WEIGHTING METHOD FOR CONVEX VECTOR INTERVAL-VALUED OPTIMIZATION PROBLEMS

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In this paper, a nonlinear vector optimization problem with the multiple interval-valued objective function is considered. We use the weighting method for finding its weakly type-I Pareto and type-I Pareto solutions. Therefore, for the considered nonlinear interval-valued multiobjective programming problem, its associated noninterval scalar optimization problem with weights is defined in the aforesaid approach. Then, under appropriate convexity hypotheses, the equivalence between a (weakly) type-I Pareto of the considered nonlinear interval-valued multiobjective programming problem and an optimal solution of its associated noninterval scalar weighting optimization problem is established.

Keywords: nonlinear vector optimization problem with multiple interval-objective function; weighting method; (weakly) type-I Pareto solution.

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1. Introduction

One of the most used deterministic optimization models to deal with the extremum problems with uncertain data is interval-valued optimization because it does not require the specification or the assumption of probabilistic distributions (as in stochastic programming) or possibilistic distributions (as in fuzzy programming). There exist various approaches in the literature for solving interval-valued optimization problems (see, for example, [1], [2], [4], [7], [9], [10], [11], [12], [14], [16], and others).

Vector optimization problems, commonly known as multiobjective programming problems or multicriteria optimization problems, gained importance because in the real world applications we encounter such extremum problems. Scalarization techniques for solving a nonlinear multiobjective programming problem substitute the original vector optimization problem by a suitable scalar optimization problem, in such a way that minimizers of the constructed scalar extremum problem are also (weak) Pareto solutions of the original one. The main advantage of such an approach, from a practical point of view, is that a large number of fast and reliable methods developed for single-valued optimization in order to solve vector optimization problems can be used. One of the most widely used scalarization techniques in multiobjective programming is the weighting method, which consists of minimizing a weighted sum of the different objectives.

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In the paper, the weighting method is applied for solving nonlinear vector optimization problems with multiple interval-valued objective functions. This method is used for identifying weakly type-I Pareto and type-I Pareto solutions of the considered nonlinear vector optimization problem with the multiple interval-valued objective function. In this technique, at first, an associated scalar noninterval optimization problem is constructed for the considered nonlinear interval-valued multiobjective programming problem. Then the equivalence is established between a (weakly) type-I Pareto solution of the considered nonlinear vector optimization problem with multiple interval-valued objective function and a minimizer of its associated scalar noninterval extremum problem constructed in the weighting method. This result is established under assumption that the functions constituting the considered interval-valued multiobjective programming problem are convex. The example of a nonlinear convex vector optimization problem with the multiple interval-valued objective function, which is solving by the weighting method, is presented to illustrate the results established in the paper.

2. Notations and preliminaries

Let R^n be the n-dimensional Euclidean space. For any vectors $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$ in R^n, we define: (i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, \dots, n$; (ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, \dots, n$; (iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, \dots, n$; (iv) $x \geq y$ if and only if $x = y$ and $x \neq y$.

Let $I(R)$ be a class of all closed and bounded intervals in R. Throughout this paper, when we say that A is a closed interval, we mean that A is also bounded in R. If A is a closed interval, we use the notation $A = [a^L, a^U]$, where a^L and a^U mean the lower and upper bounds of A, respectively. In other words, if $A = [a^L; a^U] \in I(R)$, then $A = [a^L, a^U] = \{x \in R : a^L \leqq x \leqq a^U\}$. If $a^L = a^U = a$, then $A = [a, a] = a$ is a real number. In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another.

For $A = [a^L, a^U]$ and $B = [b^L, b^U]]$, we write $A \preccurlyeq_{LU} B$ if and only if $a^L \leqq b^L$ and $a^U \leqq b^U$ It is easy to see that $\preccurlyeq_{LU}$ is a partial ordering on $I(R)$. Also, we can write $A \prec_{LU} B$ if and only if $A \preccurlyeq_{LU} B$ and $A \neq B$. Equivalently, $A \prec_{LU} B$ if and only if $(a^L < b^L,\ a^U \leqq b^U)$ or $(a^L \leqq b^L,\ a^U < b^U)$ or $(a^L < b^L,\ a^U < b^U)$.

Let X be a nonempty subset of R^n. A function $\varphi : X \to I(R)$ is called an interval-valued function if $\varphi(x) = [\varphi^L(x), \varphi^U(x)]$ with $\varphi^L, \varphi^U : X \to R$ such that $\varphi^L(x) \leqq \varphi^U(x)$ for each $x \in X$.

Definition 2.1. [16] *Let X be a nonempty convex subset of R^n and $f : X \to I(R)$ be an interval-valued function defined on X. It is said that f is a convex interval-valued function on X if the inequality*

$$f\big(u+\alpha(x-u)\big) \preccurlyeq_{LU} \alpha f(x) + (1-\alpha) f(u) \tag{1}$$

holds for all $x, u \in X$ *and* $\alpha \in [0;1]$.

