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GENERALIZED CONVEXITY AND MATHEMATICAL PROGRAMS

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In this paper, we derive the sufficient condition for global optimality for a nonsmooth mathematical program with equilibrium constraints involving generalized convexity. We formulate the Mond-Weir type dual model and establish weak and strong duality theorems to relate the mathematical program with equilibrium constraints and the dual models in the framework of convexificators.

Keywords: Duality, Stationary point, Generalized convexity, Convexificators

1. INTRODUCTION

A mathematical program with equilibrium constraints is usually defined by complementarity system or a parametric variational inequality. There are many equilibrium phenomena that arise from economics and engineering, characterized by either a variational inequality or an optimization problem, which justifies the name mathematical program with equilibrium constraints (MPEC) for the smooth case [30, 9] and for the nonsmooth case [23, 18]. By using the standard Fritz-John conditions Flegel and Kanzow [7] obtained the optimality conditions for MPEC. Moreover, Flegel and Kanzow [8] introduced a new Slater type constraint qualification and a new Abadie type constraint qualification for the MPEC, and proved that the new Slater type constraint qualification implies a new Abadie type constraint qualification.

The concept of convexificators was introduced by Demyanov [5]. Convexificators has been employed to extend the results in optimization and nonsmooth analysis [11, 12]. It has been shown in [12] that the Clarke subdifferentials, Michel-Penot subdifferentials, and Treiman subdifferentials of a locally Lipschitz real-valued function are convexificators.

Invex (invariant convex) function is one of the important generalization of a convex functions. Invex function was introduced by Hanson [10] and later named by Craven [4]. Generalized invex functions [19, 20, 21, 22] play a vital role in optimization and related areas. For the last four decades, optimality and duality conditions in optimization have been discussed by several authors (see [3, 15]). Duality results are very useful and fruitful in the development of numerical algorithms for solving certain classes of the optimization problems. Duality theory is very important subject in the study of mathematical programming problems as weak duality gives a lower bound to the objective function of the primal problem. Mond and Weir [17] dual model is most popular in nonlinear programming problems. These dual models have been abundantly studied for semi- infinite programming problems [14], mathematical programs with vanishing constraints (MPVC) [16] and bi-level problems [29].

The organization of this paper is as follows: in Section 2, we provide some preliminary definitions and results. In Section 3, we derive the sufficient optimality condition for MPEC under generalized convexity assumptions. In Section 4, we establish weak and strong duality theorems relating to the MPEC and dual model using generalized convex functions in the framework of convexificators. In Section 5, we conclude the results of this paper.

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2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with inner product $\langle .,. \rangle$ and X is a nonempty subset of $\mathbb{R}^n$. The convex hull of X is denoted by coX .

We consider the MPEC in the following form:

$$\begin{aligned} \text{MPEC} \quad & \min \quad F(u) \\ & \text{subject to}: \quad g(u) \leqslant 0, \ h(u) = 0, \\ & \qquad H(u) \geqslant 0, \ G(u) \geqslant 0, \ \langle H(u), G(u) \rangle = 0, \end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $H : \mathbb{R}^n \to \mathbb{R}^l$ and $G : \mathbb{R}^n \to \mathbb{R}^l$ are given functions. If we take $h(u) := 0$, $H(u) := 0$, $G(u) := 0$, then, the optimization problem with equilibrium constraint coincides with the standard nonlinear programming problem, which is well studied in the literature, see e.g., Mangasarian [13].

The feasible set of the problem MPEC is denoted by X and defined by

$$X := \{u \in \mathbb{R}^n : g(u) \leqslant 0, \ h(u) = 0, \ H(u) \geqslant 0, \ G(u) \geqslant 0, \ \langle H(u), G(u) \rangle = 0\}.$$

The following index sets will be used throughout the paper:

$$\begin{aligned} I_g &:= I_g(\tilde{u}) := \{i = 1, 2, \ldots, k : g_i(\tilde{u}) = 0\}, \\ \delta &:= \delta(\tilde{u}) := \{i = 1, 2, \ldots, l : H_i(\tilde{u}) = 0, G_i(\tilde{u}) > 0\}, \\ \alpha &:= \alpha(\tilde{u}) := \{i = 1, 2, \ldots, l : H_i(\tilde{u}) = 0, G_i(\tilde{u}) = 0\}, \\ \kappa &:= \kappa(\tilde{u}) := \{i = 1, 2, \ldots, l : H_i(\tilde{u}) > 0, G_i(\tilde{u}) = 0\}, \end{aligned}$$

where $\tilde{u} \in X$ is a feasible vector for the problem MPEC and the set α denotes the degenerate set.

