U.P.B. Sci. Bull., Series A, Vol. 73, Iss. 1, 2011 ISSN 1223-7027

ON HYPERACTIONS OF HYPERGROUPS

Jianming Zhan[1], S.Sh. Mousavi[2], M. Jafarpour[3]

In this paper, we define the notion of hyperaction of a hypergroup on a nonempty set and also the notion of index of a subhypergroup in a hypergroup, as a generalization of the concept of action of a group on a nonempty set and the notion of index of a subgroup in a group, respectively. Some properties such as the generalized orbit-stabilizer theorem, are investigated. In particular, introduce a construction of a hypergroup from a hyperaction. Finally, we assign a generalized state hypergroup to a nondeterministic automata which can associated from a hyperaction.

Keywords: (semi)hypergroup, index, hyperaction, nondeterministic automata
MSC2000: 20N20, 68Q70

1. Introduction

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicians, where Marty [16] introduced the hypergroup notion as a generalization of groups and proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [3], Vougiouklis [17], Corsini and Leoreanu [7]. Hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary and n-ary relations, theory of fuzzy and rough sets, automata theory, artificial intelligence, etc. See, for example [2, 5, 11, 13, 15, 19, 20]. Some related recent work which some of them overlap the topic of this paper can be found in [1, 4, 6, 10, 12, 18]. We recall here some basic notions of hypergroup theory.

[1] Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province, 445000, P. R. China, e-mail: zhanjianming@hotmail.com

[2] Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran, e-mail: smousavi@mail.uk.ac.ir

[3] Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran, e-mail: m.j@mail.vru.ac.ir

Let H be a nonempty set and $P^*(H)$ the set of all nonempty subsets of H. Let $\cdot$ be a *hyperoperation* (or *join operation*) on H, that is, $\cdot$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $P^*(H)$ is denoted by $a \cdot b$ or ab. The join operation is extended to subsets of H in a natural way, that is $A \cdot B = \bigcup\{ab \mid a \in A, b \in B\}$. The notation aA is used for $\{a\}A$ and Aa for $A\{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called a *semihypergroup* if $a(bc) = (ab)c$ for all $a, b, c \in H$ and is called a *hypergroup* if it is a semihypergroup and $aH = Ha = H$ for all $a \in H$. A hypergroup $(H, \cdot)$ is called *regular* if it has at least an identity, that is an element e of H, such that for all $x \in H$, $x \in e \cdot x \cap x \cdot e$ and moreover each element has at least one inverse, that is if $x \in H$, then there exists $x' \in H$ such that $e \in x \cdot x' \bigcap x' \cdot x$. The set of all identities of H is denoted by $E(H)$, if $x \in H$, $i_l(x) = \{x' : e \in x' \cdot x\}$ is the set of all left inverses of x in H (resp. $i_r(x)$) and $i(x) = i_l(a) \bigcap i_r(x)$. A regular hypergroup $(H, \cdot)$ is called *reversible* if for all $(x, y, a) \in H^3$:

(i) $y \in a \cdot x$, then there exists $a' \in i(a)$ such that $x \in a' \cdot y$;

(ii) $y \in x \cdot a$, then there exists $a'' \in i(a)$ such that $x \in y \cdot a''$.

A hypergroup $(H, \cdot)$ is called feebly quasi canonical if it is regular, reversible and satisfies the condition

$$\forall x, a \in H, \forall\{u, v\} \subseteq i_l(x), \forall\{w, z\} \subseteq i_r(x), u \cdot a = v \cdot a, a \cdot w = a \cdot z.$$

Let $(H, *)$ is a hypergroup and $K \subset H$, $K \neq \emptyset$. We say that $(K, *)$ is a *subhypergroup* of H if, for any $x \in K$ we have $K * x = K = x * K$.

2. Hyperaction

In this section we consider the notion of hyperaction of a hypergroup on a nonempty set, extending the definition given by Davvaz [9] in the particular case of polygroups. Some properties such as the generalized orbit-stabilizer theorem, are found.

Definition 2.1. *Let $(H, *)$ be a hypergroup, K a nonempty subset of H. We say that K is invertible to the left if the implication $y \in K * x \Rightarrow x \in K * y$ valid. We say K is invertible if K is invertible to the right and to the left.*

Proposition 2.1. *If $(H, *)$ is a hypergroup such that $E(H) \neq \emptyset$ and K is an invertible subhypergroup of it, then $E(H) \subseteq K$.*

Proof. Suppose that $e \in E(H)$. Since $K \subseteq e * K$, we have $e \in K * K \subseteq K$, because K is an invertible subhypergroup. □

Suppose that H is a hypergroup contain at least one identity element and K is an invertible subhypergroup of H. For all $x, y \in H$ define the relation $\overset{K}{\equiv}_l$

on H as follows:

$$x \overset{K}{\equiv}_l y \Longleftrightarrow x * K = y * K.$$

Proposition 2.2. *The relation $\overset{K}{\equiv}_l$ is an equivalence relation and for all $x \in H$ the equivalence class of x which is denoted by $[x]_l$, is $x * K$ and is called the left generalized generalized coset of K.*

