THE PRIME ORDER CAYLEY GRAPH

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Let S be the set of prime order elements of the group G. In this paper we introduce the prime order Cayley graph of the group G relative to S. The structure of the prime order Cayley graph associated to the certain cyclic groups and a dihedral group is discussed under special conditions. Moreover, it is proved that the prime order Cayley graph of an abelian group G is planar if and only if $G \cong \mathbb{Z}_{2^n}, \mathbb{Z}_{3^n}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6, \mathbb{Z}_{2^n.3}, \mathbb{Z}_{2.3^n}$

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1. Introduction

The algebraic graph theory involving the use of group theory and the study of graph. Recently mathematician try to assign a graph to an algebraic structure. They hope to use the advantage of graph properties for the algebraic structures and vice versa.

Study of Cayley graphs that their properties related to the structure of the group is one of the interesting topics in this area. Cayley graph was considered for finite groups by Cayley in 1878 to explain the concept of abstract groups which are generated by a set of generators in Cayley's time. Later, many similar researches about the Cayley graph have been done by some authors for instance see [1, 2].

Let G be a finite group and $S \subseteq G$ be a subset. The corresponding Cayley graph Cayley(G, S) has the vertex set equal to G. Two vertices $g, h \in G$ are joined by a directed edge from g to h if and only if there exists $s \in S$ such that $g = sh$. Each edge is labeled to denote that it corresponds to $s \in S$. A Cayley graph Cayley (G, S) is connected if and only if $G = \langle S \rangle$, so that Cayley $(\langle S \rangle, S)$ is a component of Cayley(G, S).

The importance of the order of the elements of the groups is the subject which is clear for every group theorist. We introduce the prime order Cayley graph which is related to the elements of prime order in a group. It is a Cayley graph associated to a group G, such that S is the set of prime order elements of G. We denote this

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graph by Cayley(G, S). Under this assumption we may treat Cayley (G, S) as an undirected graph because $S = S^{-1}$. Moreover as S does not contain the identity, so that Cayley (G, S) does not contain any loops. We present the general properties of Cayley(G, S), where G is an elementary abelian p-group, non-nilpotent group of order $p^n q$, it is isomorphic to the simple group A_5, $G \cong \mathbb{Z}_{p^n}, \mathbb{Z}_{pq}, \mathbb{Z}_{pq^n}, \mathbb{Z}_{pq_r}, \mathbb{Z}_{\Pi_{i=1}^n pi}$ and D_{2n}, where n is a non-negative integer and p, q, p_i are prime numbers. The diameter, girth and clique, chromatic and independent numbers of some them are found. The planarity of the prime order Cayley graph associated to the groups of order less than 16 is verified. We observe that the prime order Cayley graph associated to an abelian group is planar if and only if $G \cong \mathbb{Z}_{2^n}, \mathbb{Z}_{3^n}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6, \mathbb{Z}_{2^n.3}, \mathbb{Z}_{2.3^n}$.

2. Preliminary notions

In fact this paper is combination of two fields of graph theory and group theory. Therefore, in this section we present some notions which are useful in sequel from these two sights.

We consider simple graphs which are undirected, with no loops or multiple edges. The degree of a vertex *v* in Γ is the number of edges incident to *v*. We denote it simply by deg(v). A simple graph of order n for which every two vertices are adjacent is called a complete graph and is denoted by K_n. A subset *X* of the vertices of Γ is called a clique if the induced subgraph on *X* is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by ω(Γ). A k-vertex coloring of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The vertex chromatic number χ(Γ) of a graph Γ, is the minimum k for which Γ has a k-vertex coloring. A subset S is called an independent set of the graph Γ if no two vertices of S are adjacent in Γ. The number of vertices in a maximum independent set is called independence number of Γ and is denoted by α(Γ). A Hamilton cycle of Γ is a cycle that contains every vertex of Γ. A graph which contains a Hamilton cycle is called Hamiltonian. If Γ is a graph such that each vertex has equal number of neighbors, then it is a regular graph. A graph is said to be embeddable in a plane or planar, if it can be drawn in the plane so that its edges intersect only at their ends. Throughout the paper, all the notations and terminologies about the graphs are found in [3, 4] and for more details one can refer to these references.