Definition 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $u \in \mathbb{R}^n$, and let $F(u)$ be finite. Then, the lower and upper Dini directional derivatives of F at u in the direction y are defined, respectively, by*

$$F_d^-(u, y) := \liminf_{t \to 0^+} \frac{F(u + ty) - F(u)}{t},$$

and

$$F_d^+(u, y) := \limsup_{t \to 0^+} \frac{F(u + ty) - F(u)}{t}.$$

Definition 2.2. *(see [11]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$,*

$$F_d^-(u, y) \leqslant \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle .$$

Definition 2.3. *(see [11]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have lower convexificators, $\partial_* F(u)$ at $u \in \mathbb{R}^n$ if $\partial_* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$,*

$$F_d^+(u, y) \geqslant \inf_{\xi \in \partial_* F(u)} \langle \xi, y \rangle .$$

The function F is said to have a convexificator $\partial^* F(u) \subseteq \mathbb{R}^n$ at $u \in \mathbb{R}^n$, iff $\partial^* F(u)$ is both upper and lower convexificators of F at u.

Definition 2.4. *(see [6]) A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper semi-regular convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and, for each $y \in \mathbb{R}^n$*

$$F_d^+(u, y) \leqslant \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle . \tag{1}$$

Based on the definitions of generalized invex functions [2], we are introducing the definition of generalized invex functions in terms of convexificators.

Definition 2.5. *Let $F:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be an extended real valued function, which admit convexificator at $\tilde{u}\in\mathbb{R}^n$ and $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a kernel function then, f is said to be*

(i) ∂^*-p-invex at $\tilde{u}$ with respect to η if for every $u\in\mathbb{R}^n$,

$$F(u)\geqslant F(\tilde{u})+\frac{1}{p}\langle\xi,e^{p\eta(u,\tilde{u})}-\mathbf{1}\rangle,\forall\,\xi\in\partial^*F(\tilde{u}),p\neq 0.$$

(ii) ∂^*-p-pseudoinvex at $\tilde{u}$ with respect to η if for every $u\in\mathbb{R}^n$,

$$\exists\,\xi\in\partial^*F(\tilde{u}),\ \frac{1}{p}\langle\xi,e^{p\eta(u,\tilde{u})}-\mathbf{1}\rangle\geqslant 0\Rightarrow F(u)\geqslant F(\tilde{u}),p\neq 0.$$

(iii) ∂^*-p-quasiinvex at $\tilde{u}$ with respect to η if for every $u\in\mathbb{R}^n$,

$$F(u)\leqslant F(\tilde{u})\Rightarrow\frac{1}{p}\langle\xi,e^{p\eta(u,\tilde{u})}-\mathbf{1}\rangle\leqslant 0,\forall\,\xi\in\partial^*F(\tilde{u}),p\neq 0.$$

We provide following examples in support of the definition of ∂^*-p-invex function and generalized ∂^*-p-invex functions respectively.
Example 2.1 Consider the function $F:\mathbb{R}\to\mathbb{R}$ is given by

$$F(x)=\begin{cases}x; & x\geqslant 0,\\ x^2; & x<0,\end{cases}$$

then the function becomes ∂^*-p-invex at $\tilde{x}=0$ with respect to the kernel function, $\eta(x,\tilde{x})=\cos x\sin\tilde{x}$ and $\partial^*F(0)=\{0,1\}$.
Example 2.2 Consider the function $F:\mathbb{R}\to\mathbb{R}$ is given by

$$F(x)=\begin{cases}1+x; & x\geqslant 0,\\ 1+x^2; & x<0,\end{cases}$$

if we take point $\tilde{x}=0$, then the function becomes strongly ∂^*-p-pseudoinvex function at $\tilde{x}=0$ with respect to the kernel function, $\eta(x,\tilde{x})=\sin x\tilde{x}$ and $\partial^*F(0)=\{0,1\}$.