Proof. It is easy to see that $\overset{K}{\equiv}_l$ is an equivalence relation. Suppose that $y \in [x]_l$ is given, so $x * K = y * K$. Since $\emptyset \neq E(H) \subseteq K$, $y * E(H) \subseteq y * K = x * K$. Therefore $y \in x * K$ and hence $[x]_l \subseteq x * K$. Now suppose that $y \in x * K$ is given, so $x \in y * K$ because of invertibility of K. Thus $x * K \subseteq y * K * K \subseteq y * K$. By $y \in x * K$ we have $y * K \subseteq x * K$. Therefore $x * K = y * K$ and hence $x \overset{K}{\equiv}_l y$. So $y \in [x]_l$. □

Remark 2.1. *If K is an invertible subhypergroup of H as the above we can define the equivalence relation $\overset{K}{\equiv}_r$ on H as follows:*

$$x \overset{K}{\equiv}_r y \Longleftrightarrow K * x = K * y.$$

In this way for all $x \in H$ the equivalence class off x that denoted by $[x]_r$ is $K * x$ and it is called the right generalized coset of K. From now on we will consider the hypergroups which have at least one identity element.

Notation 2.1. *Suppose that K is an invertible subhypergroup of H. The number of all left generalized cosets of K in H is denoted by $[H : K]_l$ and the number of all right generalized cosets of K in H is denoted by $[H : K]_r$. If $[H : K]_l = [H : K]_r = n$, then we say n is the index of K in H and denoted by $[H : K]$.*

Theorem 2.2. *Suppose that H is a feebly quasi canonical hypergroup and K is an invertible subhypergroup of H, then*

$$[H : K]_l = [H : K]_r.$$

Proof. Define $\varphi : \{x * K \mid x \in H\} \longrightarrow \{K * x \mid x \in H\}$ by $\varphi(x * K) = K * x'$ for some $x' \in i(x)$. We show that φ is well define. Suppose that $x * K = y * K$, so $y \in x * K$ and therefore there exists $a \in K$ such that $y \in x * a$. By reversibility of H we have $a \in x' * y$ for some $x' \in i(x)$ and hence $x' \in a * y'$ for some $y' \in i(y)$ thus $K * x' = K * y'$. Therefore φ is a well-defined.

As the above we can prove the following implication:

$$\forall x' \in i(x) \text{ and } \forall y' \in i(y), K * x' = K * y' \Longrightarrow x * K = y * K.$$

So φ is one-to-one. It is easy to see that φ is onto and hence φ is an invertible map. Thus $[H : K]_l = [H : K]_r$. □

Definition 2.2. *Let X be a nonempty set and $(H,*)$ be a hypergroup such that $E(H) \neq \varnothing$. A left hyperaction of H on X is a map $\centerdot : H \times X \longrightarrow P^*(X)$ such that:*

(HA1) for all $(a,b) \in H^2$ and for all $x \in X$, $a \centerdot (b \centerdot x) = (a * b) \centerdot x$ such that $A \centerdot Y :\overset{def}{=} \bigcup_{a\in A, y\in Y} a \centerdot y$ for all nonempty subsets A and Y of H and X respectively.

(HA2) for all $x \in X$ and $e \in E(H)$, $x \in e \centerdot x$.

We say X is a hyper H-set and the left hyperaction of H on X is denoted by $(H \mid X)$. Similarly the right hyperaction H on X is defined and is denoted by $(X \mid H)$.

Example 2.1. *Suppose that $(G,\cdot)$ is a group and H is the subgroup of G. Consider $G /\!\!/ H$ as the set of all left generalized cosets of H in G. Define the hyperoperation $\diamond$ on $G /\!\!/ H$ by $xH \diamond yH :\overset{def}{=} \{zH \mid z \in xHy\}$ for all xH and yH in $G /\!\!/ H$. The mapping $\centerdot : G /\!\!/ H \times G \longrightarrow P^*(G)$ defined by $\centerdot(gH,x) :\overset{def}{=} gHx$ is a left hyperaction $G /\!\!/ H$ on G.*

Proof. For all $aH, bH \in G /\!\!/ H$ and $x \in G$ we have:

$$\centerdot(aH, \centerdot(bH,x)) = \centerdot(aH, bHx) = \bigcup_{y\in bHx} aHy = aHbHx;$$

on the other side,

$$\centerdot(aH \diamond bH, x) = \bigcup_{c\in aHb} \centerdot(cH,x) = aHbHx.$$

Consequently the condition (HA1) holds.

For proving the condition (HA2), first we need to find the identities of $G /\!\!/ H$. If $eH \in E(G /\!\!/ H)$, then $xH \in eH \diamond xH \bigcap xH \diamond eH$, which means $xH = zH = z'H$, for some z, z' in eHx and xHe, respectively. Thus we conclude that $e \in H$ and therefore $E(G /\!\!/ H) = \{H\}$. Thus $x \in \centerdot(H,x) = Hx$, for all $x \in G$. □ □

Example 2.2. *Suppose that G is a graph and H the set of all vertices of G. For all h_1 and h_2 in H, consider $path(h_1,h_2)$ the set of all paths contain h_1 and h_2 and $\langle h_1,h_2\rangle$ the set of all vertices of G lie in the paths contain h_1 and h_2. Define the hyperoperation $*$ on H by $h_1 * h_2 :\overset{def}{=} \{h_1,h_2\}$ for all $h_1,h_2 \in H$. Thus $(H,*)$ is a hypergroup. The mapping $\centerdot : H \times H \longrightarrow P^*(H)$ defined by:*

$$h \centerdot v :\overset{def}{=} \begin{cases} \langle h,v\rangle & \text{if } path(h,v) \neq \varnothing, \\ \{v\} & \text{otherwise,} \end{cases}$$

is a left hyperaction of H on H.