Let p be a prime number. A group G is called a p-group if every element g of G has order p^n, $n \geq 0$. Moreover, a finite group G is a p-group if and only if G has order p^m for some non-negative integer m. An abelian group of exponent p is called an elementary abelian p-group. An elementary abelian p-group can be considered as a direct sum of cyclic groups of prime order p. We denote a cyclic group of order k by Z_k.

3. Main results

Let G be a group and S be the set of prime order elements of G. Consider the Cayley graph Cayley (G, S) associated to the group G relative to S. We call it prime order Cayley graph.

Example 3.1. *In this example we present some groups such that its prime order Cayley graphs are complete.*
(i) It is clear that if $|G| = p$ *, then Cayley (G, S)=*K_p *,where p is a prime number.*
(ii) If G is an elementary abelian p-group of order P^{α}*, then Cayley (G,S)* $=K_{p^{\alpha}}$ *.Thus prime order Cayley graph of an elementary abelian p-group is connected.*
(iii) Suppose G is a group such that order of its elements are not composite, then Cayley (G, S) is a complete graph. A finite group having all (nontrivial) elements of prime order if it is a p-group of exponent p or a non-nilpotent group of order $p^n q$ *or it is isomorphic to the simple group* A_5 *, where n is a non-negative integer and p, q are prime numbers (see[5]). Consequently prime order Cayley graph of the dihedral group of order 2p is an example, where p is an odd prime number.*

If x is an element of order p, then deg $(x) \geq p-1$, where p is a prime number. Suppose y is an element of composite order. Then deg $(y) \geq m$, where m is the number of primes which are appear in |y|.

Proposition 3.2. *Let G be a group.*
(i) If there are k distinct prime numbers greater than 2 which divides the order of the group, then at least k distinct cycles exist in the graph Cayley(G, S).
(ii) If p divides the order of G, then Cayley(G, S) is not planar, where $p \geq 5$ *is a prime number.*

Proof. (i) For each prime that divides the order of G, there is an element of that order. Since all the powers of such an element are adjacent, we have a cycle by these powers.
(ii) There is an element of order p and all its power are adjacent. Thus K_5 is induced subgraph of Cayley(G, S). □

Nathanson [6] open the way to a new class of graphs, namely, arithmetic graphs. An arithmetic graph is the graph whose vertex set is the set of first n positive integers 1, 2, 3,…,n and two vertices x and y are adjacent if and only if

$x+y \equiv s$, (mod n) where $s \in S$. The prime order element Cayley graph of cyclic groups are kind of arithmetic graphs.

Proposition 3.3. *Let G be a cyclic group of order p^n, p is a prime number, n>1. Then for Cayley(G, S) we have,*

(i) Cayley(G, S) is $(p-1)$-regular.

(ii) Cayley(G,S) is not connected. Cayley(G,S) has p^{n-1} complete components each of them contains p vertices. In particular, $\langle S \rangle$ is one of its component.

(iii) Cayley(G, S) is not planar except for $G \cong Z_{2^n}$ and $G \cong Z_{3^n}$.

(iv) Cayley $(Z_{p^n}, S) = \bigcup (p^{n-1}) K_p$.

(v) ω (Cayley(G, S)) = χ (Cayley(G, S))=p and α (Cayley(G, S))=p^{n-1}, where ω, χ and α are clique, chromatic, independent numbers of the graph.

Proof. (i) It is obvious that $S = \{\overline{p^{n-1}}, 2\overline{p^{n-1}}, \ldots, (p-1)\overline{p^{n-1}}\}$.

(ii) Since $\sum_{i=1}^{p-1} m_i \left(ip^{n-1}\right) \not\equiv \overline{1}$ (mod p^n), we observe that $\overline{1} \in \mathbf{Z}_{p^n}$ is not generated by S, where m_i are integers. Thus Cayley(G, S) is not connected. The rest is clear.

(iii) It is clear by part (ii), if $p \geq 5$ then K_5 is induced subgraph of Cayley (G,S).

(iv) and (v) follows by (ii). □

In the following proposition we present some properties of Cayley ($\mathbf{Z}_{pq}$, S).