The following result follows from (1) immediately (see [15]).

Proposition 2.1. *Let* X *be a nonempty convex subset of* R^n *and* $f: X \to I(R)$. f *is a convex interval-valued function on* X *if and only if the functions* f^L *and* f^U *are convex on* X *(in the classical sense).*

The following nonlinear theorem of the alternative is a particular case of more general results established in [3], [5] (see also [6]).

Theorem 2.1. *Let* $C \subset R^n$ *be a nonempty convex set, and* $f = (f_1, \ldots, f_p): R^n \to R^p$ *and* $g = (g_1, \ldots, g_q): R^n \to R^q$ *be convex functions. If the system*

$$f_i(x) < 0, i = 1, \ldots, p,, g_j(x) \leqq 0, j = 1, .., q, x \in C$$

has no solution, there exist $\lambda \in R^p$, $\lambda \geqq 0$, $\mu \in R^q$, $\mu \geqq 0$, $(\lambda, \mu) \neq 0$, *such that*

$$\textstyle\sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j=1}^{q} \mu_j g_j(x) \geqq 0, \forall x \in C.$$

3. Weighting method for solving convex vector optimization problems with interval-valued objective functions

In the paper, consider the following vector optimization problem with the multiple interval-valued objective function defined by

$$f(x) = \big(f_1(x), \ldots, f_r(x)\big) \to min \quad \text{(IVP)}$$
$$x \in C,$$

where C is a nonempty convex subset of R^n, each $f_k: R^n \to I(R)$, $k \in K = \{1, \ldots, r\}$ is an interval-valued function, that is, $f_k(x) = [f_k^L(x), f_k^U(x)]$, $f_k^L(x) \leqq f_k^U(x)$, $x \in R^n$, $k \in K$. We shall assume, moreover, that f_k^L, $f_k^U: R^n \to R$, $k \in K$, are continuous functions. For the purpose of simplifying our presentation, we will introduce the following notations $f^L = (f_1^L, \ldots, f_r^L)^T$, $f^U = (f_1^U, \ldots, f_r^U)^T$.

Since each of the objective values f_k is a closed interval, we need to provide an ordering relation between any two closed intervals. The most direct way is to invoke the ordering relation $\preccurlyeq_{LU}$ that was defined above. However, $\preccurlyeq_{LU}$ is a partial ordering relation, not a total ordering, on $I(R)$, and we shall follow the similar concept of a nondominated solution used in a multiobjective programming problem to investigate the solution concepts. Now, for the considered multiobjective programming problem (IVP) with multiple interval-valued objective function, we give the definitions of weakly type-I Pareto and type-I Pareto solutions introduced by Wu [15].

Definition 3.1. *A feasible point* $\bar{x}$ *is said to be a weakly type-I Pareto solution of the considered vector optimization problem (IVP) with multiple interval-valued objective function if and only if there is no other feasible solution* x *such that* $f_k(x) \prec_{LU} f_k(\bar{x})$, $k \in K$.

Definition 3.2. *A feasible point* $\bar{x}$ *is said to be a type-I Pareto solution of the considered vector optimization problem (IVP) with multiple interval-valued objective function if and only if there is no other feasible solution* x *such that* $f(x) \prec_{LU} f(\bar{x})$.

Remark 3.1. It is known in the optimization literature (see, for example, [13]) that a weakly type-I Pareto solution and a type-I Pareto solution of (IVP) are also called a weakly LU-Pareto solution and a LU-Pareto solution, respectively.

In this section, we use the weighting method in order to characterize (weakly) type-I Pareto optimality of the interval-valued multiobjective programming problem (IVP). Therefore, for the considered vector optimization problem (IVP) with interval-valued objective functions, we define in the weighting method the associated scalar optimization problem as follows:

$$\Gamma(x) = \sum_{k=1}^{r} \lambda_k^L f_k^L(x) + \sum_{k=1}^{r} \lambda_k^U f_k^U(x) \to \min \quad x \in C, \qquad (\mathrm{WOP}_\lambda)$$

where $\lambda^L = (\lambda_1^L, \dots, \lambda_r^L) \geq 0$ and $\lambda^U = (\lambda_1^U, \dots, \lambda_r^U) \geq 0$.

Now, we give the definition of an optimal solution of the scalar optimization problem (WOP_λ) defined in the weighting method.