Proof. We can easily see that $E(H) = \{H\}$ and $\centerdot(a, \centerdot(b, x)) = \centerdot(a, x) \bigcup \centerdot(b, x) = \centerdot(a * b, x)$, for all $(a, b, x) \in H^3$ and thus the conditions (HA1) and (HA2) hold. $\square$

Example 2.3. *Suppose $(H, *)$ is a hypergroup such that $E(H) \neq \varnothing$. The mapping $\centerdot : H \times H \longrightarrow P^*(H)$ defined by $h \centerdot x :\overset{def}{=} \mathcal{C}(h * x)$, where $\mathcal{C}(h * x)$ is the complete closure of $h * x$ is a left hyperaction of H on H.*

Proof. It is well known that $\mathcal{C}(h * x) = h * x * \omega_H$, for all $(h, x) \in H^2$, where ω_H is the core of the canonical projection φ_H, and therefore $\centerdot(a, \centerdot(b, x)) = \centerdot(a, b * x * \omega_H) = a * b * x * \omega_H * \omega_H = a * b * x * \omega_H = (a * b) * x * \omega_H = \centerdot(a * b, x)$, for all $(a, b, x) \in H^3$.

Now let $e \in E(H)$. Since $x \in e * x$, it follows that $x \in \mathcal{C}(x) \subseteq \mathcal{C}(e * x) = \centerdot(e, x)$. $\square$

Definition 2.3. *Suppose that $(H \mid X)$ and $x \in X$. A generalized orbit of x is denoted by Hx and defined $Hx :\overset{def}{=} \bigcup\limits_{h \in H} h \centerdot x$.*

Definition 2.4. *Suppose that X is a nonempty set, $(H, *)$ is a reversible hypergroup and $\centerdot : H \times X \longrightarrow P^*(X)$ is a left hyperaction of H on X.*

(i) We say $\centerdot$ is a quasi strong left hyperaction and denoted by $(H \mid^{qs} X)$ whenever, for all $(a, b) \in H^2$ and $(x, y) \in X^2$ if $a \centerdot x \cap b \centerdot y \neq \varnothing$, then $x \in (a' * b) \centerdot y$ and $y \in (b' * a) \centerdot x$ for all $a' \in i(a)$ and $b' \in i(b)$.

(ii) We say $\centerdot$ is a strong left hyperaction and denoted by $(H \mid^{s} X)$ whenever, $\centerdot$ is a quasi strong left hyperaction and for all $a \in H, e \in E(H)$ and $x \in X$ if $x \in (a * e) \centerdot x$, then $(a * e) \centerdot x \subseteq e \centerdot x$.

Proposition 2.3. *Suppose that $(H \mid^{qs} X)$ and there exist $x, y \in H$ such that $Hx \cap Hy \neq \varnothing$. Then $Hx = Hy$.*

Proof. Since $Hx \cap Hy \neq \varnothing$, then there exist $a, b \in H$ such that $a \centerdot x \cap b \centerdot y \neq \varnothing$. Thus we have $x \in (a' * b) \centerdot y$ and $y \in (b' * a) \centerdot x$ for all $a' \in i(a)$ and $b' \in i(b)$. Let $\centerdot$ be the left hyperaction of H on X so for all $h \in H$, we have the map $\centerdot_h : X \longrightarrow P^*(X)$ defined by $\centerdot_h(x) :\overset{\text{def}}{=} h \centerdot x$. Therefore for all $h \in H$ we have $h \centerdot x \subseteq (h * a' * b) \centerdot y$ and $h \centerdot y \subseteq (h * b' * a) \centerdot x$ and hence $Hx \subseteq Hy$ and $Hy \subseteq Hx$ and the proof is complete. $\square$

Corollary 2.1. *Suppose that $(H \mid^{qs} X)$. The relation $\smile$ on X defined by:*

$x \smile y$ if and only if x and y lie at the same generalized orbit

is an equivalence relation on X.

Proof. It is clear from the Proposition 2.3. $\square$

Definition 2.5. *Suppose that* $(H\,|^{qs}\,X)$ *and* $x \in X$*. The generalized stabilizer of* x *is denoted by* H_x *and defined:*

$$H_x :\overset{def}{=} \{h \in H \,|\, (h*e) \centerdot x \cup (h'*e) \centerdot x \subseteq e \centerdot x \text{ for all } e \in E(H) \text{ and } h' \in i(h)\}$$

Remark 2.2. *Suppose that* X *is nonempty set and* $(H \mid X)$*. It is easy to see that for all* $(h_1, h_2, h_3) \in H^3$ *we have* $(h_1 * h_2 * h_3 *) \centerdot x = h_1 \centerdot [(h_2 * h_3) \centerdot x]$.

