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STUDIES ON NBVPS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ONE-DIMENSIONAL $p-$LAPLACIAN

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Sufficient conditions for the existence of at least one solution to Neumann boundary value problems for second order nonlinear functional differential equations are established by using Mawhin fixed point theorem and Leray-Schauder's fixed point theorem, respectively. Some examples show that our results cannot be trivially deduced from the previous works, see the remark at the end of Section 2.

Keywords: Solutions; second order differential equation; Neumann boundary value problems; fixed-point theorem; growth condition

1. **Introduction**

In recent years, Neumann boundary value problems (NBVPs for short) have been investigated in a large number of papers. Atslaga in [1] studied the following NBVP

$$\begin{cases} x'' = f(x), t \in (0,1), \\ x'(0) = x'(1) = 0. \end{cases} \tag{1}$$

Under the assumptions that f is continuous, f has simple zeros at $p_1 < p_2 < p_3 < p_4 < p_5$, $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$, the multiplicity results for solutions of NBVP(1) were proved. The NBVP of the form

$$\begin{cases} -x''(t) + mx(t) = g(t)f(t, x(t)), \ \ 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

was studied in papers [3,4,8,12,14-16], where $m \in R$, $g : (0, +\infty) \to [0, +\infty)$ and $f : (0, 1) \times (0, +\infty) \to [0, +\infty)$ are continuous and may be singular at $t = 0$ or $t = 1$ and $x = 0$. The techniques involved are based on the fixed point theorems in cones in Banach spaces such as the nonlinear alternative of Leray-Schauder [8], the Krasnoselskii fixed point theorem [4,12,14,15], the Leggett-Williams fixed-point theorem [16].

In papers [5,6,7], the existence of solutions of the following NBVPs

$$\begin{cases} -(\phi(x'(t)))' = f(t, x(t), x'(t)), \ \ t \in [a, b], \\ u'(a) = u'(b) = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} -(\phi(x'(t)))' = f(t, x(t)), \ \ t \in [a, b], \\ u'(a) = u'(b) = 0, \end{cases} \tag{4}$$

were studied, where $0 < a < b$ are constants, f is a continuous or Caratheodory function and $\tau \in C[0, T]$. The upper and lower solutions method coupled with the monotone iterative technique was used in these papers.

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In paper [9], Girg studied the following problem

$$(5)\qquad \begin{cases} (\phi(u'(t)))' + g(u'(t)) + h(u(t)) = f(t), & 0 < t < T, \\ u'(0) = u'(T) = 0, \end{cases}$$

Let $f(t) = \widetilde{f} + \overline{f}$ with $\overline{f} = \frac{1}{T}\int_0^T f(t)dt$. Denote

$$\widetilde{C}[0,T] = \left\{ u \in C[0,T] : \int_0^T u(t)dt = 0 \right\}, \quad \widetilde{C_T} = C_T \cap \widetilde{C}[0,T].$$

Under the following assumptions:

(i) ϕ is an increasing homeomorphism of I_1 onto I_2, where $I_1, I_2 \subset R$ are open intervals containing zero and $\phi(0) = 0$;

(ii) g is continuous;

(iii) h is continuous, bounded real function having limits in $\pm\infty$ with

$$h(-\infty) := \lim_{\xi \to -\infty} h(\xi) < \lim_{\xi \to +\infty} h(\xi) =: h(+\infty);$$

(iv) ϕ is odd and there exist $c, \delta > 0$ and $p > 1$ such that for all $z \in (-\delta, \delta) \cap \mathrm{Dom}\phi$: $c|z|^{p-1} \le |\phi(z)|$.

It was proved that NBVP(5) has at least one solution if $s(\widetilde{f}) + h(-\infty) < \overline{f} < s(\widetilde{f}) + h(+\infty)$. and

$$\sqrt{\frac{3}{T}}b - \sqrt{T}\sup_{\xi \in R}|h(\xi)| > 0, \ ||\widetilde{f}||_{L^2} < \sqrt{\frac{3}{T}}b - \sqrt{T}\sup_{\xi \in R}|h(\xi)|.$$

Afrouzi and Moghaddam in [2] studied the following Neumann-Robin boundary value problem

$$(6)\qquad \begin{cases} -(\phi_p(x'(t)))' = \lambda f(x(t)), \ t \in [0,1], \\ x'(0) = 0, \ \ x'(1) + \alpha x(1) = 0, \end{cases}$$

where $\alpha \in R$, $\lambda > 0$ are parameters and $p > 1$, and $p' = \frac{p}{p-1}$ is the conjugate exponent of p and $p(x) := |x|^{p-2}x$ for all $x \in R$, where $(\phi_p(u'))'$ is the one dimensional p-Laplacian and $f \in C^2[0, +\infty)$ such that $f(0) < 0$, or $f(0) > 0$, and also f is increasing and concave up. The existence and multiplicity of nonnegative solutions of BVP(6) were studied.

In papers [13], the NBVPs of the form

$$(7)\qquad \begin{cases} [\phi(x'(t))]' = -f(t, x(t), x(\tau(t))), & t \in [0,T], \\ u'(0) = u'(T) = 0, \end{cases}$$

and

$$\begin{cases} x''(t) = g(t, x(t), x(\tau(t)), x'(t)), & t \in [0,T], \\ u'(0) = u'(T) = 0, \end{cases} \qquad (7)'$$

was studied, where $f : [0,T] \times R^2 \to R$ and $g : [0,T] \times R^3 \to R$ are continuous and $\tau \in C([0,T],[0,T])$, $\phi : R \to R$ is an increasing homeomorphism such that $\phi(0) = 0$. The methods used in [13] are based upon the upper and lower solutions methods and the monotone iterative technique. It was showed that the monotone technique produces two monotone sequences that converge uniformly to extremal solutions of NBVP(7) and NBVP(7)$'$, respectively.

Motivated by the paper [13], we study the following NBVP for the functional differential equation with one-dimensional $p-$Laplacian

$$(8)\qquad \begin{cases} (\phi(u'(t)))' = f(t, u(t), u(\tau(t)), u'(t)), \ t \in (0,T), \\ u'(0) = 0, \\ u'(T) = 0, \end{cases}$$

where $T > 0$, $\tau \in C([0,T],[0,T])$, $\phi : R \to R$ is an increasing homeomorphism such that $\phi(0) = 0$ whose inverse function s denoted by ϕ^{-1}, f is a L^1-Carathéodory function on $[0,T]$, i.e. $f : (0,T) \times R^3 \to R$ satisfies the following conditions:

(i) $f(\cdot,x,y,z)$ is measurable for all $(x,y,z) \in R^3$.

(ii) $f(t,\cdot,\cdot,\cdot)$ is continuous for almost all $t \in (0,T)$.

(iii) For each $K > 0$ there exists $h_K \in L^1(0,T)$ such that $|x| + |y| + |z| \leq K$ implies $|f(t,x,y,z)| \leq h_K(t)$ for almost all $t \in (0,T)$.

A function $x : [0,T] \to R$ is called a solution of NBVP(8) if $x \in C^1[0,T]$, $[\phi(x')]' \in L^1(0,T)$ and x satisfies (8).

