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G-FRAMES POTENTIAL IN HILBERT SPACES

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In this paper, by generalizing the frame potential to the g-frame, we introduce the g-frame potential and investigate some of its properties. We also generalize the Welch inequality to the g-frames.

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1. Introduction

The frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] in 1952. In 2006, g-frame as a generalization of frame was introduced and investigated by Sun [11]. In 2003, Benedeto and Fickus introduced an important tool in the frame theory so called the frame potential [1]. This gave a geometric interpretation for the tight frames which resulted in the field including a physical interpretation for the tight frames along the lines of Coulomb's law in Physics [1], [5].

Throughout this paper, H is a complex N-dimensional Hilbert space and $\{H_j\}_{j\in\mathbb{J}}$ is a finite sequence of Hilbert spaces, where $\mathbb{J}$ is a finite subset of natural numbers $\mathbb{N}$. We denote the space of all bounded linear operators from H into H_j by $B(H,H_j)$.

Definition 1.1. *A sequence of operators $\Lambda=\{\Lambda_j\in B(H,H_j): j\in\mathbb{J}\}$ is called a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if there exist two constants $0<A_\Lambda\leq B_\Lambda<\infty$, such that*

$$A_\Lambda||f||^2\leq\sum_{j\in\mathbb{J}}||\Lambda_j f||^2\leq B_\Lambda||f||^2,\quad f\in H, \tag{1.1}$$

A_Λ and B_Λ are called the lower and upper g-frame bounds, respectively.

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We call Λ a tight g-frame if $A_\Lambda = B_\Lambda$ and a Parseval g-frame if $A_\Lambda = B_\Lambda = 1$. If the right hand inequality of (1.1) holds for all $f \in H$ then we say that Λ is a g-Bessel sequence. Let us consider the space

$$\widehat{H} = \{\{f_j\}_{j\in\mathbb{J}} : f_j \in H_j, j \in \mathbb{J}\}$$

with the inner product given by $\langle \{f_j\}_{j\in\mathbb{J}}, \{g_j\}_{j\in\mathbb{J}} \rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j \rangle$. It is easy to show that $\widehat{H}$ is a Hilbert space with respect to the pointwise operations. It is proved in [9], if Λ is a g-Bessel sequence for H then the operator that

$$T_\Lambda : \widehat{H} \to H, \quad T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} \Lambda_j^*(f_j) \qquad (1.2)$$

is well defined and bounded and its adjoint is

$$T_\Lambda^* : H \to \widehat{H}, \quad T_\Lambda^* f = \{\Lambda_j f\}_{j\in\mathbb{J}}.$$

Also, a sequence Λ is a g-frame for H if and only if the operator T_Λ defined by (1.2) is bounded and onto. We call the operators T_Λ and T_Λ^*, the synthesis and analysis operators of Λ, respectively. If Λ is a g-frame for H then

$$S_\Lambda : H \to H, \quad S_\Lambda f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f,$$

is a bounded invertible positive operator [11]. S_Λ is called the g-frame operator of Λ.

Definition 1.2. *[8] Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ and $\Gamma = \{\Gamma_j \in B(H, H_j) : j \in \mathbb{J}\}$ be g-Bessel sequences for H with respect to $\{H_j\}_{j\in\mathbb{J}}$. Γ is called a dual of Λ if*

$$f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Gamma_j f, \quad f \in H.$$

Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H and $\tilde{\Lambda}_j = \Lambda_j S_\Lambda^{-1}$ for all $j \in \mathbb{J}$. Then $\tilde{\Lambda} = \{\tilde{\Lambda}_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for H with the g-frame bounds $\frac{1}{B_\Lambda}$ and $\frac{1}{A_\Lambda}$. We call $\widetilde{\Lambda}$ the canonical dual of Λ. If $\widetilde{S_\Lambda}$ is a g-frame operator of $\widetilde{\Lambda}$ then $\widetilde{S_\Lambda} = S_\Lambda^{-1}$.