Theorem 2.3. *Suppose that* H *is a feebly quasi canonical hypergroup and* $(H\,|^{qs}\,X)$ *and* $x \in X$*. Then we have:*

(i) for all $h_1, h_2 \in H_x$, $h_1 * h_2 \subseteq H_x$;

(ii) for all $h \in H_x$ *and* $h' \in i(h)$, $h' \in H_x$;

(iii) if H_x *is a nonempty set, then* H_x *is invertible and reversible subhypergroup of* H*.*

Proof. (i) Suppose that $h_1, h_2 \in H_x$ and $h \in h_1 * h_2$. So $h * e \subseteq h_1 * h_2 * e$ and hence by Remark 2.2, we have $(h*e) \centerdot x \subseteq h_1 \centerdot [(h_2 * e) \centerdot x] \subseteq (h_1 * e) \centerdot x \subseteq e \centerdot x$. So $(h * e) \centerdot x \subseteq e \centerdot x$.

By $h \in h_1 * h_2$ and H is a feebly quasi canonical we have $h' \in h'_2 * h'_1$, where $h' \in i(h), h'_1 \in i(h_1)$ and $h'_2 \in i(h_2)$. Thus $h' * e \subseteq h'_2 * h'_1 * e$. Therefor $(h' * e) \centerdot x \subseteq h'_2 * [(h'_1 * e) \centerdot x]$ and hence $(h * e) \centerdot x \subseteq e \centerdot x$. So $h_1 * h_2 \subseteq H_x$.

(ii) The proof is obvious because $h'' * e = h * e$ for all $h'', h' \in i(h')$.

(iii) Suppose that $a, b \in H$ such that $a \in H_x * b$. So there exists $h \in H_x$ such that $a \in h * b$. Since H is reversible, there exists $h' \in i(h)$ such that $b \in h' * a$. By (ii) we have $h' \in H_x$, so $b \in H_x * a$ and hence H_x is invertible to right. Similarly H_x is invertible to left. Reversibility of H_x follows from (ii) and the fact that H is reversible. For the proof H_x is a subhypergroup of H, by (i) it is enough to show that for all $h \in H_x$, $H_x \subseteq h * H_x$ and $H_x \subseteq H_x * h$. Suppose that $h_1 \in H_x$ is given, thus there exists $h_2 \in H$ such that $h_1 \in h * h_2$. Since H_x is invertible, we have H_x is close and hence $h_2 \in H_x$. Therefore $H_x \subseteq h * H_x$ and the proof is complete. □

Remark 2.3. *If* H *is a feebly quasi canonical hypergroup and* $H_x \neq \emptyset$*, then by Theorems 2.2 and 2.3, we have* $[H : H_x]_l = [H : H_x]_r$*.*

Theorem 2.4. *(generalized orbit-stabilizer theorem) Suppose that* H *is a feebly quasi canonical hypergroup and* $(H\,|^{s}\,X)$ *and* $x \in X$*. We have:*

(i) $card(\{h \centerdot x \mid h \in H\}) \geqslant [H : H_x]$ *where* $card(A)$ *is the cardinal number of the set* A*;*

(ii) if H *has scalar identity* e *and for all* $x \in X$, $e \centerdot x = \{x\}$*, then*

$$card(\{h \centerdot x \mid h \in H\}) = [H : H_x].$$

Proof. Define $\psi : \{h \centerdot x \mid h \in H\} \longrightarrow \{a * H_x \mid a \in H\}$ by $\psi(h \centerdot x) = h * H_x$. First we show that ψ is a well define map. For this reason suppose that $h_1 \cdot x = h_2 \cdot x$. Since $(H \mid^s X)$, we have $x \in (h'_2 * h_1) \cdot x$ where $h'_2 \in i(h_2)$. Therefore there exists $l \in h'_2 * h_1$ such that $x \in l \cdot x$ and hence $(l * e) \cdot x \subseteq e \cdot x$ because $l \cdot x \subseteq (l * e) \cdot x$ and $(H \mid^s X)$. Also from $x \in l \cdot x$ we have $e \cdot x \cap l \cdot x \neq \emptyset$ and so $x \in (l' * e) \cdot x$ for all $l' \in i(l)$ and hence

$$(l' * e) \cdot x \subseteq e \cdot x. \tag{1}$$

Suppose that $a \in h_1 * H_x$, so there exists $k \in H_x$ such that $a \in h_1 * k$. Since $l \in h'_2 * h_1$, there exists $h''_2 \in i(h'_2)$ such that $h \in h''_2 * l$ and so $h_1 \in h_2 * l$, because H is a feebly quasi canonical and $h_2 \in i(h'_2)$. Therefore $h_1 * k \subseteq h_2 * l * k$ and hence $a \in h_2 * (l * k)$. Now we show that $l * k \subseteq H_x$ and so $a \in h_2 * H_x$ as desired. Suppose that $s \in l * k$ is given. By Remark 2.2, and $k \in H_x$ we have $(s * e) \cdot x \subseteq (l * e) \cdot x \subseteq e \cdot x$. Let $s' \in i(s)$ since $s \in l * k$ and H is a feebly quasi canonical, we have $s' \in k' * l'$ where $s' \in i(s), k' \in i(k)$ and $l' \in i(l)$. Thus we have

$$\begin{aligned}
(s' * e) \cdot x &\subseteq [(k' * l') * e] \cdot x \\
&\subseteq k' \cdot [(l' * e) \cdot x] && \text{by Remark 2.2.} \\
&\subseteq (k' * e) \cdot x && \text{by equation (1)} \\
&\subseteq e \cdot x. && \text{by Theorem 2.3(ii) and } k \in H_x
\end{aligned}$$

