VEHICLE DYNAMICS MODELING DURING MOVING ALONG A CURVED PATH. MATHEMATICAL MODEL USAGE ON STUDYING THE ROBUST STABILITY

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Problematica deplasãrii pe traiectorie curbã a unui automobil reprezintã un subiect de studiu intens în încercarea inginerilor de profil şi automatişti de a realiza sistemul de control adecvat. Inainte de toate se impune cunoaşterea riguroasã a dinamicii autovehiculului aflat într-o mişcare circularã, pentru a se putea transpune cât mai exact realitatea fizicã în ecuații matematice. Modelul matematic trebuie sã cuprindã elementele definitorii pentru procesul respectiv, în scopul realizãrii unui compromis între complexitatea ecațiilor şi puterea de calcul necesarã procesãrii acestora. În aceastã lucrare este prezentat un model matematic neliniar cu şase grade de libertate, pe baza cãruia se studiazã influența diferiților parametri asupra stabilitãții autovehiculului în curbã (unghi de bracare, viteza şi accelerația automobilului etc.). Din studiul realizat cu ajutorul criteriului stabilitãții în sens Liapunov, reiese faptul cã, la viteze mai mari de 120 km/h, unghiuri de bracare mici pot destabiliza autovehiculul.

Rezultatele obținute se pot pune în valoare printr-un sistem de control.

The problem of the motion of an automobil on a curved path represents a subject of high interest, in the strive of the profile engineers and automatists to design the proper control system. First of all rigorous knowledge is compulsory, about the vehicle dynamics on a circular motion, in order to express as accurate as possible the physical reality into mathematical equations. The mathematical model must contain all the defining elements for the specified process, so that a compromise can be done, between the complexity of the equations and the computing power needed. This paperwork presents a mathematical nonlinear model with six freedom degrees, based on which the influence of different parameters over the stability of the vehicle on a curved path is studied (brackage angle, velocity and acceleration of the vehicle a.s.o.).From the study based on the Liapunov meaning stability criterion, we conclude that, at speed exceeding 120 km/h, small brakeage angles can destabilize the vehicle.

The obtained results can be shown to advantage through a control system.

Keywords: brackage angle, speed, robust stability

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1. Introduction

While moving along the curve, on the contact surface of each wheel, forces appear maintaining the vehicle on the trajectory. Under these forces, the tire is deformed and therefore the velocity of each wheel deviates from the tire plane under a certain angle, function of the lateral stiffness of the tire and the force value.

The movement on a curved trajectory has been debated in numerous papers of the domain, among which [4] where a three freedom degrees model is considered, subjected to the wheel forces given by the “magic formula” of H.B. Pacejka, [5] from the same conference where a two freedom degrees model is studied and the tire model proposed by Dugoff modified by the authors. In the article [3] where a three freedom degree model is presented on a self made tire model. In [8] the authors have studied the wheel dynamics on a curved path and in straight line and the result obtained is very usefull to establish the sideslip angles.

The vehicle stability is studied in [6] using coefficients to correct the forces that act upon each wheel.

In [2] a common bicycle model is made equivalent to a model in which instead of the four forces on each wheel there are considered two forces with a sliding point of application along the symmetry axes of the vehicle.

Among the many articles it is worth mentioned [1] in which the authors define a way to control the sideslip of the vehicle starting from a two freedom degree model. This method is used together with the control of the spinning speed.

The present article analyzes the curved motion using a vehicle model with six freedom degrees, which considers the displacement of the vehicle on the transverse direction, on the longitudinal direction, also the rotation movements around the transversal axis, around the longitudinal axis and the vertical displacements for each of the four wheels of the vehicle. The mathematical model offers better results comparing to the other models, considering the higher precision of the estimation and the wider spectrum of the stability analysis.

The objective of this paper is to identify the critical situations that a vehicle may encounter in a curve.

2. The mathematical model

The model with six freedom degrees takes into account the movement of the vehicle on transverse direction, on longitudinal direction, of the rotation movements around its transverse axis, around its longitudinal axis, and of the movements on the vertical for each of the four wheels of the car. Comparing to the previously studied model, other five more movements are considered: the rotation around the longitudinal axis (the rolling motion) and the vertical movements for each wheel, in order to better analyse the stability of the vehicle with a higher precision than the analyzed case based on the bicycle model.