Proposition 3.4. *Let G be a cyclic group of order pq, where p and q are distinct prime numbers. Then for Cayley(G, S) we have,*

(i) Cayley (G, S) is $(p+q-2)$-regular.

(ii) The elements of order r are adjacent, where r is a prime number.

(iii) The elements of orders p and q are not adjacent.

(iv) If x is a generator, then it joins to k, where $k = tp+1$ or $sq+1$ and $1 < t \leq q-1$, $1 < s \leq p-1$, t, $s \in N$.

(v) Cayley (G, S) is connected. Moreover, diam (Cayley(G, S))=2 and girth (Cayley(G, S))≤ 4. In particular, $G = \langle S \rangle$.

(vi) Cayley(G, S) is not planar except Cayley (Z_6, S).

Proof. (i) If $G \cong \mathbf{Z}_{pq}$, then $S = \{\overline{p}, 2\overline{p}, \ldots, (q-1)\overline{p}, \overline{q}, 2\overline{q}, \ldots, (p-1)\overline{q}\}$. Therefore $|S| = p+q-2$, and Cayley(G, S) is $(p+q-2)$-regular.

(ii) It is clear.

(iii) If $a, b \in \mathbf{Z}_{pq}$ of orders p and q respectively, then $|a-b| = pq$ and so a and b are not adjacent.

(iv) Let y be an element which is adjacent to x. As x is a generator, we have $y = kx$ and their adjacency implies that $(pq, k-1) = p$ or q. Hence the result is clear.

(v) Suppose $a, b \in V$ (Cayley(G, S)) are not adjacent. If $|a|$ and $|b|$ are distinct prime numbers, then both join 0 so $d(a,b) = 2$. Assume a and b are two non-adjacent generators. Then by (iv) there is a vertex that joins both. That means $d(a,b) = 2$. Hence diam (Cayley(G, S)) = 2. If there exist two elements of prime order which are adjacent, then it is clear that both join 0. Thus we have a triangle. But if S contains just two elements of two different prime orders, then these two prime order elements are not adjacent. This means there is no prime order elements which join. These two prime order elements join to zero and a generator. Thus we have a square. This happens for Z_6 (see Figure (1)).

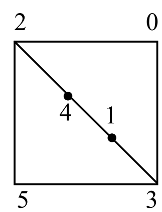

Fig. 1. Cayley($\mathbf{Z}_6$, S)

(vi) Easily one can see Cayley ($\mathbf{Z}_6$, S) is planar (see Figure (1)). Therefore we consider Cayley (Z_{pq}, S), where p or q are greater or equal than 5. By (i) we deduce the number of elements of order p or q are more than 4. Thus these elements and 0 form K_5 as induced subgraph of Cayley (Z_{pq}, S). Hence Cayley (Z_{pq}, S) is not planar whenever p or q are greater or equal than 5 . □

Proposition 3.5. *Let G be the cyclic group of order pq^n, n>1. Then*

(i) Cayley(G, S) is (p + q-2)-regular.

(ii) Cayley(G, S) is not connected and union of q^{n-1} isomorphic components of size pq.

(iii) The components of Cayley($Z_{2.3^n}$, S) and Cayley($Z_{2^n.3}$,S) are isomorphic to the graph in Figure (2), the first and second Cayley graphs have 3^{n-1} and 2^{n-1} components, respectively.

(iv) Cayley(G, S) is not planar except Cayley($Z_{2.3}{}^n$, S) and Cayley($Z_2{}^n{}_{.3}$,S).

Proof. (i) There are p-1 elements of order p and q-1 elements of order q, $S = \{q^n, 2q^n, ..., (p-1)q^n, pq^{n-1}, 2pq^{n-1}, ..., (q-1)pq^{n-1}\}$.

(ii) Since G is not generated by S, Cayley(G, S) is not connected. The identity element, elements of order p, q and elements of order pq form a component. The other components are isomorphic to this component.

(iii) and (iv) follows by the second part and Proposition 3.2 . □

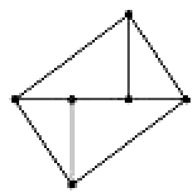

Fig. 2.