The purpose of this paper is to establish new sufficient conditions for the existence of at least one solutions of NBVP (8) by using Mawhin's fixed point theorem and Schauder's fixed point theorem [11]. It is interesting that we allow f to be sublinear, at most linear or superlinear. The methods used in this paper are different from those used in papers [1-10,11-16] and so are the assumptions and techniques new.

This paper is organized as follows. In Section 2, main results are given, and two examples are presented to illustrate them, whereas the known results in the current literature do not cover them, in Section 3, we prove the main results, i.e., Theorems 2.1-2.3.

2. Main Results and Examples

Let us list some conditions.

(A) there exists a positive constant M_0 such that

(i) $\left[\int_0^T f(t,M_0,y,0)dt\right]\left[\int_0^T f(t,-M_0,y,0)dt\right] < 0$ for all $y \in R$;

(ii) there exist $q \in L^1([0,T])$ and $\Phi : [0,+\infty) \to [0,+\infty)$ nondecreasing with $1/\Phi(x)$ integrable over bounded intervals such that $|f(t,x,y,z))| \leq q(t)\Phi(|z|)$ for all $(t,x,y) \in (0,T) \times [-M_0,M_0]^2$, $z \in R$ and

$$\int_{TM_0}^{+\infty} \frac{d\sigma}{\Phi(\phi^{-1}(\sigma))} > \int_0^T q(s)ds.$$

(B) there exists a constant $M > 0$ such that

$$xf(t,x,y,0) > 0, \ \ t \in (0,T), \ |x| > M \text{ and } |y| > M.$$

(C) there exist the Carathéodory functions $h : (0,T) \times R^3 \to R$, $g_i : (0,T) \times R \to R$, and function $r \in L^1(0,T)$ such that

(i) $f(t,x,y,z) = h(t,x,y,z)+g_1(t,x)+g_2(t,y)+g_3(t,z)+r(t)$ holds for all $(t,x,y,z) \in (0,T) \times R^3$;

(ii) there exist constants $\theta \geq 0$ and $\beta > 0$ such that

$$h(t,x,y,z)z \geq \beta|z|^{\theta+1}$$

holds for all $(t,x,y,z) \in (0,T) \times R^z$;

(iii) there exist the limits

$$\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|^\theta} = r_i \in [0,+\infty), i = 1,2,3.$$

(C′) there exist the Carathéodory functions $h : (0,T) \times R^3 \to R$, $g_i : (0,T) \times R \to R$, and function $r \in L^1(0,T)$ such that (C)(i) and (C)(iii) hold and

(ii) there exist constants $\theta \geq 0$ and $\beta > 0$ such that

$$h(t,x,y,z)z \leq -\beta|z|^{\theta+1}$$

holds for all $(t,x,y,z) \in (0,T) \times R^3$.

Theorem 2.1. Suppose that (B) and (C) hold. Then BVP(8) has at least one solution if $T^\theta(r_1+r_2)+r_3<\beta$.

Theorem 2.2. Suppose that (B) and (C') hold. Then BVP(8) has at least one solution if $T^\theta(r_1+r_2)+r_3<\beta$.

Theorem 2.3. Assume that conditions (A) holds. Then BVP(8) has at least one solution.

Now, we present examples that our results can readily apply, whereas the known results in the current literature do not cover them.

Example 2.1. Consider the NBVP

$$\begin{cases} x''(t) = \frac{[x'(t)]^{\frac{3}{5}}}{1+2[\sin x(t)]^8} + p(t)[x(t)]^{\frac{3}{5}} + q(t)[x'(t)]^{\frac{3}{5}} + r(t), \\ x'(0) = x'(T) = 0, \end{cases} \tag{9}$$

where $p,q,r \in C^0(0,T)$ with $p(t)>0$. Corresponding to NBVP(8), one sees that

$$f(t,x,y,z) = \frac{z^{\frac{3}{5}}}{1+2[\sin x]^8} + p(t)x^{\frac{3}{5}} + q(t)z^{\frac{3}{5}} + r(t),$$

we set

$$h(t,x,y,z) = \frac{z^{\frac{3}{5}}}{1+2[\sin x]^8},$$

and

$$g_1(t,x) = p(t)x^{\frac{3}{5}}, \;\; g_2(t,y)=0, \;\; g_3(t,y) = q(t)y^{\frac{3}{5}}$$

and $\beta = 1/3$, $\theta = 3/5$.

It is easy to check that $r_1 = ||p||$ and $r_2 = 0, r_3 = ||q||$, where $||p|| = \max_{t\in[0,T]}|p(t)|$ and $||q|| = \max_{t\in[0,T]}|q(t)|$. It follows that

$$f(t,x,y,z) = h(t,x,y,z) + g_1(t,x) + g_2(t,y) + g_3(t,z) + r(t),$$

$$zh(t,x,,y,z) = \frac{z^{\frac{8}{5}}}{1+2[\sin x]^8} \geq \frac{1}{3}|z|^{\frac{8}{5}} = \beta|z|^{\theta+1},$$

and

$$\lim_{x\to+\infty}\sup_{t\in[0,T]}\frac{g_i(t,x)}{|x|^\theta} = r_i, i=1,2,3.$$

On the other and, we have

$$xf(t,x,y,0) = x\left(p(t)x^{\frac{3}{5}} + r(t)\right).$$

Since $p \in C^0[0,T]$ with $p(t)>0$, there exists $k>0$ such that $p(t)>k$ for all $t\in[0,T]$. Then

$$xf(t,x,y,0) = x\left(p(t)x^{\frac{3}{5}} + r(t)\right) > kx^2 + r(t)x.$$

It is easy to see that there exists $M>0$ such that $xf(t,x,y,0)>0$ for all $t\in[0,T]$ and $(x,y)\in R^2$. Hence (B) and (C) hold.

It follows from Theorem 2.1 that NBVP(9) has at least one solution if

$$T^{\frac{3}{5}}||p|| + ||q|| < \frac{1}{3}.$$

Example 2.2. Consider the NBVP

$$(10)\qquad \begin{cases} [\phi(x'(t))]' = -\frac{[x(t)]^5}{1+2[\sin x(t)]^8} + p(t)[x(t)]^5 + q(t)[x'(t)]^5 + r(t), \\ x'(0) = x'(1) = 0, \end{cases}$$

where $\phi(x) = |x|^4 x$, $p, q, r \in C^0(0, 1)$. Corresponding to the assumptions of Theorem 2.2, we set

$$h(t, x, y, z) = -\frac{z^5}{1 + 2[\sin x]^8},$$

and

$$g_1(t, x) = p(t)x^5, \;\; g_2(t, y) = 0, \;\; g_3(t, z) == q(t)z^5$$

and $\beta = 1/3$, $\theta = 5$, $T = 1$.