Example 2.3 Consider the function $F:\mathbb{R}\to\mathbb{R}$ is given by

$$F(x)=\begin{cases}1-\frac{x}{3}; & x\geqslant 0,\\ e^x; & x<0,\end{cases}$$

if we take point $\tilde{x}=0$ then the function becomes ∂^*-p-quasiinvex function at $\tilde{x}=0$ with respect to the kernel function, $\eta(x,\tilde{x})=\cos x\sin\tilde{x}$, and $\partial^*F(0)=\{-\frac{1}{3},1\}$.

The following definitions of generalized alternatively stationary point and generalized strong stationary point are taken from Ardali *et.al.* [1].

Definition 2.6. *A feasible point $\tilde{u}$ of MPEC is called a generalized alternatively stationary (GA-stationary) point if there are vectors $\tau=(\tau^g,\tau^h,\tau^H,\tau^G)\in\mathbb{R}^{k+p+2l}$ and $\gamma=(\gamma^h,\gamma^H,\gamma^G)\in\mathbb{R}^{p+2l}$*

satisfying the following conditions

$$0 \in co\partial^* F(\tilde{u}) + \sum_{i \in I_g} \tau_i^g co\partial^* g_i(\tilde{u}) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(\tilde{u}) + \gamma_m^h co\partial^*(-h_m)(\tilde{u}) \right]$$
$$+ \sum_{i=1}^{l} \left[\tau_i^H co\partial^*(-H_i)(\tilde{u}) + \tau_i^G co\partial^*(-G_i)(\tilde{u}) \right]$$
$$+ \sum_{i=1}^{l} \left[\gamma_i^H co\partial^*(H_i)(\tilde{u}) + \gamma_i^G co\partial^*(G_i)(\tilde{u}) \right], \tag{2}$$
$$\tau_{I_g}^g \geqslant 0,\ \tau_m^h,\ \gamma_m^h \geqslant 0,\ m = 1,2,\ldots,p, \tag{3}$$
$$\tau_i^H, \tau_i^G, \gamma_i^H, \gamma_i^G \geqslant 0,\ i = 1,2,\ldots,l, \tag{4}$$
$$\tau_\kappa^H = \tau_\delta^G = \gamma_\kappa^H = \gamma_\delta^G = 0, \tag{5}$$
$$\forall\ i \in \alpha, \gamma_i^H = 0 \ or\ \gamma_i^G = 0. \tag{6}$$

Definition 2.7. *A feasible point $\tilde{u}$ of MPEC is called a generalized strong stationary (GS-stationary) point if there are vectors $\tau = (\tau^g, \tau^h, \tau^H, \tau^G) \in \mathbb{R}^{k+p+2l}$ and $\gamma = (\gamma^h, \gamma^H, \gamma^G) \in \mathbb{R}^{p+2l}$ satisfying* (2)-(5) *together with the following condition*

$$\forall\ i \in \alpha,\ \gamma_i^H = 0,\ \gamma_i^G = 0.$$

In the next section, we show that under certain MPEC generalized invexity assumptions, generalized alternatively (GA) -stationarity turns into a global sufficient optimality condition.

3. Optimality condition

We consider the following index sets:

$$\alpha_\gamma^H := \{i \in \alpha : \gamma_i^G = 0, \gamma_i^H > 0\},$$
$$\alpha_\gamma^G := \{i \in \alpha : \gamma_i^G > 0, \gamma_i^H = 0\},$$
$$\delta_\gamma^+ := \{i \in \delta : \gamma_i^H > 0\},$$
$$\kappa_\gamma^+ := \{i \in \kappa : \gamma_i^G > 0\}.$$

Theorem 3.1. *Let $\tilde{u}$ be a feasible GA-stationary point of MPEC, assume that F is ∂^*-p-pseudoinvex at $\tilde{u}$ with respect to the kernel η and $g_i(i \in I_g), \pm h_m(m = 1,2,\ldots,p), -H_i(i \in \delta \cup \alpha), -G_i(i \in \alpha \cup \kappa)$ are ∂^*-p-quasiinvex at $\tilde{u}$ with respect to the common kernel η and for the same real number $p \neq 0$. If $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then $\tilde{u}$ is a global optimal solution of MPEC.*

Proof. Let u be any arbitrary feasible point of MPEC, i.e.,

$$g_i(u) \leqslant 0 = g_i(\tilde{u}),\ \forall\ i \in I_g.$$

By ∂^*-p-quasiinvexity of g_i at $\tilde{u}$, we get

$$\frac{1}{p}\langle \xi_i^g, e^{p\eta(u,\tilde{u})} - \mathbf{1} \rangle \leqslant 0,\ \forall\ \xi_i^g \in \partial^* g_i(\tilde{u}),\ \forall\ i \in I_g. \tag{7}$$