Definition 1.3. *Let $\mathbb{I}$ be a finite subset of $\mathbb{N}$. The frame potential of a frame $\{x_i\}_{i\in\mathbb{I}}$ in H is defined by*

$$FP(\{x_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} |\langle x_i, x_j \rangle|^2.$$

2. Main Results

In this section we introduce the g-frame potential and we generalize the Welch inequality. Also, we generalize some results of [3], [5], [10] and [12] to the g-frame setting.

Definition 2.1. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Suppose that $\{f_l\}_{l=1}^{L}$ is a Parseval frame for H. The g-frame potential of $\{\Lambda_i\}_{i\in\mathbb{J}}$ with respect to $\{H_i\}_{i\in\mathbb{J}}$ is defined by*

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = \sum_{l=1}^{L} \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(f_l), \Lambda_j^*\Lambda_j(f_l)\rangle.$$

Proposition 2.2. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H and $\{f_l\}_{l=1}^{L}$ be a Parseval frame for H. Then*

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = \sum_{l=1}^{L} \langle S_\Lambda^2(f_l), f_l\rangle,$$

and the definition of the g-frame potential is independent of the choice of the Parseval frame.

Proposition 2.3. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Then*

$$A_\Lambda Tr(S_\Lambda) \le FP(\{\Lambda_i\}_{i\in\mathbb{J}}) \le B_\Lambda Tr(S_\Lambda).$$

In particular, $FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = A_\Lambda^2 N$ for a tight g-frame Λ and $FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = N$ for a Parseval g-frame Λ.

Proof. Since Λ is a g-frame, thus

$$A_\Lambda I_H \le S_\Lambda \le B_\Lambda I_H,$$

where I_H is the identity operator in H. By Theorem $A.6.5$ of [4], we have

$$A_\Lambda S_\Lambda \le S_\Lambda^2 \le B_\Lambda S_\Lambda.$$

Therefore,

$$\begin{aligned} A_\Lambda \textstyle\sum_{i\in\mathbb{J}} ||\Lambda_i f||^2 &\le \textstyle\sum_{j\in\mathbb{J}} \sum_{i\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i f, \Lambda_j^*\Lambda_j f\rangle \\ &\le B_\Lambda \textstyle\sum_{i\in\mathbb{J}} ||\Lambda_i f||^2, \quad f \in H. \end{aligned} \tag{2.1}$$

Suppose that $\{e_n\}_{n=1}^{N}$ is an orthonormal basis for H. Then by (2.1)

$$\begin{aligned} A_\Lambda \sum_{n=1}^{N} \sum_{i\in\mathbb{J}} ||\Lambda_i e_n||^2 &\le \sum_{n=1}^{N} \sum_{j\in\mathbb{J}} \sum_{i\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle \\ &\le B_\Lambda \sum_{n=1}^{N} \sum_{i\in\mathbb{J}} ||\Lambda_i e_n||^2. \end{aligned}$$

Since $Tr(S_\Lambda) = \sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2$, we have

$$A_\Lambda Tr(S_\Lambda) \leq FP(\{\Lambda_i\}_{i\in\mathbb{J}}) \leq B_\Lambda Tr(S_\Lambda).$$

□

Theorem 2.4. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ *be a* g*-frame for* H*. Then*

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) \geq \frac{(\sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2)^2}{N}. \tag{2.2}$$

Proof. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for H and

$$S_\Lambda e_n = \lambda_n e_n, \quad n = 1,2,\dots,N.$$

Then

$$\sum_{n=1}^N \lambda_n = \sum_{n=1}^N \langle S_\Lambda e_n, e_n\rangle = \sum_{n=1}^N \sum_{i\in\mathbb{J}} ||\Lambda_i(e_n)||^2 = \sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2. \tag{2.3}$$