It is easy to check that $r_1 = ||p||$, $r_2 = 0$ and $r_3 = ||q||$. Similarly to Example 2.1, we can show that (B) and (C$'$) hold. It follows from Theorem 2.2 that NBVP(10) has at least one solution if

$$\frac{1}{3} > ||p|| + ||q||.$$

Example 2.3. Consider the NBVP

$$(11)\qquad \begin{cases} [\phi(x'(t))]' = (r(t) + x^3(t))[x'(t)]^5 + x^3(t) + r(t), \; t \in (0, T), \\ x'(0) = x'(1) = 0, \end{cases}$$

where $\phi(x) = |x|^4 x$, $r \in C^0(0, 1)$

. Corresponding to the assumptions of Theorem 2.3, we set

$$\int_0^T f(t, M_0, M_0, 0)dt \int_0^T f(t, -M_0, M_0, 0)dt = \int_0^T (M_0^3 + r(t))dt \int_0^T (-M_0^3 + r(t))dt$$

and

$$|(r(t) + x^3(t))[x'(t)]^5 + x^3(t) + r(t)| \leq (M_0^3 + r(t))(|x'(t)|^5 + 1)$$

if $|x(t)| \leq M_0$ for all $t \in [0, T]$.

Choose $\Phi(x) = x^5 + 1$, and $q(t) = M_0^3 + r(t)$. Then $|f(t, x, y, z)| \leq q(t)\Phi(\phi^{-1}(z))$.

It follows from Theorem 2.3 that NBVP(11) has at least one solution if

$$\int_0^T [M_0^3 + r(t)]dt < \int_{TM_0}^{+\infty} \frac{1}{x^5 + 1} dx.$$

One sees that imply that there is a large number of functions that satisfy the conditions of Theorem 2.3. In addition, the conditions

$$\int_0^T (M_0^3 + r(t))dt \int_0^T (-M_0^3 + r(t))dt < 0$$

and

$$\int_0^T [M_0^3 + r(t)]dt < \int_{TM_0}^{+\infty} \frac{1}{x^5 + 1} dx$$

are also easy to check.

Remark. Examples 2.1-2.3 can not be coved by the theorems obtained in [8,4,14-16,12] since here f may changes sign. Comparing to the results obtained in [5,7,13,6], we do not need the existence of upper and lower solutions when establish the existence results for solutions. Our results (Theorem 2.1-2.3) are different from those ones in [1,9,10] since we do not need the assumption $|f(t, y, p)| \leq A(t, y)|p^2 + B(t, y)$.

3. **Proofs of Theorems**

To get the existence results for solutions of NBVP(8), we need two fixed point theorems, one is Mawhin's fixed point theorem and the other Schauder's fixed point theorem.

Let X and Y be Banach spaces, $L:\ D(L)\subset X\to Y$ be a Fredholm operator of index zero, $P:\ X\to X$, $Q:\ Y\to Y$ be projectors such that $\text{Im } P=\text{Ker } L$, $\text{Ker } Q=\text{Im } L$, $X=\text{Ker } L\oplus \text{Ker } P$, $Y=\text{Im } L\oplus\text{Im } Q$. It follows that

$$L|_{D(L)\cap \text{Ker } P}:\ \text{Dom } L\cap \text{Ker } P\to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega}\neq\emptyset$, the map $N:\ X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact.

Theorem GM[11]. Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:

(i). $Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(D(L)\setminus\text{Ker}L)\cap\partial\Omega]\times(0,1)$;

(ii). $Nx\notin\text{Im}L$ for every $x\in\text{Ker}L\cap\partial\Omega$;

(iii). $\deg(\wedge QN\,|_{\text{Ker}L}\,,\ \Omega\cap\text{Ker}L,0)\neq 0$, where $\wedge:\ Y/\text{Im}L\to\text{Ker}L$ is the isomorphism.

Then the operator equation $Lx=Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

Theorem LS[11]. Suppose $T:X\to X$ is completely continuous operator. If there exists a open bounded subset Ω such that $0\in\Omega\subset X$ and $x\neq\lambda Tx$ for all $x\in D(T)\cap\partial\Omega$ and $\lambda\in[0,1]$, then there is at least one $x\in\Omega$ such that $x=Tx$.

We use the Sobolev space $W^{1,1}(0,T)$ defined by

$$W^{1,1}(0,T)=\{x:[0,T]\to R|x \text{ is absolutely continuous on } [0,T] \text{ with } x'\in L^1[0,T]\}.$$

Let the Banach space be $X=C^0[0,T]\times C^0[0,T]$ with the norm

$$||(x,y)||=\max\left\{\sup_{t\in[0,T]}|x(t)|,\ \sup_{t\in[0,T]}|y(t)|\right\}$$

and $D(L)=W^{1,1}(0,T)$. Let $Y=L^1[0,T]\times L^1[0,T]\times R^2$. Define the linear operator $L:D(L)\cap X\to Y$ by

$$L\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}x'(t)\\ y'(t)\\ y(0)\\ y(T)\end{pmatrix}\text{ for all }(x,y)\in D(L)\cap X.$$

Define the nonlinear operator $N:X\to Y$, for all $(x,y)\in X$, by

$$N\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}\phi^{-1}(y(t))\\ f(t,x(t),x(\tau(t)),\phi^{-1}(y(t)))\\ 0\\ 0\end{pmatrix}.$$

It is easy to show the following results. We omit their proofs since the proofs are simple and standard.

(i). $\text{Ker}L=\{(a,0):a\in R\}$;

(ii). $\text{Im}L=\{(u,v,a,b)\in Y:\int_0^T v(t)dt=b-a\}$;

(iii). L is a Fredholm operator of index zero;

(iv). There exist projectors $P:\ X\to X$ and $Q:\ Y\to Y$ such that $\text{Ker}L=\text{Im}P$ and $\text{Ker}Q=\text{Im}L$. There is an isomorphism $\wedge:\text{Ker}L\to Y/\text{Im}L$.

(v). Let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap D(L)\neq\emptyset$, then N is $L-$compact on $\overline{\Omega}$;

(vi). $(x,y)\in D(L)$ is a solution of the operator equation $L(x,y)=N(x,y)$ implies that x is a solution of NBVP(8).

Let $F_x(t)=f(t,x(t),x(\tau(t)),\phi^{-1}(y(t)))$. In fact, we have, for $a\in R$, $(x,y)\in X$ and $(u,v)\in Y$, that

$$\begin{aligned}
P\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} &= \begin{pmatrix} x(0)\\ 0\end{pmatrix},\ K_p\begin{pmatrix} u(t)\\ v(t)\\ a\\ b\end{pmatrix}=\begin{pmatrix}\int_0^t u(s)ds\\ \int_0^t v(s)ds\end{pmatrix},\\
Q\begin{pmatrix} u(t)\\ v(t)\\ a\\ b\end{pmatrix} &= \begin{pmatrix} 0\\ \frac{1}{T}\left(\int_0^T v(t)dt-(b-a)\right)\\ 0\\ 0\end{pmatrix},\\
K_p(I-Q)N\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} &= K_p(I-Q)\begin{pmatrix}\phi^{-1}(y(t))\\ F_x(t)\end{pmatrix}\\
&= \begin{pmatrix}\int_0^t\phi^{-1}(y(s))ds\\ \int_0^t F_x(s)ds-\frac{t}{T}\int_0^T F_x(s)ds\end{pmatrix},\\
\wedge\begin{pmatrix} a\\ 0\end{pmatrix} &= \begin{pmatrix} 0\\ a\\ 0\\ 0\end{pmatrix}.
\end{aligned}$$

Lemma 3.1. Suppose that (B) and (C) hold. Let $\Omega_1=\{(x,y):L(x,y)=\lambda N(x,y),\ ((x,y),\lambda)\in[(D(L)\setminus\mathrm{Ker}L)]\times(0,1)\}$. Then Ω_1 is bounded if $T^\theta(r_1+r_2)+r_3<\beta$.