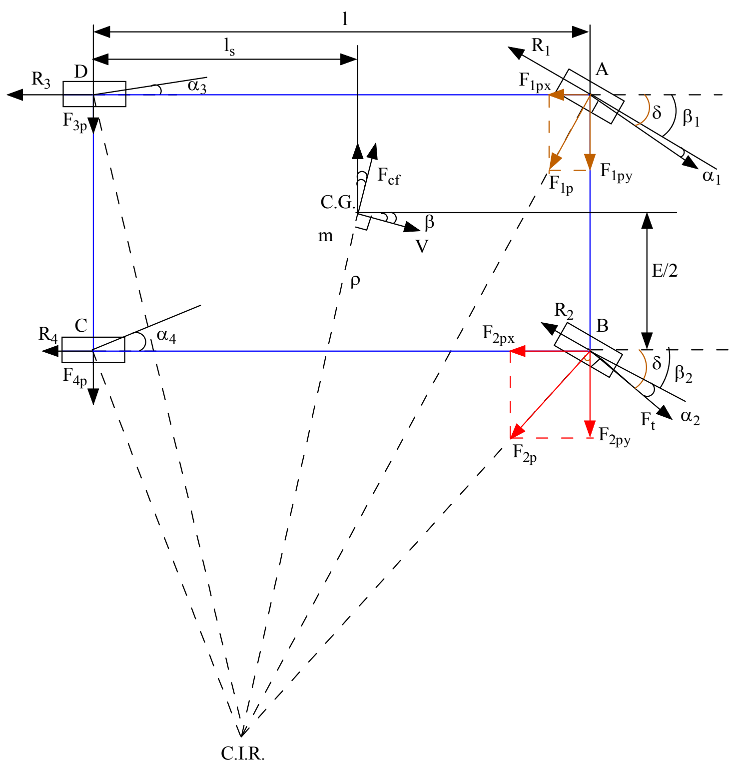

Fig. 1. The forces representation, acting on the wheels of the vehicle

In Fig.1. the following elements appear:

m - the mass of the vehicle;

l - the length of the vehicle;

l_s - the distance between the center of mass and the rear axel axis;

E - the front wheelbase (the distance between the front wheels);

ρ - the curvature radius;

α_i - the sliding angle of the wheel;

δ - the brackage angle;

β_i - the deviation angle;

F_{ip} - the perpendicular force on the direction of the wheel, due to the sliding of the wheel, determined by the sliding angle α_i;

F_t - the driving force;
F_{cf} - the centrifugal force;
$F_{cf} = \frac{mV^2}{\rho}$;
R_i - the resistive force to rolling;
$R_i = R_1 + R_2$;
V - the velocity of the vehicle.

As it can be noticed in Fig.1., three pairs of forces act upon the front wheels, along the wheel (F_t and R_i) and perpendicular on it (F_{ip}).

For the front wheels, the force along the wheel is the driving force (F_t) generated by the engine minus the resistive force to rolling (R_i).

For the rear wheels the force from the wheel plane represents the resistance to rolling (R_i).

In the center of mass the centrifugal force act along the curvature radius (F_{cf}).

Let us write the equation of the moment around the vertical axis, crossing through the intersection point between the longitudinal axis and the rear axel:

$$\left(F_{1py} + F_{2py}\right) l + \left(F_t - R_1 - R_2\right) l \sin\delta - \left(F_{1p} - F_{2p}\right)\frac{E}{2}\sin\delta + \left(R_4 - R_3\right)\frac{E}{2} = F_{cf} l_s \cos\beta \tag{1}$$

The torque given by the resistances to rolling for the rear wheels $\left(R_4 - R_3\right)\frac{E}{2}$ is much smaller than any other terms and can be neglected.

The force perpendicular on the direction of the wheel, for small deformations of the tire, is expressed as:

$$F_{ip} = c_{\alpha i}\alpha_i \tag{2}$$

where:

$c_{\alpha i}$ - the rigidity coefficient of the tire adapted to brackage, described by the formulas:

$$c_{\alpha 1} = \frac{1}{1-\lambda} c_\alpha \left[\frac{1}{H_1} - \frac{1}{4\left(H_1\right)^2}\right] \tag{3}$$

where:
λ – the sliding of the wheel;
c_α – the transversal elasticity coefficient of the tire (experimentally determined).