Thus $h_1 * H_x \subseteq h_2 * H_x$. Similarly we can show that $h_2 * H_x \subseteq h_1 * H_x$. Therefore $h_1 * H_x = h_2 * H_x$ and hence ψ is a well define map. It is easy to see that ψ is onto and so $card(\{h \cdot x \mid h \in H\}) \geqslant [H : H_x]$.

(ii) By part (i) it is enough to show that ψ is one-to-one. Suppose that $h_1 * H_x = h_2 * H_x$ since e is an scalar identity, we have $e \in H_x$ and hence $h_2 \in h_1 * H_x$. Thus there exists $k \in H_x$ such that $h_2 \in h_1 * k$ and hence $e \in (h'_2 * h_1) * k$, where $h'_2 \in i(h_2)$. By Remark 2.2, we have $e \cdot x \subseteq (h'_2 * h_1) \cdot x$ and so $x \in (h'_2 * h_1) \cdot x$. Therefore there exists $r \in h'_2 * h_1$ such that $x \in r \cdot x$. Since $(H \mid^{qs} X)$, $r \cdot x \subseteq e \cdot x = \{x\}$ and hence

$$r \cdot x = \{x\}. \tag{2}$$

From $r \in h'_2 h_1$ we have $h_1 \in h''_2 * r$ where $h''_2 \in i(h'_2)$ and since $h_2 \in i(h'_2)$ and H is feebly quasi canonical, then $h''_2 * r = h_2 * r$ and so $h_1 \in h_2 * r$, thus by Remark 2.2 and (2) we have $h_1 \cdot x \subseteq h_2 \cdot x$. From the equation (2) we have $r' \cdot x = (r' * r) \cdot x$ for all $r' \in i(r)$. So $x \in r' \cdot x$ and similarly we have $r' \cdot x = \{x\}$ for all $r' \in i(r)$. Since $r \in h'_2 * h_1$ and H is feebly quasi canonical we have $h_2 \in h_1 * r'$ and as above $h_2 \cdot x \subseteq h_1 \cdot x$. Therefore $h_1 \cdot x = h_2 \cdot x$ and hence ψ is a one-to-one map. □

3. **A construction of hypergroups from hyperactions**

In this section, we give a hyperstructure on the nonempty set X derived from the strongly hyperaction of some hypergroups H on X.

Let $(H \mid X)$ and $\mathbf{Orb}(X) :\overset{\text{def}}{=} \{Hx \mid x \in X\}$ is the set of all orbits in X and C be a choice function on $\mathbf{Orb}(X)$, that is, $C : \mathbf{Orb}(X) \longrightarrow X$ such that $c_x :\overset{\text{def}}{=} C(Hx) \in Hx$. Then we denote the image of C by C_X and call it a class mark of X. For all $x \in X$ the subset $s_{_C}(x)$ of H is defined by $s_{_C}(x) :\overset{\text{def}}{=} \{h \in H \mid e \centerdot x \cap h \centerdot c_x \neq \varnothing \text{ for all } e \in E(H)\}$.

Theorem 3.1. *Suppose that $(H, *)$ is a feebly quasi canonical hypergroup with scalar identity e . If $(H \mid^{qs} X)$, then for all $x \in X$ and $h \in H$, $s_{_C}(h \centerdot x) = h * s_{_C}(x)$ where $s_{_C}(h \centerdot x) = \bigcup\limits_{t \in h \centerdot x} s_{_C}(t)$.*

Proof. let $a \in s_{_C}(h \centerdot x)$ so there exists $t \in h \centerdot x$ such that $a \in s_{_C}(t)$ and hence

$$e \centerdot t \cap a \centerdot c_t \neq \varnothing. \tag{3}$$

also we have:

$$\begin{aligned}
t \in h \centerdot x \Rightarrow\ & e \centerdot t \subseteq (e * h) \centerdot x \\
\Rightarrow\ & a \centerdot c_t \cap (e * h) \centerdot x \neq \varnothing && \text{, by (3)} \\
\Rightarrow\ & a \centerdot c_t \cap h \centerdot x \neq \varnothing && (3.1.1) \\
\Rightarrow\ & Hc_t = Hx && \text{, by Proposition 2.3} \\
\Rightarrow\ & c_t = c_x && (3.1.2) \\
\Rightarrow\ & a \centerdot c_x \cap h \centerdot x \neq \varnothing && \text{, by (3.1.1) \& (3.1.2))} \\
\Rightarrow\ & x \in (h' * a) \centerdot c_x && \text{where } h' \in i(h) \\
\Rightarrow\ & x \in h_1 \centerdot c_x \quad \text{for some} \quad h_1 \in h' * a \\
\Rightarrow\ & e \centerdot x \cap h_1 \centerdot c_x \neq \varnothing && \text{, because } x \in e \centerdot x \\
\Rightarrow\ & h_1 \in s_{_C}(x). && (3.1.3)
\end{aligned}$$

