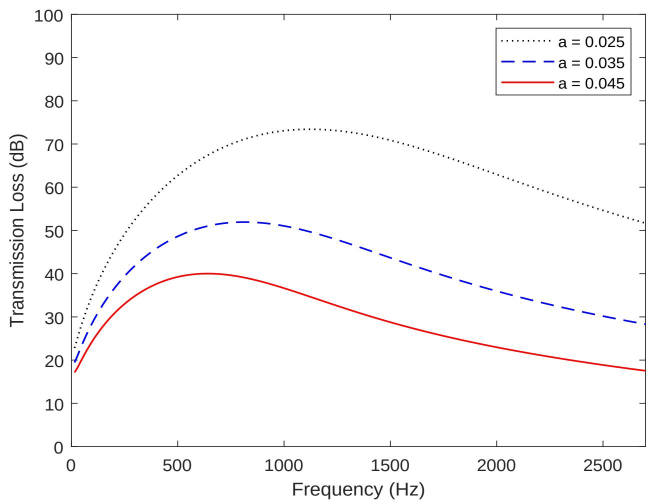

Fig. 6. Transmission loss analysis for different values of duct radius a while $b = 2a$, $l = 0.5$ m, $\eta^{-1} = 1 - 1i$.

5. Conclusions

The propagation of plane acoustic waves along an infinite coaxial duct whose inner surface of the outer wall is lined partially by acoustically absorbing material has been examined by using an analytic mode-matching technique. Analytical derivations have been compared with Wiener-Hopf technique numerically and excellent agreement is obtained. This agreement is very important since the mode-matching technique, whose implementation is more simple than Wiener-Hopf method, can be preferable in more complicated problems. Moreover, several numerical results are obtained in Section 4 to demonstrate the affects of different parameters on the transmission and reflection coefficients and transmission loss.

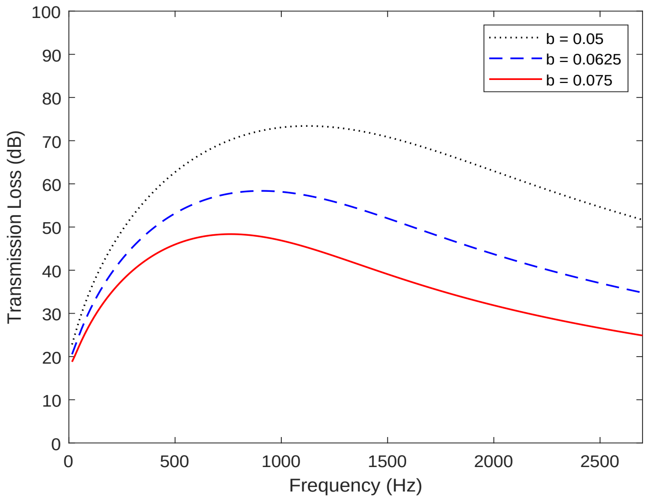

Fig. 7. Transmission loss analysis for different values of duct radius b while $b = 2a$, $l = 0.5$ m, $\eta^{-1} = 1 - 1i$.

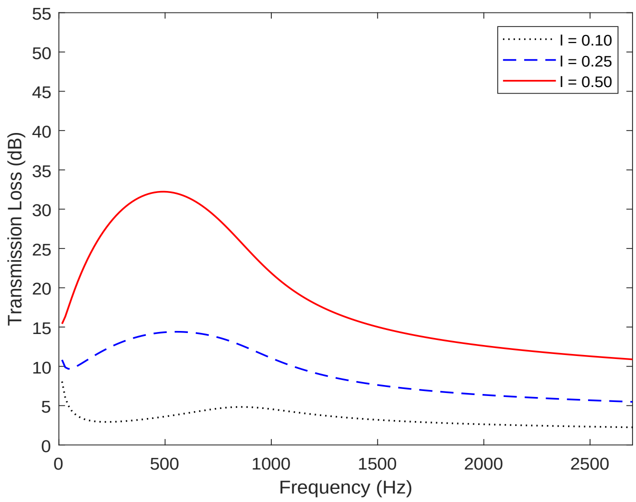

Fig. 8. Transmission loss analysis for various values of lining length while $a = 0.025$ m, $b = 2a$, $\eta^{-1} = 1 - 3i$.

These results demonstrated of the effect of partial lining on sound absorption. As shown in Figs. 4 through 10, we should note that sound absorption is fairly good by properly changing the parameters. Some particular values can provide maximum noise reduction. Thus, these investigations can be beneficial in constructing dissipative silencers.

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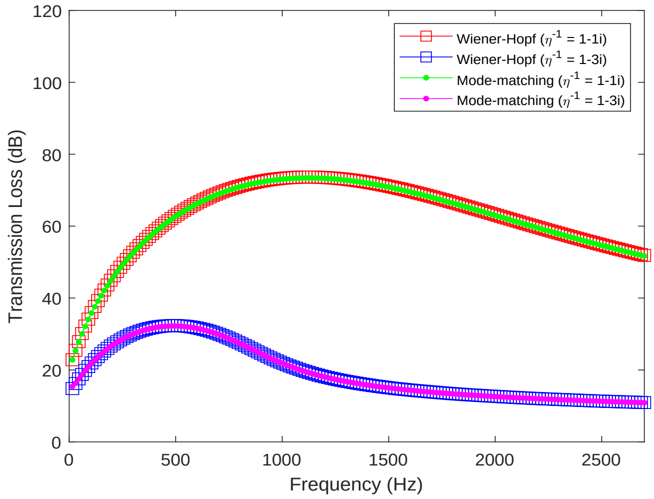

Fig. 9. Comparison of the transmission loss when $a = 0.025$ m, $b = 2a$, $l = 0.5$ m.

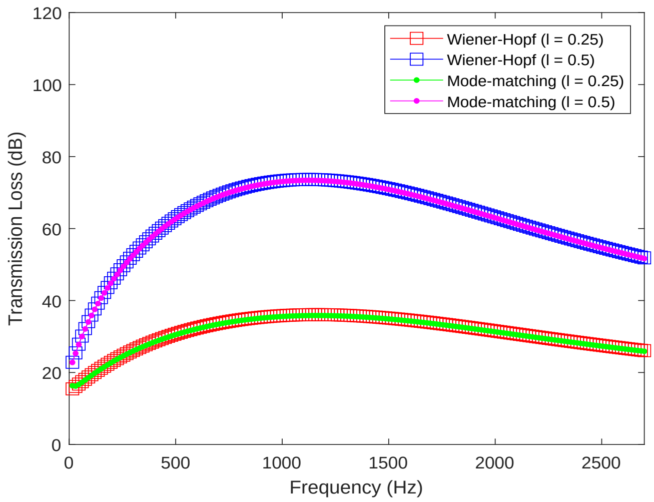

Fig. 10. Comparison of the transmission loss when $a = 0.025$ m, $b = 2a$, $\eta^{-1} = 1 - 1i$.

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