Proof. For $(x,y)\in\Omega_1$, we have $L(x,y)=\lambda N(x,y)$, $\lambda\in(0,1)$, i.e.

$$\begin{cases} x'(t)=\lambda\phi^{-1}(y(t)),\\ y'(t)=\lambda f(t,x(t),x(\tau(t)),\phi^{-1}(y(t))),\\ y(0)=0,\quad y(T)=0.\end{cases}\tag{12}$$

Thus

$$\lambda\phi(\lambda)\int_0^T f(t,x(t),x(\tau(t)),\frac{1}{\lambda}x'(t))x'(t)dt \quad=\quad 0.$$

We get from $C(i)$ that

$$\begin{aligned}
&\int_0^T h\left(t,x(t),x(\tau(t)),\frac{1}{\lambda}x'(t)\right)x'(t)dt+\int_0^T g_1(t,x(t))x'(t)dt\\
&+\int_0^T g_2(t,x(\tau(t))x'(t)dt+\int_0^T g_3(t,\frac{1}{\lambda}x'(t))x'(t)dt+\int_0^T r(t)x'(t)dt=0.
\end{aligned}$$

Then $C(ii)$ implies that

$$\begin{aligned}
\lambda\beta\int_0^T\left(\frac{|x'(t)|}{\lambda}\right)^{\theta+1}dt &\le \lambda\int_0^T h\left(t,x(t),x(\tau(t)),\frac{1}{\lambda}x'(t)\right)\frac{x'(t)}{\lambda}dt\\
&\quad-\int_0^T g_1(t,x(t))x'(t)dt-\int_0^T g_2(t,x(\tau(t))x'(t)dt\\
&\quad-\int_0^T g_3(t,\frac{1}{\lambda}x'(t))x'(t)dt-\int_0^T r(t)x'(t)dt.
\end{aligned}$$

It follows from $\theta \geq p-1$ that

$$\beta \int_0^T |x'(t)|^{\theta+1}dt \leq \int_0^T |g_1(t,x(t))||x'(t)|dt + \int_0^T |g_2(t,x(\tau(t))||x'(t)|dt + \lambda^\theta \int_0^T \left| g_3\left(t, \frac{1}{\lambda}x'(t)\right)\right| |x'(t)|dt + \int_0^T |r(t)||x'(t)|dt.$$

Now, we prove that there exists $\xi \in [0,T]$ such that $|x(\xi)| \leq M$. It follows from (B) that there exists $M > 0$ such that

$$xf(t,x,y,0) > 0,\ \ t \in [0,T],\ \ |x| > M \text{ and } |y| > M.$$

Suppose $\max_{t\in[0,T]} x(t) = x(t_1)$ and $\min_{t\in[0,T]} x(t) = x(t_2)$ with $t_1, t_2 \in [0,T]$. It is easy to see that $x'(t_i) = 0 (i = 1, 2)$ and $[\phi(x'(t))]'|_{t=t_1} \leq 0$ and $[\phi(x'(t))]'|_{t=t_2} \geq 0$. We consider three cases.

Case (i). $x(t_1) \geq 0$ and $x(t_2) \leq 0$. At this case, there exists $\xi \in [0,T]$ such that $x(\xi) = 0$. The claim is true.

Case (ii). $x(t_1) < 0$ and $x(t_2) < 0$. At this case, if $x(t_1) < -M$, then $x(t_2) < -M$. Hence $x(t) < -M$ for all $t \in [0,T]$. Then $f(t, x(t_2), x(\tau(t_2), 0)x(t_2) > 0$ implies that

$$f(t_2, x(t_2), x(\tau(t_2)), 0) < 0.$$

It follows that $0 \leq [\phi(x'(t))]'|_{t=t_2} = \lambda f(t, x(t_2), x(\tau(t_2)), 0) < 0$, a contradiction. Hence we get that $-M \leq x(t_1) < 0$. The claim is true.

Case (iii). If $x(t_1) > 0$ and $x(t_2) > 0$. At this case, if $x(t_2) > M$, then $x(t_1) > M$. Hence $x(t) > M$ for all $t \in [00,T]$. Then $x(t_1)f(t, x(t_1), x(\tau(t_1)), 0) > 0$ implies that $f(t, x(t_1), x(\tau(t_1)), 0) > 0$. It follows that $0 \geq [\phi(x'(t))]'|_{t=t_1} = \lambda f(t, x(t_1), x(\tau(t_1)), 0) > 0$, a contradiction. Hence we get that $M \geq x(t_2) > 0$.

It follows from Cases 1, 2, and 3 that there exists $\xi \in [0,T]$ such that $|x(\xi)| \leq M$.

Now, we have, for $t \in [0,T]$, that

$$|x(t)| = \left| x(\xi) + \int_\xi^t x'(t)dt \right| \leq M + \int_0^T |x'(t)|dt.$$

Choose $\epsilon > 0$ such that

$$\beta > (r_1 + r_2 + 2\epsilon)T^\theta + r_3 + \epsilon. \tag{13}$$

From (C)(iii), there exists $\delta > 0$ such that

$$|g_i(t,x)| \leq |x|^\theta (r_i + \epsilon),\ |x| > \delta,\ t \in [0,T],\ i = 1,2,3.$$

Then

$$\begin{aligned}
&\beta \int_0^T |x'(t)|^{\theta+1}dt \\
\leq\ & (r_1 + r_2 + 2\epsilon)\left(M + \int_0^T |x'(t)|dt\right)^\theta \int_0^T |x'(t)|dt + (r_3+\epsilon)\int_0^T |x'(t)|^{\theta+1}dt \\
&+ \Bigg(\max_{t\in[0,T],|x|\leq\delta} |g_1(t,x)| + \max_{t\in[0,T],|x|\leq\delta} |g_2(t,x)| \\
&+ \max_{t\in[0,T],|x|\leq\delta} |g_3(t,x)|\Bigg) T^{\frac{\theta}{\theta+1}} \left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&+ \left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}} \left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.
\end{aligned}$$

We claim that there is a constant $\sigma \in (0,1)$, independent of λ, such that $(1+x)^\theta \le 1+(\theta+1)x$ for all $x \in (0,\sigma]$. In fact, let $q(x)=(1+x)^\theta-(1+(\theta+1)x)$, we see $q(0)=0$, and $q'(0)=-1<0$ implies that there exists $\sigma>0$ such that $q(x)<0$ for all $x\in(0,\sigma]$, so the claim is valid.

Now, we prove that there exists $\overline{M}>0$ such that $\int_0^T |x'(t)|^{\theta+1}ds \le \overline{M}$, we consider two cases.