Similarly, we have

$$\frac{1}{p}\left\langle \zeta_m, e^{p\eta(u,\tilde{u})} - \mathbf{1} \right\rangle \leqslant 0,\ \forall\ \zeta_m \in \partial^* h_m(\tilde{u}), \forall\ m = \{1,2,\ldots,p\}, \tag{8}$$
$$\frac{1}{p}\left\langle \nu_m, e^{p\eta(u,\tilde{u})} - \mathbf{1} \right\rangle \leqslant 0,\ \forall\ \nu_m \in \partial^*(-h_m)(\tilde{u}), \forall\ m = \{1,2,\ldots,p\}, \tag{9}$$
$$\frac{1}{p}\left\langle \xi_i^H, e^{p\eta(u,\tilde{u})} - \mathbf{1} \right\rangle \leqslant 0, \forall\ \xi_i^H \in \partial^*(-H_i)(\tilde{u}), \forall\ i \in \delta \cup \alpha, \tag{10}$$
$$\frac{1}{p}\left\langle \xi_i^G, e^{p\eta(u,\tilde{u})} - \mathbf{1} \right\rangle \leqslant 0, \forall\ \xi_i^G \in \partial^*(-G_i)(\tilde{u}), \forall\ i \in \alpha \cup \kappa. \tag{11}$$

If $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (7)-(11) by $\tau_i^g \geqslant 0\ (i \in I_g)$, $\tau_m^h > 0\ (m = 1,2,\dots,p)$, $\gamma_m^h > 0\ (m = 1,2,\dots,p)$, $\tau_i^H > 0\ (i \in \delta \cup \alpha)$, $\tau_i^G > 0\ (i \in \alpha \cup \kappa)$, respectively and adding, we obtain

$$\frac{1}{p}\left\langle \left(\sum_{i\in I_g} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^{l} \tau_i^H \xi_i^H + \sum_{i=1}^{l} \tau_i^G \xi_i^G \right), e^{p\eta(u,\tilde{u})} - \mathbf{1} \right\rangle \leqslant 0,$$

for all $\xi_i^g \in co\partial^* g_i(\tilde{u})$, $\zeta_m \in co\partial^* h_m(\tilde{u})$, $\nu_m \in co\partial^*(-h_m)(\tilde{u})$, $\xi_i^H \in co\partial^*(-H_i)(\tilde{u})$, $\xi_i^G \in co\partial^*(-G_i)(\tilde{u})$. Thus by GA-stationarity of $\tilde{u}$, we can select $\xi \in co\partial^* F(\tilde{u})$, so that,

$$\frac{1}{p}\langle \xi, e^{p\eta(u,\tilde{u})} - \mathbf{1}\rangle \geqslant 0.$$

By ∂^*-pseudoinvexity of F at $\tilde{u}$ with respect to the common kernel η and for the same real number $p \neq 0$, we get

$$F(u) \geqslant F(\tilde{u})$$

for all feasible points u. Hence $\tilde{u}$ is a global optimal solution of MPEC. □

The following example illustrates Theorem 3.1.

Example 3.1 Consider the following MPEC problem

MPEC

$$min\ F(u) = \begin{cases} u; & u \geqslant 0, \\ u^2; & u < 0, \end{cases}$$

$$\text{subject to}: \begin{aligned}[t] & g(u) = -u^2 \leqslant 0, \\ & H(u) = u^2 \geqslant 0, \\ & G(u) = |u| \geqslant 0, \\ & \langle H(u), G(u)\rangle = \langle u^2, |u|\rangle = 0. \end{aligned}$$

Here F is ∂^*-p-pseudoinvex at $\tilde{u} = 0$ and $p = 1$ with respect to the kernel, $\eta(u.\tilde{u}) = \cos u\tilde{u}$. Further, $g, -H$ and $-G$ are ∂^*-p-quasiinvex at $\tilde{u} = 0$ with respect to the common kernel, $\eta(u,\tilde{u}) = \cos u\tilde{u}$ and $p = 1$. The feasible point for the given MPEC is $\tilde{u} = 0$. We have $co\partial^* F(0) = [0,1]$, $co\partial^* g(0) = \{0\}$, $co\partial^*(-H)(0) = \{0\}$ and $co\partial^*(-G)(0) = [-1,1]$. One can easily verify that there exist $\tau^g = 1, \tau^H = 1$, and $\tau^G = 1$ such that $\tilde{u} = 0$ is a GA- stationary point, and $\tilde{u} = 0$ is a global optimal solution for the given primal problem MPEC. Hence, the assumptions of the Theorem 3.1 are satisfied.