Since $h_1 \in h' * a$ and H is a feebly quasi canonical, $a \in h * h_1$ and by (3.1.3) we have $a \in h * s_{_C}(x)$ and hence $s_{_C}(h \centerdot x) \subseteq h * s_{_C}(x)$. Now suppose that $a \in h * s_{_C}(x)$ so there exists $b \in s_{_C}(x)$ such that $a \in h * b$. Thus $b \in h' * a$ and hence,

$$b \centerdot c_x \subseteq (h' * a) \centerdot c_x. \tag{4}$$

also we have:

$$
\begin{aligned}
b \in s_C(x) &\Rightarrow e \centerdot x \cap b \centerdot c_x \neq \varnothing \\
&\Rightarrow e \centerdot x \cap (h' * a) \centerdot c_x \neq \varnothing && \text{, by (4)} \\
&\Rightarrow e \centerdot x \cap h' \centerdot (a \centerdot c_x) \neq \varnothing \\
&\Rightarrow e \centerdot x \cap h' \centerdot s \neq \varnothing \quad \text{for some} \quad s \in a \centerdot c_x && (3.1.4) \\
&\Rightarrow Hx = Hs && \text{, by Proposition 2.3} \\
&\Rightarrow c_x = c_s.
\end{aligned}
$$

From (3.1.4) we have:

$$
\begin{aligned}
e \centerdot x \cap h' \centerdot s \neq \varnothing &\Rightarrow s \in (h'' * e) \centerdot x && h'' \in i(h') \\
&\Rightarrow s \in (h * e) \centerdot x && \text{because } h \in i(h') \\
&\Rightarrow s \in (e * h) \centerdot x && \text{because } h * e = e * h = \{h\} \\
&\Rightarrow s \in e \centerdot (h \centerdot x) \\
&\Rightarrow s \in e \centerdot t \quad \text{for some} \quad t \in h \centerdot x && (3.1.5) \\
&\Rightarrow e \centerdot s \cap e \centerdot t \neq \varnothing && \text{, because } s \in e \centerdot s \\
&\Rightarrow Hs = Hx && \text{, by Proposition 2.3} \\
&\Rightarrow c_s = c_t.
\end{aligned}
$$

Thus $c_x = c_t$ and by (3.1.4) and (3.1.5) we have $e \centerdot t \cap a \centerdot c_t \neq \varnothing$ and hence $a \in s_C(t)$ where $t \in h \centerdot x$. Therefore $h * s_C(x) \subseteq s_C(h \centerdot x)$ and the proof is complete. □

Theorem 3.2. *Suppose that $(H, *)$ is a feebly quasi canonical hypergroup with scalar identity e(i.e., $e * x = x = x * e$ for all $x \in H$). If $(H \,|^{qs}\, X)$, then the mapping $\circ_c : X \times X \longrightarrow P^*(X)$ defined by $x \circ_c y :\overset{def}{=} s_C(x) \centerdot c_x \bigcup s_C(y) \centerdot c_y$ is a hyperoperation on X and $(X, \circ_c)$ is a hypergroup.*

Proof. First we show that for all $x \in X$, $x \in s_C(x) \centerdot c_x$. For this reason suppose that $x \in X$ is given. Since $c_x \in Hx$, then by Proposition 2.3, $Hx = Hc_x$ and hence there exists $h \in H$ such that $x \in h \centerdot c_x$. Thus $e \centerdot x \cap h \centerdot c_x \neq \varnothing$ and so $h \in s_C(x)$. Therefore we have $x \in s_C(x) \centerdot c_x$. Thus $\{x, y\} \subseteq x \circ_c y$. It is easy to see that "$\circ_c$" is a well define map now we prove "$\circ_c$" is associative. Suppose that x , y and z in X are given so $(x \circ_c y) \circ_c z = \bigcup_{t \in x \circ_c y} (s_C(t) \centerdot c_t) \bigcup s_C(z) \centerdot c_z$ and $x \circ_c (y \circ_c z) = s_C(x) \centerdot c_x \bigcup \bigcup_{s \in y \circ_c z} (s_C(s) \centerdot c_s)$. Let $w \in (x \circ_c y) \circ_c z$ be given if $w \in \bigcup_{t \in x \circ_c y} s_C(t) \centerdot c_t$, then there exists $t \in x \circ_c y$ such that

$$w \in s_C(t) \centerdot c_t. \tag{5}$$

By $t \in x \circ_c y$ we have $t \in s_C(x) \centerdot c_x$ or $t \in s_C(y) \centerdot c_y$. Let $t \in s_C(x) \centerdot c_x$, so

$$\begin{aligned} t \in s_C(x) \centerdot c_x &\Rightarrow s_C(t) \subseteq s_C(x) * s_C(c_x) && \text{, by Theorem 3.1} \\ &\Rightarrow s_C(t) \centerdot c_t \subseteq s_C(x) * s_C(c_x) \centerdot c_x && \text{, because by (5) } , c_t = c_x \\ &\Rightarrow s_C(t) \centerdot c_t \subseteq s_C(x) \centerdot (s_C(c_x) \centerdot c_x) \\ &\Rightarrow s_C(t) \centerdot c_t \subseteq s_C(x) \centerdot (e \centerdot c_x) \\ &\Rightarrow s_C(t) \centerdot c_t \subseteq (s_C(x) * e) \centerdot c_x \\ &\Rightarrow s_C(t) \centerdot c_t \subseteq s_C(x) \centerdot c_x. \end{aligned}$$