The intermediate measure H_i is described by the formula:

$$H_1 = \sqrt{\left(\frac{c_\lambda}{\mu N_1}\frac{\lambda}{\lambda - 1}\right)^2 + \left(\frac{1}{\lambda - 1}\frac{c_\alpha}{\mu N_1}tg\left(\alpha_1\right)\right)^2} \tag{4}$$

where:
c_λ – the longitudinal elasticity coefficient of the tire (experimentally determined);
μ – the friction coefficient between the tire and the the rolling path;
N_i - the soil reaction for each wheel;

In order to determine the N_i reactions from the road, the vertical movement was considered equal to the displacement of the joining point of the body with the suspension. The N_i reaction is considered proportional to the movement, which is almost exact if the tire vertical deformation is neglected, compared to the deformation of suspension spring.

$$\begin{cases} \left(N_1 + N_3\right)E - G\dfrac{E}{2} - \dfrac{mV^2}{\rho}h_g \cos\beta = 0 & (5) \\ \left(N_1 + N_2\right)l - Gl_s - \dfrac{mV^2}{\rho}h_g \sin\beta = 0 & (6) \\ N_1 + N_2 + N_3 + N_4 = G & (7) \end{cases}$$

For solving the equations system a fourth relation is necessary, between the 4 unknowns.

If we consider:

$$N_1 + N_4 = N_2 + N_3 \tag{8}$$

from the four equations the forces Ni can be calculated.

For computing the traction force F_t the following formula will be used:

$$F_t = \frac{M_m i i_0 \eta_t}{r} = ct. \tag{9}$$

where:

M_m - the engine moment corresponding to the vehicle velocity and to the coupled gear;

i - the current gear ratio;

i_0 - the main transmission ratio;

η_t - the total mechanical efficiency of the transmission;

r - the rolling radius of the wheel.

R_i - the rolling resistance is described by the expression $R_i = fN_i$ (10)

where:

f – the rolling resistance coefficient.

The sliding rear angles, as they result from the fig.1, have the formulas :

$$\alpha_3 = arctg\left(\frac{l_s - \rho\sin(\beta)}{\frac{E}{2} + \rho\cos(\beta)}\right); \qquad \alpha_4 = arctg\left(\frac{l_s - \rho\sin(\beta)}{-\frac{E}{2} + \rho\cos(\beta)}\right)$$

The sliding front angles have the formulas:

$$\alpha_1 = arctg\left(\frac{2l}{2\sqrt{\rho^2 - l_s^2} - E}\right) \quad \text{and} \quad \alpha_2 = arctg\left(\frac{2l}{2\sqrt{\rho^2 - l_s^2} + E}\right)$$

respectively, according to the results obtained in the work [13] where the authors study these parameters starting from the general plane movement equations of a body, particularly for the circular motion. In the same work the result for the sliding angle of the vehicle is presented:

$$\beta = 2arctg\left(\frac{\rho - \sqrt{\rho^2 - l_s^2}}{l_s}\right)$$

From the equation (1) results the curvature radius of the trajectory, ρ. The radius is calculated using Mathcad software. These results will be used to determine the stability by the help of the robust stability criterion.

In order to determine the numerical value of the Liapunov function, which will be used to study the robust stability, it is necessary to compute the curvature radius for each case separately.

The equation (1) is graphically solved. For each pair of values of the brakeage angle (δ) or of the velocity (V) the function $h(\rho)$ is graphically represented, and the ρ value for which the function becomes null, is the solution of the equation.

The graphical solving method is used because of the complexity of the resulting equation when one unknown is replaced.

All the necessary data for the numerical computation of the curvature radius are to be found in the table below:

Table.1

The necessary values for the numerical calculus of the curvature radius ρ

Fix parameters				Variable parameters	
Masses [kg] and Momentum[Nm]	Dimensions [m]	Ratios	Coefficients	Angle$[^o]$	Velocity $[km/h]$
$m = 1350$	$l_f = 1,4$	$i_0 = 4,1$	$c_\alpha = 8500[N/rad]$	δ	V
$m_r = 10$	$l_s = 1,6$	$\eta_t = 0,9$	$c_\lambda = 10500[N/rad]$		
$M_m = 150$	$l = l_f + l_s$	$\lambda = 0,1$	$\mu = 0,9$		
	$E = 1,5$	$i = 0,26$	$f = 0,015$		
	$h_g = 0,4$				
	$r = 0,291$				

The results of the numerical calculus are shown in the table below.