Proposition 3.6. *Let G be a cyclic group of order pqr such that p<q<r. Then*

(i) Cayley(G, S) is $(p+q+r-3)$-regular.

(ii) If there are k, k', k'' such that $k'q = k - k''p$, then the elements of order pq and p are adjacent, where k, k', k'' are integers satisfies $(k, pq) = 1$, $(k', p) = 1$ and $(k'', q) = 1$. For instance, the elements of order pq and p are adjacent, whenever $p = 2$ and $q = 3$.

(iii) If $k''q = k - k'p$, then the elements of order pq and q are adjacent, where k, k', k'' are integers such that $(k, pq) = 1$, $(k', q) = 1$ and $(k'', p) = 1$. Suppose $p = 2$ and $q = 3$. Then there are x_1, x_2 and y_1, y_2 elements of order 3 and 6, respectively. Moreover x_i and y_i are adjacent but x_i does not join y_i, i, j=1, 2, $i \neq j$.

(iv) The elements of order pq and r are not adjacent.

(v) The elements of order pr and p are adjacent whenever there are integers k, k' and k'' such that $k - k''p = k'r$, where $(k, pr) = 1$, $(k', p) = 1$ and $(k'', r) = 1$.

(vi) The elements of order pr and q are not adjacent.

(vii) The elements of order pr and r are adjacent whenever there are integers k, k' and k'' such that $k - k'p = k''r$, where $(k, pr) = 1$, $(k', r) = 1$ and $(k'', \mathrm{p}) = 1$.

(viii) The elements of order qr and p are not adjacent.

(ix) Suppose there are integers k, k' and k'', such that $k - k''q = k'r$, where $(k, qr) = 1$, $(k', q) = 1$ and $(k'', r) = 1$. Then the elements of order qr and q are adjacent.

(x) The elements of order qr and r are adjacent whenever there are integers k, k' *and* k'' *such that* $k - k'q = k''r$, *where* $(k, qr) = 1$, $(k', r) = 1$ *and* $(k'', q) = 1$.

(xi) The elements of prime order are not adjacent to the generators.

(xii) If x, y *are the generators and* $x = kpq + y, k'pr + y,$ *or* $k''qr + y$, *then they are adjacent, where* $(k, r) = 1, (k', q) = 1$ *and* $(k'', p) = 1$.

Proof. (i) We can observe that $x = kqr, y = k'pr$ and $t = k''pq$ is of orders p, q and r respectively, where $(k, p) = 1$, $(k', q) = 1$ and $(k'', r) = 1$. It is enough to count such elements. For instance the possible cases for x are $qr, 2qr, ..., (p-1)qr$. Therefore we have $p-1$, $q-1$ and $r-1$ elements of order p, q and r. Hence the assertion is clear.

(ii) kr and k' qr are elements of order pq and p respectively, where $(k, pq) = 1$ and $(k', p) = 1$. Clearly the order of $kr - k'\ qr$ is not p and r. It is possible that $|kr - k'qr| = q$ this means $kr - k'qr = k''qr$. For instance, if $q = p + 1$, then it is possible that the order of $r - qr$ become q and consequently these two vertices are adjacent.

(iii) Similar to the (ii) the first part follows. Let G=$\mathbf{Z}_{6r}$, where r is a prime number greater than 3. Clearly there are two elements of order 3, $x_1 = 2r$ and $x_2 = 4r$. Moreover there are just two elements $y_1 = 5r$ and $y_2 = r$ of order 6. Hence the assertion follows.

(iv) Follows immediately.

(v), (vi), (vii), (viii), (ix) and (x) are deduced similar to the previous parts.

(xi) Let x be a generator. Then $|x|=$ pqr on the other hand $|x|= |\overline{1}|/(|\overline{1}|, x)$.

Therefore $(pqr, x) = 1$. Consider an element of order p, say kqr, where $(k, p) = 1$. Thus $|kqr - x|$ is not p, q or r as x does not have factors rq, rp or pq, respectively.