Case 1. $\int_0^T |x'(s)|ds \le \frac{M}{\sigma}$. We get

$$\begin{aligned}\beta\int_0^T |x'(t)|^{\theta+1}dt \le\ & (r_1+r_2+2\epsilon)\left(M+\frac{M}{\sigma}\right)^\theta T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\ & +(r_3+\epsilon)\int_0^T |x'(t)|^{\theta+1}dt\\ & +\left(\max_{t\in[0,T],|x|\le\delta}|g_1(t,x)|+\max_{t\in[0,T],|x|\le\delta}|g_2(t,x)|\right.\\ & \left.+\max_{t\in[0,T],|x|\le\delta}|g_3(t,x)|\right)T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\ & +\left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.\end{aligned}$$

Since $\beta>r_3+\epsilon$, there is a constant $M_1>0$ such that $\int_0^T |x'(t)|^{\theta+1}dt \le M_1$.

Case 2. $\int_0^T |x'(s)|ds > \frac{M}{\sigma}$. At this case, $0<\frac{M}{\int_0^T |x'(s)|ds}<\sigma$. We get

$$\begin{aligned}\beta\int_0^T |x'(t)|^{\theta+1}dt \le\ & (r_1+r_2+2\epsilon)T^\theta\int_0^T |x'(t)|^{\theta+1}dt\\ & +(r_1+r_2+2\epsilon)(\theta+1)MT^{\frac{\theta^2}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{\theta}{\theta+1}}\\ & +(r_3+\epsilon)\int_0^T |x'(t)|^{\theta+1}dt\\ & +\left(\max_{t\in[0,T],|x|\le\delta}|g_1(t,x)|+\max_{t\in[0,T],|x|\le\delta}|g_2(t,x)|\right.\\ & \left.+\max_{t\in[0,T],|x|\le\delta}|g_3(t,x)|\right)T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\\ & +\left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.\end{aligned}$$

Since $\beta>(r_1+r_2+2\epsilon)T^\theta+(r_3+\epsilon)$, there exists a constant $M_2>0$ such that $\int_0^T |x'(t)|^{\theta+1}dt \le M_2$.

Hence we get

$$\int_0^T |x'(t)|^{\theta+1}dt \le \max\{M_1,M_2\} =: \overline{M}.$$

So, for all $t\in[0,T]$, we get

$$|x(t)| \le M+\int_0^T |x'(t)|dt \le M+T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x'(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \le M+T^{\frac{\theta}{\theta+1}}\overline{M}^{\frac{1}{\theta+1}}.$$

It follows that $|x(t)| \le M + T^{\frac{\theta}{\theta+1}}\overline{M}^{\frac{1}{\theta+1}}$ for all $t \in [0,T]$. Then

$$\max_{t\in[0,T]} |x(t)| \le M + T^{\frac{\theta}{\theta+1}}\overline{M}^{\frac{1}{\theta+1}}. \tag{14}$$

It is easy to show, for $t \in [0,T]$, from the second equation in (12), that

$$\int_t^T y'(s)\phi^{-1}(y(s))ds = \lambda \int_t^T f(s, x(s), x(\tau(s)), \phi^{-1}(y(s)))\phi^{-1}(y(s))ds. \tag{15}$$

Denoted by $G(x) = \int_0^x \phi^{-1}(s)ds$. One sees

$$G(y(t)) = -\lambda \int_t^T f(s, x(s), x(\tau(s)), \phi^{-1}(y(s))\phi^{-1}(y(s))ds. \tag{16}$$

So

$$\int_0^T f(s, x(s), x(\tau(s)), \phi^{-1}(y(s))\phi^{-1}(y(s))ds = 0$$

and (C)(i)-(ii) imply that

$$\begin{aligned}
& \beta \int_0^T |\phi^{-1}(y(s))|^{\theta+1}ds \\
\le & \int_0^T h(s, x(s), x(\tau(s)), \phi^{-1}(y(s))\phi^{-1}(y(s))ds \\
= & -\int_0^T g_1(s, x(s))\phi^{-1}(y(s))ds - \int_0^T g_2(s, x(\tau(s))\phi^{-1}(y(s))ds \\
& -\int_0^T g_3(s, \phi^{-1}(y(s))\phi^{-1}(y(s)ds - \int_0^T r(s)\phi^{-1}(y(s))dt \\
\le & \int_0^T |g_1(s, x(s))||\phi^{-1}(y(s))|ds + \int_0^T |g_2(s, x(\tau(s))||\phi^{-1}(y(s))|ds \\
& + \int_0^T |g_3(s, \phi^{-1}(y(s))||\phi^{-1}(y(s))|ds + \int_0^T |r(s)||\phi^{-1}(y(s))|dt
\end{aligned}$$

Since (B) implies that there exists $\xi \in [0,T]$ such that $|x(\xi)| \le M$, we get that

$$|x(t)| \le M + \left|\int_t^\xi x'(s)ds\right| \le M + \int_0^T |\phi^{-1}(y(s))|ds.$$

Then

$$\begin{aligned}
&\beta \int_0^T |\phi^{-1}(y(s))|^{\theta+1} ds \\
\le\ & (r_1+r_2+2\epsilon)\left(M+\int_0^T |\phi^{-1}(y(t))|dt\right)^{\theta}\int_0^T |\phi^{-1}(y(t))|dt \\
&+(r_3+\epsilon)\int_0^T |\phi^{-1}(y(t))|^{\theta+1}dt \\
&+\left(\max_{t\in[0,T],|x|\le\delta}|g_1(t,x)|+\max_{t\in[0,T],|x|\le\delta}|g_2(t,x)|\right. \\
&\left.+\max_{t\in[0,T],|x|\le\delta}|g_3(t,x)|\right)T^{\frac{\theta}{\theta+1}}\left(\int_0^T |\phi^{-1}(y(t))|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&+\left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |\phi^{-1}(y(t))|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.
\end{aligned}$$

Similarly we can prove that there exists $M_3>0$ such that

$$\int_0^T |\phi^{-1}(y(s))|^{\theta+1} ds \le M_3.$$

Then using (C), we get similarly that

$$\begin{aligned}
G(y(t)) \quad \le\ & (r_1+r_2+2\epsilon)T^{\theta}M_3+(r_1+r_2+2\epsilon)(\theta+1)MT^{\frac{\theta^2}{\theta+1}}M_3^{\frac{\theta}{\theta+1}} \\
&+(r_3+\epsilon)M_3 \\
&+\left(\max_{t\in[0,T],|x|\le\delta}|g_1(t,x)|+\max_{t\in[0,T],|x|\le\delta}|g_2(t,x)|\right. \\
&\left.+\max_{t\in[0,T],|x|\le\delta}|g_3(t,x)|\right)T^{\frac{\theta}{\theta+1}}M_3^{\frac{1}{\theta+1}}+\left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}M_3^{\frac{1}{\theta+1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
|G(y(t))| \quad \le\ & (r_1+r_2+2\epsilon)T^{\theta}M_3+(r_1+r_2+2\epsilon)(\theta+1)MT^{\frac{\theta^2}{\theta+1}}M_3^{\frac{\theta}{\theta+1}} \\
&+(r_3+\epsilon)M_3+\left(\int_0^T |r(t)|^{\frac{\theta+1}{\theta}}dt\right)^{\frac{\theta}{\theta+1}}M_3^{\frac{1}{\theta+1}} \\
&+\left(\max_{t\in[0,T],|x|\le\delta}|g_1(t,x)|+\max_{t\in[0,T],|x|\le\delta}|g_2(t,x)|\right. \\
&\left.+\max_{t\in[0,T],|x|\le\delta}|g_3(t,x)|\right)T^{\frac{\theta}{\theta+1}}M_3^{\frac{1}{\theta+1}}.
\end{aligned}$$

It is easy to see that there exists a constant $\widetilde{M}>0$ such that $G(||y||)\le\widetilde{M}$. Then there exists a constant $\widetilde{M_0}>0$ such that $||y||\le\widetilde{M_0}$.