Table.2

The curvature radius for different brackage angles $\delta[^o]$ and velocities $V[km/h]$

V [km/h] \ $\delta[^o]$	5	10	15	20	25
50	1161,1	585,66	396,2	303,25	276,54
60	1931,05	972,24	655,68	500	461
70	2846,27	1431,56	963,8	732,84	680
80	3909,57	1965,12	1321,64	1003,5	906
90	5124,25	2574,56	1730,3	1312,47	1178,7
100	6494	3261,74	2190,98	1660,72	1454,54
110	8023,15	4028,8	2705,2	2049,26	1759,55
120	9716,55	4878,12	3274,28	2479,25	2093,8

The model is nonlinear and in order to analyze the stability, the criterion of the robust stability is used. This implies to choose a function called Liapunov, variable in time, and its first order derivative is always positive and which fulfils the condition L(0)=0. The stability is obtained only when the Liapunov function (L) becomes negative.

In order to define the Liapunov function, the case of accelerated motion will be studied.

For the Liapunov function two variables dependent of time are chosen such that the first and second derivatives by time, to be determined.

The derivatives derive from two of the equations of the dynamical equilibrium of the vehicle, which are determined from the Fig.2:

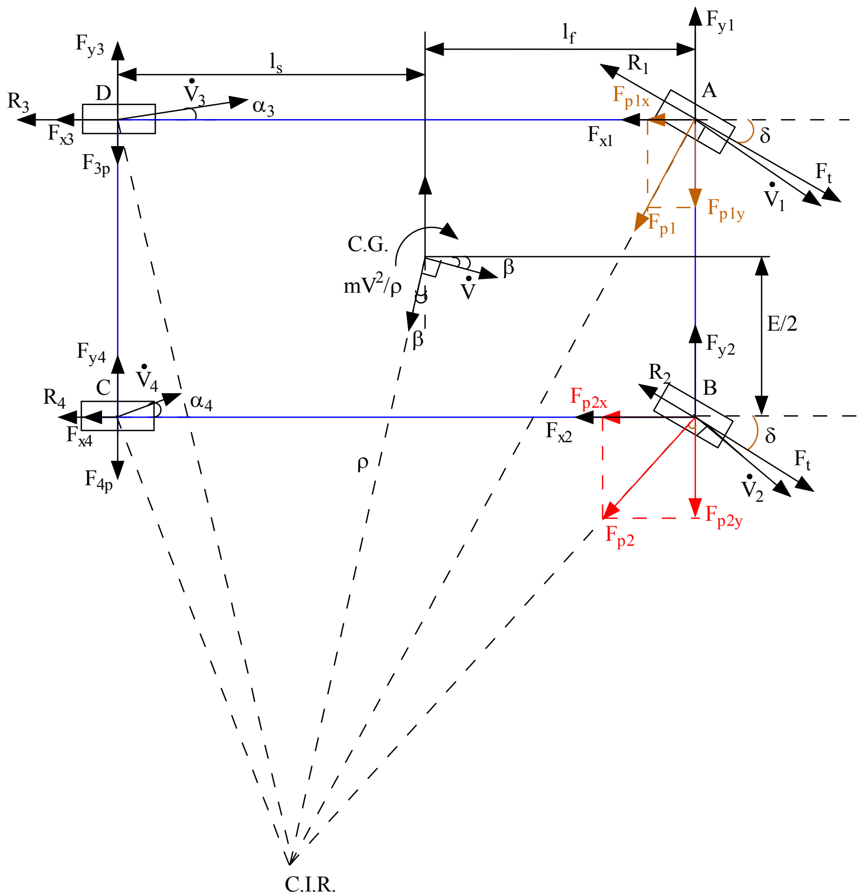

Fig. 2. The forces acting the wheels of the vehicle, in the case of the curved accelerated motion in horizontal plane (x,y)

In Fig. 2. the following terms are used:

F_{pi} - the forces perpendicular on the wheel due to the sliding angle of the vehicle;

Ft - the traction force for the front tyres

$$Ft = \frac{M_r}{r}$$

where:

M_r - the pulling force on the wheels;

r - the tire radius.