Remark 3.1. *Based on the Definition 2.5, the definitions of generalized invex functions can also be given in terms of upper semi-regular convexificators.*

4. **Duality**

Now, we formulate the Mond-Weir type dual problem (MWD) for the problem MPEC and establish duality theorems using convexificators.

$$\text{MWD} \qquad \max_{v,\tau,\gamma}\{F(v)\}$$

subject to:

$$
\begin{aligned}
0 \in co\partial^* F(v) &+ \sum_{i\in I_g} \tau_i^g co\partial^* g_i(v) + \sum_{m=1}^{p} \Big[\tau_m^h co\partial^* h_m(v) + \gamma_m^h co\partial^*(-h_m)(v)\Big] \\
&+ \sum_{i=1}^{l} \left[\tau_i^H co\partial^*(-H_i)(v) + \tau_i^G co\partial^*(-G_i)(v)\right], \\
& g_i(v) \geqslant 0 \ (i \in I_g), \ \ h_m(v) = 0 \ (m = 1,2,\ldots,p), \\
& H_i(v) \leqslant 0 \ (i \in \delta\cup\alpha), \ \ G_i(v) \leqslant 0 \ (i \in \alpha\cup\kappa), \\
& \tau_{I_g}^g \geqslant 0, \ \tau_m^h, \gamma_m^h \geqslant 0, \ m = 1,2,\ldots,p, \\
& \tau_i^H, \tau_i^G, \gamma_i^H, \gamma_i^G \geqslant 0, \ i = 1,2,\ldots,l, \\
& \tau_\kappa^H = \tau_\delta^G = \gamma_\kappa^H = \gamma_\delta^G = 0, \forall\, i \in \alpha, \gamma_i^H = 0, \gamma_i^G = 0,
\end{aligned}
\tag{12}
$$

where, $\tau = (\tau^g, \tau^h, \tau^H, \tau^G) \in \mathbb{R}^{k+p+2l}$ and $\gamma = (\gamma^h, \gamma^H, \gamma^G) \in \mathbb{R}^{p+2l}$.

Theorem 4.1. *(Weak Duality) Let $\tilde{u}$ be feasible for the problem MPEC, (v,τ) be feasible for the dual MWD and the index sets $I_g, \delta, \alpha, \kappa$ be defined accordingly. Suppose that F, g_i $(i \in I_g), \pm h_m$ $(m = 1,2,\ldots,p), -H_i$ $(i \in \delta\cup\alpha), -G_i$ $(i \in \alpha\cup\kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at v, with respect to the common kernel η and for the same real number $p \neq 0$. If $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any u feasible for the problem MPEC, we have*

$$F(u) \geqslant F(v).$$

Proof. Since f is ∂^*-p-invex at v, with respect to the kernel η, then, we have

$$F(u) - F(v) \geqslant \frac{1}{p}\big\langle \xi, e^{p\eta(u,v)} - \mathbf{1}\big\rangle, \forall\, \xi \in \partial^* F(v). \tag{13}$$

Similarly, we have

$$g_i(u) - g_i(v) \geqslant \frac{1}{p}\langle \xi_i^g, e^{p\eta(u,v)} - \mathbf{1}\rangle, \qquad \forall\, \xi_i^g \in \partial^* g_i(v), \forall\, i \in I_g, \tag{14}$$

$$h_m(u) - h_m(v) \geqslant \frac{1}{p}\langle \zeta_m, e^{p\eta(u,v)} - \mathbf{1}\rangle, \quad \forall\, \zeta_m \in \partial^* h_m(v), \forall\, m = \{1,2,\ldots,p\}, \tag{15}$$

$$-h_m(u) + h_m(v) \geqslant \frac{1}{p}\langle \nu_m, e^{p\eta(u,v)} - \mathbf{1}\rangle, \ \forall\, \nu_m \in \partial^*(-h_m)(v), \forall\, m = \{1,2,\ldots,p\}, \tag{16}$$

$$-H_i(u) + H_i(v) \geqslant \frac{1}{p}\langle \xi_i^H, e^{p\eta(u,v)} - \mathbf{1}\rangle, \quad \forall\, \xi_i^H \in \partial^*(-H_i)(v), \forall\, i \in \delta\cup\alpha, \tag{17}$$

$$-G_i(u) + G_i(v) \geqslant \frac{1}{p}\langle \xi_i^G, e^{p\eta(u,v)} - \mathbf{1}\rangle, \quad \forall\, \xi_i^G \in \partial^*(-G_i)(v), \forall\, i \in \alpha\cup\kappa. \tag{18}$$