It follows that, for $(x,y)\in\Omega_1$, there is $H>0$ such that $||(x,y)||\le H$. Hence Ω_1 is bounded. The proof is complete.

Lemma 3.2. Suppose (B) holds. Then $\Omega_2=\{(x,y)\in \text{Ker}L:\ N(x,y)\in \text{Im}L\}$ is bounded.

Proof. For $(a,0)\in \mathrm{Ker}L$, we have $N(a,0)=(0,f(t,a,a,0),0,0)$. $Nx\in \mathrm{Im}L$ implies that

$$\int_0^T f(t,a,a,0)dt=0.$$

From condition (B), there is $M>0$ such that

$$xf(t,x,y,0)>0,\ \ t\in[0,T],\ \ |x|,|y|>M.$$

If $a>M$ then $f(t,a,a,0)>0$. It follows that $\int_0^T f(t,a,a,0)dt>0$, a contradiction to $\int_0^T f(t,a,a,0)dt=0$. If $a<-M$, similar contradiction can be induced. Thus Ω_2 is bounded. The proof is complete.

Lemma 3.3. Suppose (B) holds. Let either $\Omega_3=\{(x,y)\in \mathrm{Ker}L:\ \lambda\wedge(x,y)+(1-\lambda)QN(x,y)=0,\ \lambda\in[0,1]\}$. Then Ω_3 is bounded, where $\wedge:\mathrm{Ker}L\to Y/\mathrm{Im}L$ defined by $\wedge(a,0)=(0,a,0,0)$.

Proof. Consider

$$\Omega_3=\{(x,y)\in \mathrm{Ker}L:\ \lambda\wedge(x,y)+(1-\lambda)QN(x,y)=0,\ \lambda\in[0,1]\}.$$

We will prove that Ω_3 is bounded. For $(a,0)\in\Omega_3$, and $\lambda\in[0,1]$, we have

$$-(1-\lambda)\int_0^T f(t,a,a,0)dt=\lambda aT.$$

Then

$$-(1-\lambda)\int_0^T af(t,a,a,0)dt=\lambda a^2T.$$

If $\lambda=1$, then $a=0$. If $\lambda\in[0,1)$, from condition (B), there is $M>0$ such that $xf(t,x,y,0)>0,\ \ t\in[0,T],\ \ |x|,|y|>M$. If $a>M$ then $f(t,a,a,0)>0$. Then we get

$$0>-(1-\lambda)\int_0^T af(t,a,a,0)dt=\lambda a^2T\geq 0,$$

a contradiction. If $a<-M$, similar contradiction can be induced. Hence $|a|\leq M$. Thus Ω_3 is bounded. The proof is complete.

Proof of Theorem 2.1. We know that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Since (x,y) is a solution of $L(x,y)=N(x,y)$ implies that x is a solution of equation (5). It suffices to get a solution (x,y) of $L(x,y)=N(x,y)$. To do this, we construct an open bounded set Ω such that (i), (ii) and (iii) of Theorem GM hold.

Set Ω be a open bounded subset of X such that $\Omega\supset\cup_{i=1}^3\overline{\Omega_i}$. By the definition of Ω, we have $\Omega\supset\overline{\Omega_1}$ and $\Omega\supset\overline{\Omega_2}$, thus, from Lemma 2.1 and Lemma 2.2, that $L(x,y)\neq\lambda N(x,y)$ for $(x,y)\in D(L)\setminus \mathrm{Ker}L)\cap\partial\Omega$ and $\lambda\in(0,1)$; $N(x,y)\notin \mathrm{Im}L$ for $(x,y)\in \mathrm{Ker}L\cap\partial\Omega$.

In fact, let $H((x,y),\lambda)=\pm\lambda\wedge(x,y)+(1-\lambda)QN(x,y)$. According the definition of Ω, we know $\Omega\supset\overline{\Omega_3}$, thus $H((x,y),\lambda)\neq 0$ for $(x,y)\in\partial\Omega\cap \mathrm{Ker}L$, thus, from Lemma 2.3, by homotopy property of degree,

$$\begin{aligned}&\deg(QN|_{\mathrm{Ker}L},\Omega\cap \mathrm{Ker}L,0)=\deg(H(\cdot,0),\Omega\cap \mathrm{Ker}L,0)\\ =\ &\deg(H(\cdot,1),\Omega\cap \mathrm{Ker}L,0)=\deg(\pm\wedge,\Omega\cap \mathrm{Ker}L,0)\neq 0.\end{aligned}$$

Thus by Theorem GM, $L(x,y)=N(x,y)$ has at least one solution (x,y) in $D(L)\cap\overline{\Omega}$, then x is a solution of equation (8). The proof is completed.

Proof of Theorem 2.2. The proof is similar to that of the proof of Theorem 2.1 and is omitted.

Now we begin to prove Theorem 2.3. Set $y(t) = \phi(x'(t))$, then BVP(8) is transformed into

$$\begin{cases} x'(t) = \phi^{-1}(y(t)), \\ y'(t) = f(t, x(t), x(\tau(t)), \phi^{-1}(y(t))), \quad 0 < t < T \\ y(0) = y(T) = 0\,. \end{cases} \tag{17}$$

It is easy to see that x is a solution of BVP(8) if (x, y) is a solution of problem (17).

Note that the homogeneous problem $x'(t) = 0, y'(t) = 0, y(0) = y(T) = 0$ has non-trivial solutions. So, we shall consider the following problem, for $m > 1$:

$$\begin{cases} x'(t) = \phi^{-1}(y(t)), \\ y'(t) - \frac{1}{m}x(t) = f(t, x(t), x(\tau(t)), \phi^{-1}(y(t))), \quad 0 < t < T \\ y(0) = y(T) = 0\,. \end{cases} \tag{18}$$

and consider BVP(17) as a limiting case when $m \to +\infty$.

Our aim is to provide sufficient conditions on f that which make BVP(18) solvable. First, we show that solutions to BVP(18) are uniformly bounded, independently of m. Then, we use the Arzela-Ascoli theorem to obtain the solvability of BVP(17).