Thus by (5), $w \in s_C(x) \centerdot c_x$ and hence $w \in x \circ_c (y \circ_c z)$. Let $t \in s_C(y) \centerdot c_y$ similarly we have $s_C(t) \centerdot c_t \subseteq s_C(y) \centerdot c_y$ and hence by (5), $w \in s_C(y) \centerdot c_y$. Since $w \in s_C(w) \centerdot c_w$, then $w \in \bigcup_{s \in s_C(y) \centerdot c_y} s_C(s) \centerdot c_s \subseteq \bigcup_{s \in y \circ_c z} s_C(s) \centerdot c_s$ and so $w \in x \circ_c (y \circ_c z)$. If $w \in s_C(z) \centerdot c_z$, then $w \in \bigcup_{s \in s_C(z) \centerdot c_z} s_C(s) \centerdot c_s \subseteq \bigcup_{s \in y \circ_c z} s_C(s) \centerdot c_s$ and hence $w \in x \circ_c (y \circ_c z)$. Therefore $(x \circ_c y) \circ_c z \subseteq x \circ_c (y \circ_c z)$ and similarly by above we can prove $x \circ_c (y \circ_c z) \subseteq (x \circ_c y) \circ_c z$. Thus "$\circ_c$" is associative and since for all $x \in X$, $X \circ_c x = x \circ_c X = X$, then $(X, *)$ is a hypergroup. □

Example 3.1. *Let the hyperaction* $\mathbb{Z}_2 = \{[0], [1]\}$ *(the cyclic group of order 2) on* $X = \{a, b, c, d, f\}$ *be as follows:*

$$[0] \centerdot a = [1] \centerdot b = \{a\}, [0] \centerdot b = [1] \centerdot a = \{b\}$$
$$[0] \centerdot c = [0] \centerdot d = [1] \centerdot f = \{c, d\}, [0] \centerdot f = [1] \centerdot c = [1] \centerdot d = \{f\}.$$

Now let $C_X = \{b, d\}$ *be a classes mark of* X*, then we have:*

$S(a) = S(f) = \{[1]\}, S(b) = S(c) = S(d) = \{[0]\}$ *and the commutative hypergroup* $(X, \circ_c)$ *associated from the hyperaction is as the following figure:*

$\circ_c$	a	b	c	d	f
a	$\{a\}$	$\{a, b\}$	$\{a, c, d\}$	$\{a, c, d\}$	$\{a, f\}$
b		$\{b\}$	$\{b, c, d\}$	$\{b, c, d\}$	$\{b, f\}$
c			$\{c, d\}$	$\{c, d\}$	$\{c, d, f\}$
d				$\{c, d\}$	$\{c, d, f\}$
f					$\{f\}$

Figure 1. The hyperoperation of X

4. Generalized state hypergroups

In the papers [15], [8] there are described construction of some hyperstructure on sets of words formed the given input alphabets and on the state sets of corresponding automata. In this section, we assign a commutative hypergroup to any nondeterministic automaton with out inputs. In accordance with [14] and other publications, by an nondeterministic automata we mean a third $\mathbb{A} = (S, A, \delta)$, where S, A are arbitrary sets ($A \neq \emptyset$), which are called set of states (or a state set), a set of input symbols (or input alphabet) and $\delta : S \times A^* \to P(S)$ is a mapping which satisfies these two conditions: $\delta(s, e) = s$ for any state $s \in S$ and $\delta(s, ab) = \delta(\delta(s, a), b)$ for any state $s \in S$ and any pair of words $a, b \in A^*$.

Proposition 4.1. *Suppose that S is a nonempty set, $(H, *)$ is a hypergroup with the scalar identity e and $\centerdot : S \times H \longrightarrow P^*(S)$ is a right hyperaction of H on S such that $s \centerdot e = s$ for all $s \in S$. Then the third $\mathbb{H} = (S, H, \delta)$ is a nondeterministic automata, where $\delta(s, h_1h_2...h_k) = s \centerdot (h_1 * h_2 * ... * h_k)$ for all $(h_1, h_2, ..., h_k) \in H^k$ and $k \geq 1$.*

Theorem 4.1. *Let $\mathbb{H} = (S, H, \alpha)$ be a nondeterministic automata. For any $(x, y) \in S^2$, we define*

$$x \bullet y = \alpha(x, H^*) \cup \alpha(y, H^*),$$

where $\alpha(z, H^*) = \bigcup\{\alpha(z, h) | h \in H^*\}$. Then $(S, \bullet)$ is a commutative hypergroup, called the generalized state hypergroup of $\mathbb{H}$.