We have

$$\sum_{n=1}^N \lambda_n^2 = \sum_{n=1}^N \langle S_\Lambda^2 e_n, e_n\rangle = \sum_{n=1}^N \langle \sum_{i\in\mathbb{J}} \Lambda_i^*\Lambda_i(e_n), \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j(e_n)\rangle$$

$$= \sum_{n=1}^N \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle. \tag{2.4}$$

So, by the Cauchy-Schwarz inequality and (2.4) we have

$$\frac{1}{N}(\sum_{n=1}^N \lambda_n)^2 \leq \sum_{n=1}^N \lambda_n^2 = \sum_{n=1}^N \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle.$$

Therefore by (2.3),

$$\frac{1}{N}(\sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2)^2 \leq \sum_{n=1}^N \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle = FP(\{\Lambda_i\}_{i\in\mathbb{J}}).$$

□

Theorem 2.5. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ *be a* g*-frame for* H*. Then the followings are equivalent:*

(1) We have equality in (2.2), i.e.,

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = \frac{(\sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2)^2}{N}.$$

(2) There is a representation

$$f = \frac{N}{K} \sum_{i\in\mathbb{J}} \Lambda_i^*\Lambda_i(f), \quad f \in H,$$

where

$$K=\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2.$$

(3) Λ is a A_Λ-tight g-frame with respect to $\{H_i\}_{i\in\mathbb{J}}$.

Proof. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for H and

$$S_\Lambda e_n=\lambda_n e_n,\quad n=1,2,\ldots,N.$$

Let (1) holds; then by the proof of Theorem 2.4 we have

$$\frac{1}{N}(\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2)^2=\frac{1}{N}(\sum_{n=1}^N\lambda_n)^2\le\sum_{n=1}^N\lambda_n^2=FP(\{\Lambda_i\}_{i\in\mathbb{J}}).$$

There is equality in $(\sum_{n=1}^N\lambda_n)^2\le N\sum_{n=1}^N\lambda_n^2$ if and only if $\lambda_1=\lambda_2=\cdots=\lambda_N=\lambda\neq 0$ and in this case $S_\Lambda=\lambda I_H$. We have

$$\lambda N=Tr(S_\Lambda)=\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2=K.$$

Hence,

$$f=\frac{1}{\lambda}S_\Lambda f=\frac{N}{K}\sum_{i\in\mathbb{J}}\Lambda_i^*\Lambda_i(f),\quad f\in H,$$

so (2) holds.

Now, assume that (2) is true. For each $f\in H$,

$$||f||^2=\langle f,f\rangle=\langle\frac{N}{K}\sum_{i\in\mathbb{J}}\Lambda_i^*\Lambda_i(f),f\rangle=\frac{N}{K}\sum_{i\in\mathbb{J}}||\Lambda_i f||^2.$$

By letting $A_\Lambda=\frac{N}{K}$, we obtain (3).

Let (3) holds; we have

$$A_\Lambda^{-1}\langle f,f\rangle=\langle S_\Lambda f,f\rangle=\sum_{i\in\mathbb{J}}||\Lambda_i f||^2,\quad f\in H,$$

therefore $S_\Lambda=A_\Lambda^{-1}I_H$ and this implies that $\lambda_1=\lambda_2=\ldots=\lambda_N=A_\Lambda^{-1}$, hence $(\sum_{n=1}^N\lambda_n)^2=N\sum_{n=1}^N\lambda_n^2$ and by the proof of Theorem 2.4, the proof is completed. □

Proposition 2.6. *If $\Lambda=\{\Lambda_j\in B(H,H_j):j\in\mathbb{J}\}$ is a A_Λ-tight g-frame for H then*

$$\max_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2\le A_\Lambda N=\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2.$$

In particular, if $||\Lambda_i||_{HS}=1$ for all $i\in\mathbb{J}$ and $|\mathbb{J}|=M$, then $A_\Lambda=\frac{M}{N}$.

Lemma 2.7. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j): j \in \mathbb{J}\}$ *be a* g*-frame for* H. *Then*

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = Tr(S_\Lambda^2).$$

Proof. It is clear that by (2.4).