We use the Sobolev space $W^{1,1}(0, T)$ defined by

$$W^{1,1}(0, T) = \{x : [0, T] \to R | x \text{ is absolutely continuous on } [0, T] \text{ with } x' \in L^1[0, T]\}.$$

Let the Banach space be $X = C^0[0, T] \times C^0[0, T]$ with the norm

$$||(x, y)|| = \max \left\{ \sup_{t \in [0,T]} |x(t)|, \sup_{t \in [0,T]} |y(t)| \right\}$$

and $D(L) = \{(x, y) \in W^{1,1}(0, T) : x' \in L^1[0, T], y' \in L^1[0, T]\}$. Let $Y = L^1[0, T] \times L^1[0, T] \times R^2$. Define the linear operator $L : D(L) \cap X \to Y$ by

$$L \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) - \frac{1}{m}x(t) \\ y(0) \\ y(T) \end{pmatrix} \text{ for all } (x, y) \in D(L) \cap X.$$

Define the nonlinear operator $N_f : X \to Y$, for all $(x, y) \in X$, by

$$N_f \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), x(\tau(t)), \phi^{-1}(y(t))) \\ 0 \\ 0 \end{pmatrix}.$$

It follows that BVP(18) is equivalent to

$$L(x, y) = N_f(x, y), \tag{19}$$

in the sense that every solution of BVP(18) is a solution of (19) and vice-versa.

It is easy to show that the operator L is invertible and the operator $N_f(\cdot, \cdot)$ is continuous and completely continuous.

Since our arguments are based on Theorem LS, we shall consider a one-parameter family of problems related to (19). For $0 \leq \lambda \leq 1$, consider

$$L(x, y) = \lambda N_f(x, y). \tag{20}$$

Proof of Theorem 2.3. We divide the proof into two steps.

Step 1. For $\lambda \in (0,1]$, prove that any possible solution (x,y) of

$$\left\{\begin{array}{l} x'(t) = \lambda\phi^{-1}(y(t)), \\ y'(t) - \frac{1}{m}x(t) = \lambda f_1\left(t, x(t), x(\tau(t)), \phi^{-1}(y(t))\right) \quad 0 < t < T, \\ y(0) = y(T) = 0, \end{array}\right. \tag{21}$$

where

$$f_1(t,x,y,z) = \left\{\begin{array}{ll} \max\left\{f(t,x,y,z), -\frac{M_0}{m} + \int_0^T f(t,M_0,y,0)dt\right\} & x > M_0 \\ f(t,x,y,z) & -M_0 \le x \le M_0 \\ \min\left\{f(t,x,y,z), \frac{M_0}{m} + \int_0^T f(t,-M_0,y,0)dt\right\} & x < -M_0 \end{array}\right.$$

satisfies $|x(t)| \le M_0$ for all $t \in I$.

Let (x,y) be a solution of (21). Use (A)(i), without loss of generality, we prove only the case when $\int_0^T f(t,M_0,M_0,0)dt > 0$ and $\int_0^T f(t,-M_0,M_0,0)dt < 0$. The other case is similar.

We remark that any solution (x,y) of (21) that satisfies $|x(t)| \le M_0$ is a solution of (20), because in this case

$$f_1(t,x(t),x(\tau(t)),\phi(y(t)) \equiv f(t,x(t),x(\tau(t)),\phi(y(t))).$$

Let $t_0 \in [0,T]$ be a value where x achieves its positive maximum. Then $x'(t_0) = 0$. We prove that $x(t_0) \le M_0$, then $x(t) \le M_0$ for all $t \in [0,T]$. Three cases are considered.

Case 1. $x(t_0) > M_0$ and $t_0 \in (0,T)$. Then there exists $a > 0$ such that $x(t) > M_0$ for all $t \in [t_0, t_0 + a] \subset [0,T]$. It follows from the differential equation in (21) and the definition of f_1 that for all $t \in [t_0, t_0 + a]$,

$$\begin{aligned} [\phi(x'(t))]' &\ge \phi(\lambda)\left(\frac{1}{m}x(t) - \lambda\frac{M_0}{m} + \lambda\int_0^T f(t,M_0,x(\tau(t)),0)dt\right) \\ &\ge \phi(\lambda)\left(\frac{x(t)-M_0}{m} + \lambda\int_0^T f(t,M_0,x(\tau(t)),0)dt\right) > 0. \end{aligned}$$

This implies that $\phi(x'(t)) = \int_{t_0}^t [\phi(x'(s)]'ds > 0$ for all $t \in [t_0, t_0 + a]$, which yields

$$x(t) - x(t_0) = \int_{t_0}^t x'(\tau)d\tau > 0 \quad \text{for all } t \in [t_0, t_0 + a].$$

This contradicts that $x(t_0)$ is the maximum of x. Hence $x(t) \le M_0$ for all $t \in [0,T]$.

Case 2. $t_0 = 0$. Similarly to Case 1, we can prove that $x(t_0) \le M_0$. Then $x(t) \le M_0$ for all $t \in [0,T]$.

Case 3. $t_0 = T$. Then there exists $b > 0$ such that $x(t) > M_0$ for all $t \in [t_0 - b, t_0] \subset [0,T]$. It follows from the differential equation in (21) and the definition of f_1 that for all $t \in [t_0, t_0 + a]$,

$$\begin{aligned} [\phi(x'(t))]' &\ge \phi(\lambda)\left(\frac{1}{m}x(t) - \lambda\frac{M_0}{m} + \lambda\int_0^T f(t,M_0,x(\tau(t)),0)dt\right) \\ &\ge \phi(\lambda)\left(\frac{x(t)-M_0}{m} + \lambda\int_0^T f(t,M_0,x(\tau(t)),0)dt\right) > 0. \end{aligned}$$

This implies that $\phi(x'(t)) = -\int_t^{t_0} [\phi(x'(s)]'ds < 0$ for all $t \in [t_0 - b, t_0]$, which yields

$$x(t) - x(t_0) = -\int_t^{t_0} x'(\tau)d\tau > 0 \quad \text{for all } t \in [t_0 - b, t_0].$$

This contradicts that $x(t_0)$ is the maximum of x. Hence $x(t) \leq M_0$ for all $t \in [0, T]$.

Now, in case x achieves a negative minimum at $t = \tau_0$ such that $x(\tau_0) < -M_0$ and $\tau_0 \in (0, T)$ then there exists $b > 0$ such that $y(t) < -M_0$ for all $t \in [\tau_0, \tau_0 + b]$. It follows from the differential equation in (21) and the definition of f_1 that for all $t \in [\tau_0, \tau_0 + b]$,

$$[\phi(x'(t))]' \quad \leq \quad \phi(\lambda)\left(\frac{x(t)+M_0}{m} + \lambda\int_0^T f(t, M_0, x(\tau(t)), 0)dt\right) < 0.$$

which leads to $\phi(x'(t)) = \int_{\tau_0}^t [\phi(x''(s))]' ds < 0$ for all $t \in [\tau_0, \tau_0 + b]$ and

$$x(t) - x(\tau_0) = \int_{\tau_0}^t x'(s)ds < 0 \quad \text{for all } t \in [\tau_0, \tau_0 + b]\,. \tag{22}$$

This contradicts that $x(\tau_0)$ is the minimum of x on $[0, T]$.

We can handle the case of a minimum at $\tau_0 = 0$ or $\tau_0 = T$ in a similar way as above. Hence, we have proved that

$$-M_0 \leq x(t) \leq M_0 \text{ for all } t \in [0, T], \tag{23}$$

which completes the proof of Step 1.