F_{xi} - the inertial forces on x axis for the i wheel;

F_{yi} - the inertial forces on y axis for the i wheel;

$\dot{V}$ - the tangent acceleration of the vehicle;

$\dot{V}_i$ - the tangent acceleration for each wheel;

$\ddot{\psi}$ - the angular acceleration of the vehicle around the vertical axis crossing the center of mass.

The equation of the momentum around the vertical axis crossing the center of mass is described as:

$$\begin{aligned} J_z\ddot{\psi} &= 2Ftl_f \sin\delta + \frac{E}{2}\left(F_{p2} - F_{p1}\right)\sin\delta + \left(F_{p1} + F_{p2}\right)l_f \cos\delta \\ &-\left(F_{p3} + F_{p4}\right)l_s - \left(F_{x1} - F_{x2}\right)\frac{E}{2} - \left(F_{y1} + F_{y2}\right)l_f - \left(F_{x3} - F_{x4}\right)\frac{E}{2} \\ &-\left(F_{y3} + F_{y4}\right)l_s - \left(R_1 + R_2\right)l_f \sin\delta - \left(R_1 - R_2\right)\frac{E}{2}\cos\delta - \left(R_3 - R_4\right)\frac{E}{2} \end{aligned} \tag{11}$$

and

The equilibrium force equation on the transversal direction is:

$$\begin{aligned} m\dot{V}\sin\beta + \frac{mV^2}{\rho}\cos\beta &= 2Ft\sin\delta + \left(F_{p1} + F_{p2}\right)\cos\delta + \left(F_{p3} + F_{p4}\right) \\ &-\left(F_{y1} + F_{y2}\right) - \left(R_1 + R_2\right)\sin\delta + \left(F_{y3} + F_{y4}\right) \end{aligned} \tag{12}$$

The derivatives of the above mentioned variables are in this case $\ddot{\psi}$ and $\dot{V}$, from which it results that the variables are $\dot{\psi}$ and V. The derivatives $\ddot{\psi}$ and $\dot{\psi}$ represent the acceleration, respectively the angular velocity for the gyration motion around the vertical axis from the center of mass, CG , while $\dot{V}$ and V represent the acceleration and the angular velocity tangent to the trajectory of the vehicle respectively.

The terms containing the variables $\ddot{\psi}$ and $\dot{V}$ are grouped together:

$$\begin{aligned} \ddot{\psi}\left[J_z + 2m_r E^2 + 4m_r\left(l_f^2 + l_s^2\right)\right] &= \dot{V}\left[-4m_r \sin\beta\left(l_f + l_s\right)\right] \\ &+2Ftl_f \sin\delta + \frac{E}{2}\left(F_{p2} - F_{p1}\right)\sin\delta + \left(F_{p1} + F_{p2}\right)l_f \cos\delta \\ &-\left(F_{p3} + F_{p4}\right)l_s - f\left(N_1 + N_2\right)l_f \sin\delta - f\left(N_1 - N_2\right)\frac{E}{2}\cos\delta \\ &-f\left(N_3 - N_4\right)\frac{E}{2} \end{aligned} \tag{13}$$

$$\begin{aligned} \ddot{\psi}\left[4m_r\left(l_f - l_s\right)\right] &= \dot{V}m\sin\beta - \frac{mV^2}{\rho}\cos\beta + 2Ft\sin\delta \\ &+\left(F_{p1} + F_{p2}\right)\cos\delta + \left(F_{p3} + F_{p4}\right) - f\left(N_1 + N_2\right)\sin\delta \end{aligned} \tag{14}$$

The terms following $\ddot{\psi}$ and $\dot{V}$ in the equations (13) and (14) are constant.

The equation system, from which the Liapunov function resides according to the robust stability theory, is:

$$\begin{cases} a_1\dot{x}_1 = b_1\dot{x}_2 + c_1 \\ a_2\dot{x}_1 = b_2\dot{x}_2 + d_2x_2^2 + c_2 \end{cases} \tag{15}$$

which in this case, is expressed as:

$$\begin{cases} a_1\ddot{\psi} = b_1\dot{V} + c_1 \\ a_2\ddot{\psi} = b_2\dot{V} + d_2V^2 + c_2 \end{cases}$$

The chosen function will be:

$$L = a_2x_1 - b_2x_2 - c_2t$$

and, accoding to the relations between velocity and acceleration from the clasical mechanics, the Liapunov function becomes:

$$L = a_2\frac{\dot{V}t}{\rho} - b_2\dot{V}t - c_2t \tag{16}$$

It can be easily noticed that if $t = 0$ (the Liapunov function argument) then $L = 0$, that is $L(0) = 0$, which represents one of the conditions for the system to be stable.