□

Here we bring an example of g-frame potential.

Example 2.8. *Let* $H = \mathbb{C}^2$ *and we define*

$$\Lambda_1: \mathbb{C}^2 \to \mathbb{C}, \quad \Lambda_1(x, y) = 3x - 2y,$$
$$\Lambda_2: \mathbb{C}^2 \to \mathbb{C}, \quad \Lambda_2(x, y) = 2x + 3y.$$

Then $\Lambda = \{\Lambda_j \in B(\mathbb{C}^2, \mathbb{C}): j = 1,2\}$ is a tight g-frame for $\mathbb{C}^2$ with the g-frame bound 13. Therefore, the g-frame operator is

$$S_\Lambda: \mathbb{C}^2 \to \mathbb{C}^2, \quad S_\Lambda(x, y) = (13x, 13y). \tag{2.5}$$

Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Then

$$FP(\{\Lambda_i\}_{i=1}^2) = \sum_{n=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle = 338.$$

By (2.5) and Lemma 2.7 we have

$$FP(\{\Lambda_i\}_{i=1}^2) = Tr(S_\Lambda^2) = Tr(169I) = 338.$$

where I is the identity operator in $\mathbb{C}^2$. Λ is a tight g-frame for $\mathbb{C}^2$, hence by Theorem 2.5 we have

$$FP(\{\Lambda_i\}_{i=1}^2) = \frac{(\sum_{i=1}^2 \|\Lambda_i\|_{HS}^2)^2}{2} = \frac{676}{2} = 338.$$

Proposition 2.9. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j): j \in \mathbb{J}\}$ *be a* g*-frame for* H. *Then the canonical dual* g*-frame of* Λ *has the minimum value of the* g*-frame potential with respect to other duals of* Λ.

Proof. Let $\Gamma = \{\Gamma_j \in B(H, H_j): j \in \mathbb{J}\}$ be an arbitrary dual of Λ and let $T_{\widetilde{\Lambda}}$ and T_Γ denote the synthesis operators of $\widetilde{\Lambda}$ and Γ, respectively. We have

$$T_\Gamma T_\Gamma^* = [T_{\widetilde{\Lambda}} + (T_\Gamma - T_{\widetilde{\Lambda}})][T_{\widetilde{\Lambda}} + (T_\Gamma - T_{\widetilde{\Lambda}})]^*$$
$$= T_{\widetilde{\Lambda}}T_{\widetilde{\Lambda}}^* + (T_\Gamma - T_{\widetilde{\Lambda}})(T_\Gamma - T_{\widetilde{\Lambda}})^*.$$

By Lemma 2.7 we have

$$FP(\{\Gamma_i\}_{i\in\mathbb{J}}) = Tr((T_\Gamma T_\Gamma^*)^2) = Tr((T_{\widetilde{\Lambda}}T_{\widetilde{\Lambda}}^* + (T_\Gamma - T_{\widetilde{\Lambda}})(T_\Gamma - T_{\widetilde{\Lambda}})^*)^2)$$
$$= Tr((T_{\widetilde{\Lambda}}T_{\widetilde{\Lambda}}^*)^2) + 2Tr(T_{\widetilde{\Lambda}}T_{\widetilde{\Lambda}}^*(T_\Gamma - T_{\widetilde{\Lambda}})(T_\Gamma - T_{\widetilde{\Lambda}})^*)$$

$$+Tr(((T_\Gamma - T_{\tilde{\Lambda}})(T_\Gamma - T_{\tilde{\Lambda}})^*)^2). \qquad (2.6)$$