If $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (14)-(18) by $\tau_i^g \geqslant 0$ $(i \in I_g)$, $\tau_m^h > 0$ $(m = 1,2,\ldots,p)$, $\gamma_m^h > 0$ $(m = 1,2,\ldots,p)$, $\tau_i^H > 0$ $(i \in \delta\cup\alpha)$, $\tau_i^G > 0$ $(i \in \alpha\cup\kappa)$, respectively and adding (13)-(18), we obtain

$$
\begin{aligned}
F(u) - F(v) &+ \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^H H_i(u) + \sum_{i=1}^{l} \tau_i^H H_i(v) - \sum_{i=1}^{l} \tau_i^G G_i(u) + \sum_{i=1}^{l} \tau_i^G G_i(v) \\
&\geqslant \frac{1}{p}\left\langle \xi + \sum_{i\in I_g} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \Big[\tau_m^h \zeta_m + \gamma_m^h \nu_m\Big] + \sum_{i=1}^{l} \left[\tau_i^H \xi_i^H + \tau_i^G \xi_i^G\right], e^{p\eta(u,v)} - \mathbf{1} \right\rangle.
\end{aligned}
$$

From (12), $\exists\, \tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g_i(v), \tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^H \in co\partial^*(-H_i)(v)$ and $\tilde{\xi}_i^G \in co\partial^*(-G_i)(v)$, such that

$$\tilde{\xi} + \sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tau_i^H \tilde{\xi}_i^H + \tau_i^G \tilde{\xi}_i^G\right] = 0.$$

Therefore,

$$F(u) - F(v) + \sum_{i\in I_g} \tau_i^g g_i(u) - \sum_{i\in I_g} \tau_i^g g_i(v) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u)$$
$$+ \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^H H_i(u) + \sum_{i=1}^{l} \tau_i^H H_i(v) - \sum_{i=1}^{l} \tau_i^G G_i(u) + \sum_{i=1}^{l} \tau_i^G G_i(v) \geqslant 0.$$

Now using the feasibility of u and v for MPEC and MWD, it follows that $F(u) \geqslant F(v)$. Hence, the proof is completed. □

Theorem 4.2. *(Strong Duality) Let $\tilde{u}$ be a local optimal solution of the problem MPEC and let F be locally Lipschitz near $\tilde{u}$. Suppose that F, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1,2,\ldots,p)$, $-H_i$ $(i \in \delta \cup \alpha)$, $-G_i$ $(i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at $\tilde{u}$ with respect to the common kernel η and for the same real number $p \neq 0$. If GS-ACQ* [1] *holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u},\tilde{\tau})$ is an optimal solution of the dual MWD and the corresponding objective values of MPEC and MWD are equal.*

Proof. Since, $\tilde{u}$ is a local optimal solution of MPEC and the GS-ACQ is satisfied at $\tilde{u}$, now using Corollary 4.6 [1], $\exists\ \tilde{\tau} = (\tilde{\tau}^g, \tilde{\tau}^h, \tilde{\tau}^H, \tilde{\tau}^G) \in \mathbb{R}^{k+p+2l}, \tilde{\gamma} \in (\tilde{\gamma}^h, \tilde{\gamma}^H, \tilde{\gamma}^G) \in \mathbb{R}^{p+2l}$, such that the GS-stationarity conditions for MPEC are satisfied, that is, $\exists\ \tilde{\xi} \in co\partial^* F(\tilde{u})$, $\tilde{\xi}_i^g \in co\partial^* g(\tilde{u})$, $\tilde{\zeta}_r \in co\partial^* h_r(\tilde{u})$, $\tilde{\nu}_r \in co\partial^*(-h_r)(\tilde{u})$, $\tilde{\xi}_i^H \in co\partial^*(-H_i)(\tilde{u})$ and $\tilde{\xi}_i^G \in co\partial^*(-G_i)(\tilde{u})$, such that