In the relation (16) $\dot{V}$ is not known, this variable is measured using an acceleration sensor.

The second condition for the system to be stable is that the first order derivative of the Liapunov function to be strictly positive:

$$\frac{dL}{dt} = a_2\dot{x}_1 - b_2\dot{x}_2 - c_2 > 0 \tag{17}$$

In the relation (17) the second equation of the system (15) is introduced:

$$\frac{dL}{dt} = b_2\dot{x}_2 + d_2x_2^2 + c_2 - b_2\dot{x}_2 - c_2 > 0 \Rightarrow \frac{dL}{dt} = d_2x_2^2 > 0(A)(\forall)x_2 \in \Re$$

The third condition of the stability of the system is that the Liapunov to become negative.

The stability of the vehicle will be checked under certain relevant conditions, in order to define the behavior of the vehicle in curves. The variable elements are the brakeage angle and the initial velocity in the curve. For this, five brakeage angles and seven velocities will be chosen. The ends of the angles and velocities ranges are considered as the lowest or highest limits, which these elements may have in real conditions.

The values of the Liapunov function, computed using expression (16) are shown in the table 3:

Table.3

The Liapunov function values

V (km/h) \ δ(°)	5	10	15	20	25
50	-351	-333	-302	-262	-221
60	-334	-312	-281	-243	-203
70	-305	-280	-257	-214	-175
80	-269	-241	-213	-173	-101
90	-217	-193	-165	-98	8
100	-158	-137	-81	1	53
110	-85	-66	3	42	95
120	-3	15	31	74	131

The mathematical model gives better good results compared to the ones obtained in [6] where a linear mathematical model was used. The improvement consists of the higher precision of the estimation but also the larger width of the spectrum of the stability analysis.

The precision of the stability estimation, for this vehicle, can be noticed if small calculus steps are used, for the brackage angle and velocity also.

According to the robust stability criterion, it results that the vehicle is stable as it is shown in the fig. bellow:

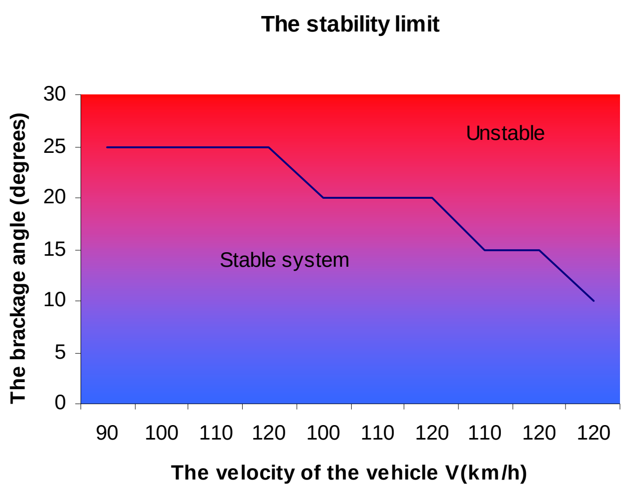

Fig. 3. The stability limit of the vehicle on a curved path

The curve from the fig. 3 represents the stability limit of the studied vehicle; that means if the vehicle has a small velocity and a brakeage angle of the wheel such that the point corresponding to the respective coordinates is under the curve, then the motion on the specified trajectory is stable and if the point is above the curve then the motion is unstable.

3. Conclusions

From the performed analysis, it results that the vehicle is stable until a 110 km/h velocity when the brakeage angle reaches the value of 10^o, at the value of 15° of the brakeage angle the stability is at its limit for the same velocity, and when the wheels have more than 15^o the instability tendency grows more, until the value 25^o of the brakeage angle when the stability is lost at velocities less than 90 km/h.

The original mathematical model with six freedom degrees is without approximations this contributing to the exactness of the results obtained. The stability criterion applied is also an original approach regarding this type of application due to the higher degree of difficulty of the Liapunov function finding.

It is important to notice that this criterion shows the tendency of loosing the stability, this means that it indicates exactly the conditions when the vehicle starts loosing adherence or contact with soil for at least one of the wheels.

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