Since $T_{\tilde{\Lambda}}T_{\tilde{\Lambda}}^*$ and $(T_\Gamma - T_{\tilde{\Lambda}})(T_\Gamma - T_{\tilde{\Lambda}})^*$ are positive operators, by Theorem 2.2.1 of [7], there exist the positive square root of $T_{\tilde{\Lambda}}T_{\tilde{\Lambda}}^*$ and $(T_\Gamma - T_{\tilde{\Lambda}})(T_\Gamma - T_{\tilde{\Lambda}})^*$ and we denote them by K and U, respectively. By Theorem 2.4.14 of [7], we have

$$Tr(T_{\tilde{\Lambda}}T_{\tilde{\Lambda}}^*(T_\Gamma - T_{\tilde{\Lambda}})(T_\Gamma - T_{\tilde{\Lambda}})^*) = Tr(KKUU) = Tr(UKKU)$$
$$= Tr((KU)^*KU) \geq 0.$$

Then by (2.6) $FP(\{\Gamma_i\}_{i\in\mathbb{J}}) \geq Tr((T_{\tilde{\Lambda}}T_{\tilde{\Lambda}}^*)^2) = FP(\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}})$.

□

Proposition 2.10. *A g-frame* $\Lambda = \{\Lambda_j \in B(H, H_j): j \in \mathbb{J}\}$ *is a minimizer of*

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) + FP(\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}}),$$

if and only if Λ is a Parseval g-frame for H.

Proof. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for H and

$$S_\Lambda e_n = \lambda_n e_n, \quad n = 1,2,\ldots,N.$$

Since the g-frame operator of $\tilde{\Lambda} = \{\tilde{\Lambda}_j \in B(H, H_j): j \in \mathbb{J}\}$ is S_Λ^{-1}, by Lemma 2.7 we have

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) + FP(\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}}) = Tr(S_\Lambda^2) + Tr(S_\Lambda^{-2}) = \sum_{n=1}^N (\lambda_n^2 + \lambda_n^{-2}). \quad (2.7)$$

Now, first we assume that $FP(\{\Lambda_i\}_{i\in\mathbb{J}}) + FP(\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}})$ is minimized, so by (2.7), $\sum_{n=1}^N (\lambda_n^2 + \lambda_n^{-2})$ is minimized. Thus for any n, $\lambda_n^2 + \lambda_n^{-2}$ is minimized and this minimum is obtained when $\lambda_n = 1$. In this case, $S_\Lambda = I_H$ and so Λ is a Parseval g-frame for H.

Conversely, we assume that Λ is a Parseval g-frame for H, so $S_\Lambda = I_H$. Therefore, for any n, $\lambda_n = 1$. Thus $\sum_{n=1}^N (\lambda_n^2 + \lambda_n^{-2}) = 2N$ and this is the minimum value of $\sum_{n=1}^N (\lambda_n^2 + \lambda_n^{-2})$ and by (2.7), $FP(\{\Lambda_i\}_{i\in\mathbb{J}}) + FP(\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}})$ is minimized.

□

Proposition 2.11. *Let* H *be a Hilbert space and* $|\mathbb{J}| = M$, $\lambda > 0$ *and let*

$$W = \{\{\Lambda_i\}_{i\in\mathbb{J}} | \{\Lambda_i\}_{i\in\mathbb{J}}\ is\ a\ g - frame\ and \sum_{i\in\mathbb{J}} ||\Lambda_i||_{HS}^2 = \lambda\}.$$

If $M \geq N$, the minimum value of the g-frame potential on W is $\frac{\lambda^2}{N}$ and the minimizers are the tight g-frames with the tight g-frame bound $\frac{\lambda}{N}$.

Proof. By Lemma 2.7, minimizing the g-frame potential under our condition means minimizing $Tr(S_\Lambda^2)=\sum_{n=1}^{N}\lambda_n^2$ under the condition

$$\sum_{n=1}^{N}\lambda_n=\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2=\lambda.$$