(xii) It is clear that if $x = kpq + y, k'pr + y$ *or* $k''pq + y$, then order of $x - y$ is r, q or p, respectively. □

Similar result can be proved for a cyclic group G of order $\Pi_{i=1}^{n} p_i$, where p_i's are distinct prime numbers $1 \le i \le n$. For instance, $S = (P_1 \cup P_2 \cup ... \cup P_n) - \{e\}$ and Cayley(G, S) is $\left(\sum_{i=1}^{n}(p_i - 1)\right)$-regular, where P_i are Sylow p_i-subgroups of G. The adjacency in Cayley(G, S) is similar to the graph which is clarify in Proposition 3.6.

Example 3.7. *In this example we present some groups and associated prime order Cayley graphs.*

(i) Cayley(S_3, S) is complete 5-regular graph K_6 ,where S_3 is symmetric group of order 6 and S = {(1 2), (1 3), (2 3), (1 2 3), (1 3 2)}.

(ii) Let $D_8=\langle a, b : a^4 = b^2=1, a^b = a^{-1} \rangle$ be dihedral group of order 8. Clearly Cayley (D_8, S) is union of complete 2-partite graph (with 4 vertices in each part) and the edges $\{1, a^2\}$, $\{a, a^3\}$, $\{b, a^2b\}$, $\{ab, a^3b\}$ where $S = \{ a^2, b, ab, a^2b, a^3b \}$ (see Figure (3)). Moreover, Cayley(D_8, S).

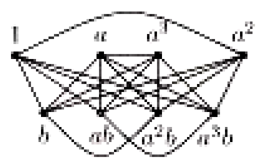

Fig. 3.

*is a Hamiltonian graph. The cycle which pass through vertices $1, a^2, a^3b, a^3, ab, a, a^2b, b, 1$ is a Hamiltonian cyc*le.

Proposition 3.8. *Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ be dihedral group of order 2n, where $n \geq 4$.*

(i) If $n = \Pi_{i=1}^{k} p_i$, then Cayley (D_{2n}, S) is $\left(n + \sum_{i-1}^{k}(p_i - 1)\right)$ -regular, where P_i are distinct prime numbers $1 \leq i \leq k$.

(ii) Cayley (D_{2n}, S) is a connected graph. Moreover diam (Cayley(D_{2n}, S))=2 and girth (Cayley(D_{2n}, S))=3.

(iii) Let $n = \Pi_{i=1}^{k} p_i^{\alpha_i}$, where p_i are distinct prime numbers. Then $\{a^j b : 0 \leq j \leq n-1\} \cup \{a^s : s = P_1^{\alpha_1} ... P_{s-1}^{\alpha_{s-1}} p_s^{\alpha_s - 1} P_{s+1}^{\alpha_{s+1}} ... P_k^{\alpha_k}\} \subseteq S$. Moreover $|S| \geq n+k$ and also we have S is the set of all elements as follows $\{a^j b : 0 \leq j \leq n-1\} \cup \{a^s : s = (P_1^{\alpha_1} ... P_{s-1}^{\alpha_{s-1}} P_s^{\alpha_s - 1} P_{s+1}^{\alpha_{s+1}} ... P_k^{\alpha_k})^{\gamma}\}$, where $1 < s \leq n-1$.

(iv) Cayley(D_{2n}, S) is union of complete 2-partite graph and edges $\{a^i b, a^j b\}$, $\{e, a^t\}$ and $\{a^i, a^j\}$ such that $a^t, a^{i-j} \in S$. Furthermore, there are n vertices in each part $\{a^i b : 0 \leq i \leq n-1\}$ and $\{a^j : 0 \leq j \leq n-1\}$.

(v) Cayley(D_{2n}, S) is not planar.

(vi) As mentioned in (iv) consider two parts in the graph Cayley(D_{2n}, S). If we use c colors for coloring of one part, then we require c different colors for the second part. Moreover, the chromatic number of Cayley(D_{2n}, S) is an even number.

(vii) Cayley(D_{2n}, S) is Hamiltonian graph.

Proof. (i) It is enough to count the prime order elements in the cyclic subgroup $\langle a \rangle$. The elements of prime order in the group $\langle a \rangle$ belongs to Sylow p_i-subgroups and since it is abelian they are normal and unique for each prime p_i. As $|P_i| = P_j$ the assertion is clear.