Proof. It is obvious that $x \bullet y = y \bullet x$, for any $x, y \in X$. Now we prove the associativity: $(x \bullet y) \bullet z = x \bullet (y \bullet z)$, for any $x, y, z \in X$. Let $u \in (x \bullet y) \bullet z$; there exists $t \in \alpha(x, H^*) \cup \alpha(y, H^*)$ such that $u \in \alpha(t, H^*) \cup \alpha(z, H^*)$. If $u \in \alpha(z, H^*)$, then $u \in \bigcup\{\alpha(v, H^*) \mid v \in \alpha(y, H^*) \cup \alpha(z, H^*)\} \subset x \bullet (y \bullet z)$. If $u \in \alpha(t, H^*)$, with $t \in \alpha(x, H^*)$ for example, then there exist $h_t, h_u \in H$ such that $t \in \alpha(x, h_t)$ and $u \in \alpha(t, h_u)$. It follows that $u \in \alpha(\alpha(x, h_t), h_u) = \alpha(x, h_th_u) \subset \alpha(x, H^*) \subset x \bullet (y \bullet z)$. Thus we obtain the first inclusion and similarly we obtain also the second inclusion.

It remains to prove the reproducibility: $x \bullet S = S = S \bullet x$, for any $x \in S$. Indeed, for any $x, y \in S$, there exists $z = y \in S$ such that $y \in x \bullet z$ and therefore we can conclude that $(S, \bullet)$ is a commutative hypergroup. □

Remark 4.1. *If $\mathbb{H} = (S, H, \alpha)$ is a nondeterministic automata such that $|\alpha(s, h)| = 1$, then the hypergroup $(S, \bullet)$ is called the state hypergroup of $\mathbb{H}$.*

Proposition 4.2. *Every generalized state hypergroup $(S, \bullet)$ is a quasi-ordering hypergroup (i.e., $x \in x \bullet x = x \bullet x \bullet x$ for any $x \in S$).*

Acknowledgements

The authors are extremely grateful to the referees for giving them many valuable comments and helpful suggestions which helps to improve the presentation of this paper.

This research is partially supported by a grant of the National Natural Science Foundation of China (60875034); a grant of the Natural Science Foundation of Education Committee of Hubei Province, China (D20092901) and also the support of the Natural Science Foundation of Hubei Province, China (2009CDB340).

REFERENCES

[1] R. Ameri, M.M. Zahedi, *Fuzzy subhypermodules over fuzzy hyperrings*, In: Sixth International Congress on AHA, 1996, 1-14.

[2] Z. Bavel, J. Grzymala, K. Soo Hong, *On the conectivity of the product of automata*, Fundam. Informatics 7(2) (1984), 225-265.

[3] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, (1993).

[4] P. Corsini, *Hypergraphs and Hypergroups, Algebra Univers*, 35 (1996), no.4, 548-555.

[5] P. Corsini, I. Cristea, *Fuzzy grade of i.p.s. hypergroups of order* 7, Iranian Journal of Fuzzy Systems, 1 (2004), no.2, 15–32.

[6] P. Corsini, I. Tofan, *On fuzzy hypergroups.* Pure Math. Appl. 8(1997), 29-37.

[7] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory, in: Advances in Mathematics*, Kluwer Academic Publishers, Dordrecht, (2003).

[8] J. Chvalina, L. Chvalinova, *State hypergroups of automata*, Acta Mathematica Universitatis Ostraviensis, 4(1996), 105-119.

[9] B. Davvaz, *On Polygroups and Weak Polygroups*, South. Asian Bull. Math. 25 (2001), 87-95.

[10] W. Dorfler, *Halbgruppen und Automaten*, Rend. Sem. Mat. Univ. Padova, 50(1973), 1-18.

[11] I. Cristea, *Hyperstructures and fuzzy sets endowed with two membership functions*, Fuzzy Sets and Systems, 160 (2009), 1114-1124.

[12] Ath. Kehagias, *An example of L-fuzzy join space*, Rendiconti di Circolo Matematico di Palermo. 52(2002), 322-350.

[13] V. Leoreanu-Fotea, *A new type of fuzzy n-ary hyperstructures*, Information Sciences: an International Journal, 179 (2009), no.15, 2710-2718.

[14] J.N. Mordeson, M.S. Malik, *Fuzzy Commutative Algebra*, World Scientific Publishers, (1998).

[15] C. Massouros, J. Mittas, *Languages-automata and hypercompositional structure*, Alg. Hyperstructure and Appl. (T. Vougiouklis, ed.) Proc. 4th Inter. Congress Xanthi, Greece 1990, World Scientific, Singapore 1991, 137-147.

[16] F. Marty, *Sur une generalization de la notion de group*, Eight Congress Math. Scandenaves, Stockholm, (1934), 45-49.

[17] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Palm Harbor, FL, (1994).

[18] M.M. Zahedi, M. Bolurian and A. Hasankhani, *On polygroups and fuzzy subpolygroups*, J. Fuzzy Math. 3(1995), 1-15.

[19] J. Zhan, B. Davvaz, K.P. Shum, *A new view of fuzzy hypermodules*, Acta Math. Sin.(Engl. Ser.), **23**(2007), no.8, 1345-1356.

[20] J. Zhan, B. Davvaz, K.P. Shum, *A new view of fuzzy hypernear-rings*, Inform. Sci., 178 (2008), 425-438.