A standard application of Lagrange multipliers yields that the minimizers satisfy $\lambda_n=\frac{\lambda}{N}$, for all $1\le n\le N$. In fact, if

$$f(\lambda_1,\lambda_2,\dots,\lambda_n):=\sum_{n=1}^{N}\lambda_n^2=\lambda_1^2+\lambda_2^2+\cdots+\lambda_N^2,$$

and

$$g(\lambda_1,\lambda_2,\dots,\lambda_n):=\lambda-\lambda_1-\lambda_2-\cdots-\lambda_N,$$

there exists $\mu\neq 0$ such that $\nabla f(\lambda_1,\lambda_2,\dots,\lambda_n)=\mu\nabla g(\lambda_1,\lambda_2,\dots,\lambda_n)$, that is,

$$(2\lambda_1,2\lambda_2,\dots,2\lambda_N)=(-\mu,-\mu,\dots,-\mu).$$

Therefore, $\lambda_n=\lambda_m, 1\le m,n\le N$ and since $\sum_{n=1}^{N}\lambda_n=\lambda$, we have $N\lambda_n=\lambda, 1\le n\le N$. So, $\lambda_n=\frac{\lambda}{N}$, $1\le n\le N$. Hence, the g-frame operator for $\{\Lambda_i\}_{i\in\mathbb{J}}$ is $\frac{\lambda}{N}I_H$. Therefore a minimizer of the g-frame potential is a tight g-frame with the tight g-frame bound $\frac{\lambda}{N}$. So

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}})=Tr(\frac{\lambda^2}{N^2}I_H^2)=\frac{\lambda^2}{N}.$$

□

Proposition 2.12. *Let* $\Lambda=\{\Lambda_j\in B(H,H_j): j\in\mathbb{J}\}$ *be a* g*-frame for* H *and* $\mathbb{I}\subseteq\mathbb{J}$. *Then*

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c})=FP(\{\Lambda_i\}_{i\in\mathbb{J}})+FP(\{\Lambda_i\}_{i\in\mathbb{I}})-\sum_{l=1}^{L}\sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{J}}\langle\Lambda_i^*\Lambda_i(f_l),\Lambda_j^*\Lambda_j(f_l)\rangle$$
$$-\sum_{l=1}^{L}\sum_{i\in\mathbb{J}}\sum_{j\in\mathbb{I}}\langle\Lambda_i^*\Lambda_i(f_l),\Lambda_j^*\Lambda_j(f_l)\rangle,$$

where $\{f_l\}_{l=1}^{L}$ is a Parseval frame for H and $\mathbb{I}^c$ is the complement of $\mathbb{I}$ with respect to $\mathbb{J}$.

Corollary 2.13. *Let* $\Lambda=\{\Lambda_j\in B(H,H_j): j\in\mathbb{J}\}$ *be a tight* g*-frame for* H *with* $||\Lambda_i||_{HS}=1$, *for all* $i\in\mathbb{J}$. *If* $\mathbb{I}\subseteq\mathbb{J}$ *with* $|\mathbb{J}|=M$ *and* $|\mathbb{I}|=k$, *then*

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \frac{M^2}{N} - 2k\frac{M}{N} + FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$

Therefore,

$$1. If |\mathbb{I}| > |\mathbb{I}^c| \ then \ FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) < FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$
$$2. If |\mathbb{I}| < |\mathbb{I}^c| \ then \ FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) > FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$
$$3. If M \ is \ even \ and |\mathbb{I}| = |\mathbb{I}^c| then \ FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$

Proof. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for H and

$$S_\Lambda e_n = \lambda_n e_n, \quad n = 1,2,\ldots,N.$$

By Theorem 2.5 we have

$$FP(\{\Lambda_i\}_{i\in\mathbb{J}}) = \frac{(\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2)^2}{N} = \frac{M^2}{N}. \tag{2.8}$$

Also, by Proposition 2.6,

$$A_\Lambda = \frac{\sum_{i\in\mathbb{J}}||\Lambda_i||_{HS}^2}{N} = \frac{M}{N}.$$

On the other hand, by assumption Λ is a tight g-frame and so $S_\Lambda = \frac{M}{N} I_H$. We have

$$\sum_{n=1}^N \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{I}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle = \frac{M}{N}\sum_{j\in\mathbb{I}} ||\Lambda_j||_{HS}^2 = k\frac{M}{N}. \tag{2.9}$$