(ii) Since b, ab$\in$S we deduce that bab=$a^{-1} \in \langle S \rangle$. Hence b, a$\in \langle S \rangle$ and so G=$\langle S \rangle$ which implies Cayley(D_{2n}, S) is a connected graph. Suppose x and y are two vertices which are not adjacent. If both belong to S, then both join to the identity element and d(x, y)=2. If order of x and y are not of prime number, then they are powers of a. Let $x = a^i$ and $y = a^j$ .It is clear that $a^i b a^{-i} = a^{2i} b$ and $a^j b a^{-i} = a^{j+i} b$ so a^i and a^j join $a^i b$. Thus diam(Cayley(D_{2n},S))=2. Further {1,$a^{n/2}$,b} is a cycle, where n is an even number. If n is an odd number, then {1,a^t,b} is a cycle, where t is such that the ratio n/(n, t) is a prime number.

(iii) It is clear that $a^j b,\ 0 \le j \le n-1$ are of order 2 and a^s is of order p_s where $S = P_1^{\alpha_1} ... P_{s-1}^{\alpha_{s-1}} P_{s-1}^{\alpha_s - 1} P_{s+1}^{\alpha_{s+1}} ... P_k^{\alpha_k}$ .The other elements of D_{2n} are not of prime order. The rest follows clearly.

(iv) It is obvious.

(v) Since $n \ge 4$ by the previous part Cayley(D_{2n}, S) has $K_{3,3}$ as its induced subgraph.

(vi) Clearly follows by presentation of D_{2n} and finiding the adjacent vertices in each part.

(vii) Since degree of each vertices is more than n+k, Cayley(D_{2n}, S) is Hamiltonian by Dirac theorem. □

Proposition 3.9. *Let D_{2n} be a dihedral group of order 2n, where $n \ge 4$ is an even integer. Then*

(i) For Cayley(D_{2n}, S), |S|$\ge$n+1. Moreover, |S|=n+1 where D_{2n} is dihedral group such that n=2^{m-1}.

(ii) ω(Cayley(D_{2^m} , S)) = χ(Cayley(D_{2^m} , S)) = 4.

Proof. (i) Suppose $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$. It is clear that $\{a^j b : 0 \le j \le n-1\} \cup \{a^{n/2}\} \subseteq S$. Hence the first part is clear. Since $(2^{m-1}, i) = 2^{m-2}$ whenever $i = 2^{m-2}$ we conclude that $|a^{2^{m-2}}| = 2$ and S is $\{a^j b : 0 \le j \le 2^{m-1} - 1\} \cup \{a^{2^{m-2}}\}$.

(ii) We use two different colors for identity element and $a^{2^{m-2}}$. The other powers of a can be colored by these 2 colors. Some of them join to 1 and some of them not so we use the suitable color out of these two colors. Thus here we use 2 colors. Moreover for $b, ab, a^2b, ..., a^{2^{m-1}}b$ we use two other different colors. Therefore χ(Cayley(D_{2^m} , S))=4. As ω(Cayley(D_{2^m} , S)) $\leq \chi$(Cayley(D_{2^m} , S))=4 and vertices 1, $a^{2^{m-2}}, b, a^{2^{m-2}}b$ form K_4 ,the assertion follows . □

Cayley(G, S) is $\left(\sum_{i-1}^{k} n_{pi}(p_i - 1)\right)$ -regular, for a non-abelian group of order $\Pi_{i=1}^{k} p_i$,where $p_i(1 \leq i \leq k)$ are distinct prime numbers and n_{p_i} is the number of Sylow p_i-subgroups.

Let G be a nilpotent group of order $\Pi_{i=1}^{k} p_i$,where $p_i(1 \leq i \leq k)$ are distinct prime numbers. Then Cayley(G, S) is connected. Since G is generated by its prime order elements the assertion follows.

Proposition 3.10. *Let G be a simple group which contains an element of order p. Then Cayley(G, S) is connected*.