$$\tilde{\xi} + \sum_{i=1}^{m} \tilde{\tau}_i^g \tilde{\xi}_i^g + \sum_{r=1}^{p} \left[\tilde{\tau}_r^h \tilde{\zeta}_r + \tilde{\gamma}_r^h \tilde{\nu}_r\right] + \sum_{i=1}^{l} \left[\tilde{\tau}_i^H \tilde{\xi}_i^H + \tilde{\tau}_i^G \tilde{\xi}_i^G\right] = 0,$$
$$\tilde{\tau}_i^g \geqslant 0, (i = 1,2,\ldots,m),\ \tilde{\tau}_r^h,\ \tilde{\gamma}_r^h \geqslant 0,\ \ (r = 1,2,\ldots,p),$$
$$\tilde{\tau}_i^H, \tilde{\tau}_i^G, \tilde{\gamma}_i^H, \tilde{\gamma}_i^G \geqslant 0,\ (i = 1,2,\ldots,l),$$
$$\tilde{\tau}_\kappa^H = \tilde{\tau}_\delta^G = \tilde{\gamma}_\kappa^H = \tilde{\gamma}_\delta^G = 0, \forall\, i \in \alpha,\ \tilde{\gamma}_i^H = 0,\ \tilde{\gamma}_i^G = 0.$$

Since $\tilde{u}$ is an optimal solution for SIMPEC, we have

$$\sum_{i=1}^{m} \tilde{\tau}_i^g g(\tilde{u}) = 0, \sum_{i=1}^{p} \tilde{\tau}_i^h h_i(\tilde{u}) = 0, \sum_{i=1}^{l} \tilde{\tau}_i^H H_i(\tilde{u}) = 0, \sum_{i=1}^{l} \tilde{\tau}_i^G G_i(\tilde{u}) = 0.$$

Therefore $(\tilde{u},\tilde{\tau})$ is feasible for MWD. By Theorem 4.3, for any feasible (v,τ), we have $F(\tilde{u}) \geqslant F(v)$.

It follows that $(\tilde{u},\tilde{\tau})$ is an optimal solution for MWD and the respective objective values are equal. This completes the proof. □

Next, we establish weak and strong duality theorems for MPEC and its Mond-Weir type dual problem (MWD) under generalized ∂^*-p-invexity assumptions.

Theorem 4.3. *(Weak Duality) Let $\tilde{u}$ be feasible for the problem MPEC, (v,τ) be feasible for the dual MWD and the index sets $I_g, \delta, \alpha, \kappa$ are defined accordingly. Suppose that F is ∂^*-p-pseudoinvex at v, with respect to the kernel η and g_i $(i \in I_g)$, $\pm h_m$ $(m = 1,2,\ldots,p)$, $-H_i$ $(i \in \delta \cup \alpha)$, $-G_i$ $(i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-quasiinvex functions at v, with respect to the common kernel η and for the same real number $p \neq 0$. If $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any u feasible for the problem MPEC, we have*

$$F(u) \geqslant F(v).$$

Proof. Assume that, for some feasible point u, such that $F(u) < F(v)$, then by ∂^*-p-pseudoinvexity of F at v, with respect to the kernel η, we get

$$\frac{1}{p}\langle \xi, e^{p\eta(u,v)} - \mathbf{1}\rangle < 0, \forall\ \xi \in \partial^* F(v). \tag{19}$$

From (12), $\exists\, \tilde{\xi} \in co\partial^* F(v),\ \tilde{\xi}_i^g \in co\partial^* g_i(v),\ \tilde{\zeta}_m \in co\partial^* h_m(v),\ \tilde{\nu}_m \in co\partial^*(-h_m)(v),\ \tilde{\xi}_i^H \in co\partial^*(-H_i)(v)$ and $\tilde{\xi}_i^G \in co\partial^*(-G_i)(v)$, such that

$$-\sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g - \sum_{m=1}^{p}\left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] - \sum_{\delta\cup\alpha} \tau_i^H \tilde{\xi}_i^H - \sum_{\alpha\cup\kappa} \tau_i^G \tilde{\xi}_i^G \in \partial^* F(v). \tag{20}$$

Using (12) and (19), we get

$$\frac{1}{p}\left\langle \left(\sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p}\left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{\delta\cup\alpha} \tau_i^H \tilde{\xi}_i^H + \sum_{\alpha\cup\kappa} \tau_i^G \tilde{\xi}_i^G\right), e^{p\eta(u,v)} - \mathbf{1}\right\rangle > 0. \tag{21}$$