Similarly, we have

$$\sum_{n=1}^N \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle = k\frac{M}{N}. \tag{2.10}$$

By Proposition 2.12, (2.8), (2.9) and (2.10) imply that

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \frac{M^2}{N} - 2k\frac{M}{N} + FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$

□

Example 2.14. *Let* $H = \mathbb{C}^2$ *and we define*

$$\Lambda_1: \mathbb{C}^2 \to \mathbb{C}, \quad \Lambda_1(x,y) = \frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y,$$
$$\Lambda_2: \mathbb{C}^2 \to \mathbb{C}, \quad \Lambda_2(x,y) = \frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y.$$

Then $\Lambda = \{\Lambda_j \in B(\mathbb{C}^2, \mathbb{C}): j = 1,2\}$ is a Parseval g-frame for $\mathbb{C}^2$ and $\| \Lambda_1 \|_{HS} = \| \Lambda_2 \|_{HS} = 1$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ and put $\mathbb{I} = \{1\} \subset \mathbb{J} = \{1,2\}$. Then

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \sum_{n=1}^{2} \langle \Lambda_2^*\Lambda_2(e_n), \Lambda_2^*\Lambda_2(e_n)\rangle = 1,$$

and

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}}) = \sum_{n=1}^{2} \langle \Lambda_1^*\Lambda_1(e_n), \Lambda_1^*\Lambda_1(e_n)\rangle = 1.$$

Also, by Corollary 2.13, we have

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \frac{2^2}{2} - 2\times 1\times\frac{2}{2} + FP(\{\Lambda_i\}_{i\in\mathbb{I}}) = FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$

Example 2.15. *Let* $H=\mathbb{C}^3$ *and we define*

$$\Lambda_1:\mathbb{C}^3\to\mathbb{C},\quad \Lambda_1(x,y,z)=zi,$$

$$\Lambda_2:\mathbb{C}^3\to\mathbb{C},\quad \Lambda_2(x,y,z)=\frac{4}{5}x+\frac{3}{5}y,$$

$$\Lambda_3:\mathbb{C}^3\to\mathbb{C},\quad \Lambda_3(x,y,z)=-\frac{3}{5}x+\frac{4}{5}y.$$

Then $\Lambda=\{\Lambda_j\in B(\mathbb{C}^3,\mathbb{C}): j=1,2,3\}$ is a Parseval g-frame for $\mathbb{C}^3$ and $\| \Lambda_1 \|_{HS}=\| \Lambda_2 \|_{HS}=\| \Lambda_3 \|_{HS}=1$. Let $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$ and put $\mathbb{I}=\{1,2\}\subset\mathbb{J}=\{1,2,3\}$. We have

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \sum_{n=1}^{3} \langle \Lambda_3^*\Lambda_3(e_n), \Lambda_3^*\Lambda_3(e_n)\rangle = 1,$$

and

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}}) = \textstyle\sum_{n=1}^{3}\sum_{i=1}^{2}\sum_{j=1}^{2} \langle \Lambda_i^*\Lambda_i(e_n), \Lambda_j^*\Lambda_j(e_n)\rangle = 2.$$

So,

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = 1 < FP(\{\Lambda_i\}_{i\in\mathbb{I}}) = 2.$$

Also, by Corollary 2.13,

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) = \frac{3^2}{3} - 2\times 2\times\frac{3}{3} + FP(\{\Lambda_i\}_{i\in\mathbb{I}}) = -1 + FP(\{\Lambda_i\}_{i\in\mathbb{I}}),$$

therefore

$$FP(\{\Lambda_i\}_{i\in\mathbb{I}^c}) < FP(\{\Lambda_i\}_{i\in\mathbb{I}}).$$

Proposition 2.16. *Suppose that* $\Lambda=\{\Lambda_j\in B(H,H_j): j\in\mathbb{J}\}$ *is a* g*-frame for* H *with the optimal* g*-frame bounds* $0<A_\Lambda\le B_\Lambda<\infty$. *Then*

$$(N-1)A_\Lambda^2 + B_\Lambda^2 \le FP(\{\Lambda_i\}_{i\in\mathbb{J}}) \le (N-1)B_\Lambda^2 + A_\Lambda^2,\quad N\neq 1.$$

If $N=1$, we should have $A_\Lambda=B_\Lambda$.