Proof. By the graph definition S contains all the elements of prime order. Since it contains an element of prime order p, G is generated by all the elements of order p (see[7, Proposition 2.5]) and consequently by S. Hence the assertion is clear.□

Every non-abelian finite simple group has even order, hence contains an involution. Thus its prime order Cayley graph is connected.

The alternating groups A_n for $n \geq 5$ are generated by involutions. Therefore Cayley(G, S) is connected.

Theorem 3.11. *Let G be a group. Then*

(*i) Suppose G is a group of order less or equal than 15. Then Cayley(G, S) is a planar graph if and only if* $G \cong \mathbf{Z}_{2^3}, \mathbf{Z}_{3^2}, \mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_6, \mathbf{Z}_{2^2.3}, \mathbf{Z}_{2.3}, \mathbf{Q}_8$.

(ii) If G is an abelian group. Then Cayley(G, S) is a planar graph if and only if $G \cong \mathbf{Z}_{2^n}, \mathbf{Z}_{3^n}, \mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_6, \mathbf{Z}_{2^n.3}, \mathbf{Z}_{2.3^n}$.

Proof. Suppose Cayley(G, S) is a planar graph. By the second part of Proposition 3.2 we deduce that the order of G is $2^{\alpha}.3^{\beta}$, where α, β are non-negative integers. If $\alpha = 0$ or $\beta = 0$, then $G \cong Z_{2^n}, Z_{3^n}$ by third part of Proposition 2.3 or $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Now let α or β be greater than one. If G is a non-nilpotent group of order 2.3^{β} or $2^{\alpha}.3$ then all its elements are of prime order by [5]. Therefore all its elements are join and since $|G| \geq 6$ we have K_5 as induced subgraph of Cayley(G,S) so they are not planar.

Assume G is of order 2.3^{β} or $2^{\alpha}.3$ and nilpotent. Initially suppose $|G| = 2.3^{\beta}$. There are one Sylow 3-subgroup and 3^{β} Sylow 2-subgroup. Thus we have 3^{β} elements of order 2 which are adjacent. As Cayley(G, S) is planar $\beta = 1$ and $|G| = 6$. G is not symmetric group of order 6 because its prime order Cayley graph is not planar or we can say S_3 is not nilpotent. If $G \cong Z_6$, then Cayley(G,S) is planar. If $|G| = 2^{\alpha}.3$, then the number of Sylow 2-subgroups of G, n_2, are 1 or 3 and the number of Sylow 3-subgroups, n_3 , are 2^{α} or 1, respectively. If $n_2 = 1$ and $n_3 = 2^{\alpha}$ then we have $2^{\alpha+1}$ elements of order 3 which are adjacent. As Cayley(G, S) is planar we conclude that $2^{\alpha+1} \leq 4$ and so $\alpha = 1$. Thus $|G| = 6$ which we discuss about it before. Suppose $n_2 = 3$ and $n_3 = 1$. With out loss of generality we can assume $|G| > 6$. If G contains 4 elements of order 2, then Cayley(G, S) is not planar. Therefore, G is a group which has at most 3 elements of order 2. The group of order $|G| = 7,11,13,14,15$ or $G \cong D_{10}$ are not acceptable by the second part of Proposition. 2.2. if $G \cong D_8$, then Cayley(G, S) is not planar by Proposition 3.8. Clearly Cayley $(Z_3 \times Z_3, S) \cong K_9$ is not planar. By these conditions we must cheque the planarity of prime order Cayley graph of $G_1 = Z_{12}, G_2 = Z_2 \times Z_2 \times Z_3, A_4, D_{12}$ and $T = \langle a, b : a^6 = 1, b^2 = a^3, a^b = a^{-1} \rangle$. Immediately, we omit D_{12} and T, because they are not nilpotent. Cayley(G_1, S) is 3-regular graph which is planar by Proposition 3.5. G_2 has 5 elements of prime order. Thus the number of edges of the Cayley (G_2 ,S) is 30. Now by [3, Corollary 9.5.2] Cayley(G_2, S) is not planar. Similarly Cayley (A_4, S) is 11-regular which is not planar.

Let G be a group of order $2^{\alpha}.3^{\beta}$, where $\alpha, \beta > 1$ Thus $|G| \geq 36$ Hence the assertion is clear. □

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