Step 2. Prove that there exists $M_1 > 0$ such that $|y(t)| \leq M_1$ for all $t \in [0, T]$ for any solution y of (21) with $|x(t)| \leq M_0$ for all $t \in [0, T]$.

Let (x, y) be a solution of (21) such that $|x(t)| \leq M_0$ for all $t \in [0, T]$. Condition (A2) implies

$$\begin{aligned}\left|\frac{[\phi(x'(t))]'}{\phi(\lambda)}\right| &\leq \frac{|x(t)|}{m} + \lambda q(t)\Phi\left(\frac{|x'(t)|}{\lambda}\right)\\ &\leq \frac{|x(t)|}{m} + q(t)\Phi\left(\frac{|x'(t)|}{\lambda}\right)\\ &\leq M_0 + q(t)\Phi\left(\frac{|x'(t)|}{\lambda}\right) \quad \text{for all } t \in [0, T].\end{aligned}$$

On the other hand,

$$\left|\frac{x'(t)}{\lambda}\right| = \phi^{-1}\left(\Big|\int_0^t \left[\frac{\phi(x'(s))}{\phi(\lambda)}\right]' ds\Big|\right) \leq \phi^{-1}\left(\int_0^t \left|\left[\frac{\phi(x'(s))}{\phi(\lambda)}\right]'\right| ds\right) \quad \text{for all } t \in [0, T].$$

Hence

$$\left|\frac{x'(t)}{\lambda}\right| \leq \phi^{-1}\left(M_0 t + \int_0^t q(s)\Phi\left(\frac{|x'(s)|}{\lambda}\right)ds\right) \quad \text{for all } t \in [0, T]\,.$$

Since $0 \leq t \leq T$, we infer that

$$\left|\frac{x'(t)}{\lambda}\right| \leq \phi^{-1}\left(M_0 T + \int_0^t q(s)\Phi\left(\frac{|x'(s)|}{\lambda}\right)ds\right) \quad \text{for all } t \in [0, T]\,.$$

Let

$$u(t) = M_0 T + \int_0^t q(s)\Phi\left(\frac{|x'(s)|}{\lambda}\right)ds \quad \text{for all } t \in [0, T]\,.$$

Then

$$u(t) \geq TM_0,\ \left|\frac{x'(t)}{\lambda}\right| \leq \phi^{-1}(u(t)),\ u'(t) = q(t)\Phi\left(\frac{|x'(t)|}{\lambda}\right) \text{ for all } t \in [0, T].$$

Since ϕ is nondecreasing, we get

$$\frac{u'(t)}{\Phi(\phi^{-1}(u(t)))} \leq q(t) \quad \text{for all } t \in [0, T].$$

It follows that
$$\int_0^t \frac{u'(s)ds}{\Phi(\phi^{-1}(u(s)))} \le \int_0^t q(s)ds \le \int_0^T q(s)ds.$$
This implies
$$\int_{TM_0}^{u(t)} \frac{d\sigma}{\Phi(\phi^{-1}(\sigma))} = \int_{u(0)}^{u(t)} \frac{d\sigma}{\phi(\sigma)} \le \int_0^T q(s)ds.$$
The condition (A)(ii) implies that there exists $M_1 > 0$ such that $u(t) \le M_1$ for all $t \in [0, T]$. Therefore, $\left|\frac{x'(t)}{\lambda}\right| \le M_1$ for all $t \in [0, T]$. Hence $|y(t)| \le M_1$ for all $t \in [0, T]$.

We have seen in the above discussion that any possible solution (x, y) of (21) satisfies
$$|x(t)| \le M_0 \quad \text{and} \quad |y(t)| \le M_1 \qquad \text{for all } t \in [0, T]\,.$$
Let $M := M_0 + M_1$. Then $\|(x, y)\| \le M$. It is clear that problem (21) is equivalent to
$$(x, y) = \lambda L^{-1} N_{f_1}(x, y) \tag{24}$$
Let $U := \{(x, y) \in X; \|(x, y)\| < 1 + M\}$. Then we can easily show that for any λ, the operator $L^{-1}N_{f_1}(\cdot, \cdot)$ is a completely continuous operator (see [6]) and
$$(x, y) = \lambda L^{-1} N_{f_1}(\cdot, \cdot)$$
has no fixed point on ∂U, the boundary of U.

Therefore, Theorem LS implies that
$$(x, y) = L^{-1} N_{f_1}(., .)$$
has at least one solution in U, i.e., there exists $(x, y) \in U$ such that
$$(x, y) = L^{-1} N_{f_1}(x, y)$$
, which means that (x, y) is a solution of (21) for $\lambda = 1$. But, we have seen that any solution of (21), satisfying $|x(t)| \le M_0$ is also a solution of (19). Hence (21), with $\lambda = 1$, has at least one solution. But (19) is exactly (21) for $\lambda = 1$. Hence, we have proved that for each $m > 1$, problem (19) has at least one solution, which we denote by (x_m, y_m). Moreover, (x_m, y_m), satisfies the estimates
$$|x_m(t)| \le M_0 \quad \text{and} \quad |y_m(t)| \le M_1 \quad \text{for all} t \in [0, T]. \tag{25}$$
Furthermore, M_0 and M_1 are independent of m. This shows that the sequences $\{(x_m, y_m)\}$ is uniformly bounded.

Now,
$$y_m(t) = \int_0^t y'(s)ds = \frac{1}{m}\int_0^t x_m(s)ds + \lambda \int_0^t f(s, x_m(s), x_m(\tau(s)), y_m(s)ds.$$
This implies
$$y_m(t_2) - y(t_1) = \frac{1}{m}\int_{t_1}^{t_2} x_m(s)ds + \lambda \int_{t_1}^{t_2} f(s, x_m(s), x_m(\tau(s)), y_m(s))ds\,.$$
Since $m > 1$ and $f \in \mathrm{Car}([0, T] \times R^2)$, we have
$$|y_m(t_2) - y_m(t_1)| \le M_0|t_2 - t_1| + \int_{t_1}^{t_2} h_{M_0}(s)ds\,.$$
This shows that $\{y_m\}$ is equicontinuous.

Also, $x_m(t) = x_m(0) + \int_0^t x_m'(s)ds$ implies
$$x_m(\tau_2) - x_m(\tau_1) = \int_{\tau_1}^{\tau_2} x_m'(s)ds = \lambda \int_{\tau_1}^{\tau_2} \phi^{-1}(y(s))ds$$

By Step 2 we have $|y_m(t) \leq M_1$ for all $t \in [0, T]$. Thus $|y_m(\tau_2) - y_m(\tau_1)| \leq M_1|\tau_2 - \tau_1|$ So that $\{y_m\}$ is also equicontinuous.

By the Arzela-Ascoli theorem, we can extract from $\{(x_m, y_m)\}$ subsequences, which we label the same, and that are uniformly convergent on $[0, T]$. Let $x(t) = \lim_{m\to+\infty} x_m(t)$ and $y(t) = \lim_{m\to+\infty} y_m(t)$. Then (x, y) is a solution of (17). Hence x is a solution of BVP(8). This completes the proof of Theorem 2.3.

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