Proof. The proof is clear and we remove it.

□

In 2014, the mixed frame potential was introduced by Carrizo and Heineken [2]. Here we generalize this concept to the g-frames. Also, we generalize the result of [2] to the g-frames.

Definition 2.17. *Suppose that* $\Lambda = \{\Lambda_j \in B(H, H_j): j \in \mathbb{J}\}$ *and* $\Gamma = \{\Gamma_j \in B(H, H_j): j \in \mathbb{J}\}$ *are* g*-frames for* H. *Let* $\{f_l\}_{l=1}^{L}$ *be a Parseval* g*-frame for* H. *The mixed* g*-frame potential of* $(\{\Lambda_i\}_{i\in\mathbb{J}}, \{\Gamma_i\}_{i\in\mathbb{J}})$ *with respect to* $\{H_i\}_{i\in\mathbb{J}}$ *is defined by*

$$\widetilde{FP}(\{\Lambda_i\}_{i\in\mathbb{J}}, \{\Gamma_i\}_{i\in\mathbb{J}}) = \sum_{l=1}^{L} \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Gamma_i(f_l), \Gamma_j^*\Lambda_j(f_l)\rangle.$$

In particular, if $\{\Lambda_i\}_{i\in\mathbb{J}} = \{\Gamma_i\}_{i\in\mathbb{J}}$ then the mixed g-frame potential is equal to the g-frame potential.

Proposition 2.18. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j): j \in \mathbb{J}\}$ *and* $\Gamma = \{\Gamma_j \in B(H, H_j): j \in \mathbb{J}\}$ *be* g*-frames for* H. *Then*

$$\widetilde{FP}(\{\Lambda_i\}_{i\in\mathbb{J}}, \{\Gamma_i\}_{i\in\mathbb{J}}) = Tr((T_\Lambda T_\Gamma^*)^2),$$

where T_Λ and T_Γ are the synthesis operators of Λ and Γ, respectively.
In particular,

$$\widetilde{FP}(\{\Gamma_i\}_{i\in\mathbb{J}}, \{\Lambda_i\}_{i\in\mathbb{J}}) = Tr((T_\Gamma T_\Lambda^*)^2).$$

Also, if Λ is a dual of Γ, then $\widetilde{FP}(\{\Lambda_i\}_{i\in\mathbb{J}}, \{\Gamma_i\}_{i\in\mathbb{J}}) = N$.

Proof. Suppose that $\{e_n\}_{n=1}^{N}$ is the standard orthonormal basis for H. So

$$\begin{aligned}\widetilde{FP}(\{\Lambda_i\}_{i\in\mathbb{J}}, \{\Gamma_i\}_{i\in\mathbb{J}}) &= \sum_{n=1}^{N} \sum_{i\in\mathbb{J}} \sum_{j\in\mathbb{J}} \langle \Lambda_i^*\Gamma_i(e_n), \Gamma_j^*\Lambda_j(e_n)\rangle \\ &= \sum_{n=1}^{N} \langle \sum_{i\in\mathbb{J}} \Lambda_i^*\Gamma_i(e_n), \sum_{j\in\mathbb{J}} \Gamma_j^*\Lambda_j(e_n)\rangle \\ &= \sum_{n=1}^{N} \langle T_\Lambda T_\Gamma^*(e_n), T_\Gamma T_\Lambda^*(e_n)\rangle \\ &= Tr((T_\Lambda T_\Gamma^*)^2).\end{aligned}$$

□

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