For each $i \in I_g,\ g_i(u) \leqslant 0 \leqslant g_i(v)$. Hence, by ∂^*-p-quasiinvexity, we obtain

$$\frac{1}{p}\left\langle \xi_i^g, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0, \forall\ \xi_i^g \in \partial^* g_i(v),\ \ \forall\ i \in I_g. \tag{22}$$

Similarly, we have

$$\frac{1}{p}\left\langle \zeta_m, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\ \zeta_m \in \partial^* h_m(v), \forall\ m = \{1,2,\ldots,p\}, \tag{23}$$

for any feasible point v of the dual MWD, and for every $m, -h_m(v) = -h_m(u) = 0$. On the other hand, $-H_i(u) \leqslant -H_i(v), \forall i \in \delta\cup\alpha$, and $-G_i(u) \leqslant -G_i(v), \forall i \in \alpha\cup\kappa$. By ∂^*-p-quasiinvexity, we obtain

$$\frac{1}{p}\left\langle \nu_m, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\ \nu_m \in \partial^*(-h_m)(v), \forall\ m = \{1,2,\ldots,p\}, \tag{24}$$

$$\frac{1}{p}\left\langle \xi_i^H, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0, \forall\ \xi_i^H \in \partial^*(-H_i)(v), \forall\ i \in \delta\cup\alpha, \tag{25}$$

$$\frac{1}{p}\left\langle \xi_i^G, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0, \forall\ \xi_i^G \in \partial^*(-G_i)(v), \forall\ i \in \alpha\cup\kappa. \tag{26}$$

From Eqs, (22)-(26), we have

$$\frac{1}{p}\left\langle \tilde{\xi}_i^g, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0\ (i \in I_g),\ \frac{1}{p}\left\langle \tilde{\zeta}_m, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,\ \frac{1}{p}\left\langle \tilde{\nu}_m, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,$$

$$\frac{1}{p}\left\langle \tilde{\xi}_i^H, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0, \forall\ i \in \delta\cup\alpha,\ \ \frac{1}{p}\left\langle \tilde{\xi}_i^G, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0, \forall\ i \in \alpha\cup\kappa.$$

Since $\alpha_\gamma^H \cup \alpha_\gamma^G \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, we have

$$\frac{1}{p}\left\langle \sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{m=1}^{p}\left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\gamma}_m\right], e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,$$

$$\frac{1}{p}\left\langle \sum_{\delta\cup\alpha} \tau_i^H \tilde{\xi}_i^H, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{\alpha\cup\kappa} \tau_i^G \tilde{\xi}_i^G, e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0.$$

Therefore,

$$\frac{1}{p}\left\langle \left(\sum_{i\in I_g} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p}\left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{\delta\cup\alpha} \tau_i^H \tilde{\xi}_i^H + \sum_{\alpha\cup\kappa} \tau_i^G \tilde{\xi}_i^G\right), e^{p\eta(u,v)} - \mathbf{1}\right\rangle \leqslant 0.$$

which contradicts (30). Therefore $F(u) \geqslant F(v)$. Hence the proof is completed. □

Theorem 4.4. *(Strong Duality) Let $\tilde{u}$ be a local optimal solution of the problem MPEC and let F be locally Lipschitz near $\tilde{u}$. Suppose that F is ∂^*-pseudoinvex at $\tilde{u}$, with respect to the kernel η, g_i $(i \in I_g)$, $\pm h_m$ $(m = 1,2,\ldots,p)$, $-H_i$ $(i \in \delta \cup \alpha)$, $-G_i$ $(i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-quasiinvex functions at $\tilde{u}$ with respect to the common kernel η and for the same real number $p \neq 0$. If GS-ACQ* [1] *holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual MWD and the respective objective values are equal.*

Proof. The proof can be done similar to the proof of Theorem 4.2 by invoking Theorem 4.3. □

5. Conclusions

We have studied a mathematical program with equilibrium constraints (MPEC) and derived the sufficient conditions for global optimality for MPEC using generalized convexity assumptions. We have formulated the Mond-Weir type dual model for the problem MPEC in the framework of convexificators. Further we established weak and strong duality theorems relating to the problem MPEC and dual model using generalized convexity assumptions.

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