Theorem 3.1. *Let* $\bar{x} \in C$ *be an optimal solution of the weighting optimization problem* ($WOP_{\bar{\lambda}}$). *If* $\bar{\lambda} = \left(\bar{\lambda}^L, \bar{\lambda}^U\right) \geq 0$ *with* $\left(\bar{\lambda}_{k_0}^L, \bar{\lambda}_{k_0}^U\right) > 0$ *for some* $k_0 \in K$, *where* $\lambda^L = (\lambda_1^L, \dots, \lambda_r^L)$ *and* $\lambda^U = (\lambda_1^U, \dots, \lambda_r^U)$, *then* $\bar{x}$ *is a weakly type-I Pareto solution of the considered vector optimization problem (IVP) with multiple interval-valued objective function.*

Proof. Let $\bar{x} \in C$ be an optimal solution of the weighting optimization problem ($WOP_{\bar{\lambda}}$). We proceed by contradiction. Suppose, contrary to the result, that $\bar{x}$ is not a weakly type-I Pareto solution of (IVP). Hence, by Definition 3.1, there exists other feasible solution $\tilde{x}$ such that

$$f_k(\tilde{x}) \prec_{LU} f_k(\bar{x}), \quad k \in K.$$

By the definition of the relation $\prec_{LU}$, it follows that, for any $k \in K$,

$$(f_k^L(\tilde{x}) < f_k^L(\bar{x}) \ \wedge \ f_k^U(\tilde{x}) \leqq f_k^U(\bar{x})) \text{ or } (f_k^L(\tilde{x}) \leqq f_k^L(\bar{x}) \wedge f_k^U(\tilde{x}) < f_k^U(\bar{x})) \text{ or } (f_k^L(\tilde{x}) < f_k^L(\bar{x}) \ \wedge \ f_k^U(\tilde{x}) < f_k^U(\bar{x})).$$

Since $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ with $\left(\overline{\lambda}^L_{k_0}, \overline{\lambda}^U_{k_0}\right) > 0$ for some $k_0 \in K$, the system of inequalities (3) implies that the inequality

$$\sum_{k=1}^{r} \lambda_k^L f_k^L(\tilde{x}) + \sum_{k=1}^{r} \lambda_k^U f_k^U(\tilde{x}) < \sum_{k=1}^{r} \lambda_k^L f_k^L(\bar{x}) + \sum_{k=1}^{r} \lambda_k^U f_k^U(\bar{x})$$

holds. Thus, by Definition 3.3, this is a contradiction to the assumption that $\overline{x}$ is an optimal solution of the weighting optimization problem (WOP$_{\overline{\lambda}}$). Hence, the proof of this theorem is completed. ■

Theorem 3.2. *Let $\overline{x} \in C$ be an optimal solution of the weighting optimization problem ($WOP_{\overline{\lambda}}$). If $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ with $\overline{\lambda}^L \geq 0$ and $\overline{\lambda}^U \geq 0$, then $\overline{x}$ is a type-I Pareto solution of the considered vector optimization problem (IVP) with multiple interval-valued objective function.*

Proof. Let $\overline{x} \in C$ be an optimal solution of the weighting optimization problem ($WOP_{\overline{\lambda}}$). We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a type-I Pareto solution of (IVP). Hence, by Definition 3.2, there exists other feasible solution $\tilde{x}$ such that

$$f(\tilde{x}) \underset{LU}{\prec} f(\overline{x}). \tag{4}$$

By the definition of the relation $\prec_{LU}$, it follows that,

$$\left(f^L(\tilde{x}) < f^L(\overline{x}) \ \wedge \ f^U(\tilde{x}) \leqq f^U(\overline{x})\right) \text{ or } \left(f^L(\tilde{x}) \leqq f^L(\overline{x}) \wedge f^U(\tilde{x}) < f^U(\overline{x})\right) \text{ or } \left(f^L(\tilde{x}) < f^L(\overline{x}) \ \wedge \ f^U(\tilde{x}) < f^U(\overline{x})\right). \tag{5}$$

Since $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ with with $\overline{\lambda}^L \geq 0$ and $\overline{\lambda}^U \geq 0$, the system of inequalities (5) implies that the inequality

$$\sum_{k=1}^{r} \lambda_k^L f_k^L(\tilde{x}) + \sum_{k=1}^{r} \lambda_k^U f_k^U(\tilde{x}) < \sum_{k=1}^{r} \lambda_k^L f_k^L(\bar{x}) + \sum_{k=1}^{r} \lambda_k^U f_k^U(\bar{x})$$

holds, which contradicts the assumption that $\overline{x}$ is an optimal solution of (WOP$_{\overline{\lambda}}$). Hence, the proof of this theorem is completed. ■

Now, under convexity hypotheses, we prove the converse results to those ones established in Theorems

Theorem 3.3. *Let the objective functions f_k, $k \in K$, be convex on C. If $\overline{x} \in C$ is a type-I Pareto solution of the vector optimization problem (IVP) with multiple interval-valued objective function, then there exists $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$, where $\overline{\lambda}^L = \left(\overline{\lambda}^L_1, \ldots, \overline{\lambda}^L_r\right)$ and $\overline{\lambda}^U = \left(\overline{\lambda}^U_1, \ldots, \overline{\lambda}^U_r\right)$, such that $\overline{x}$ is an optimal solution of the weighting optimization problem ($WOP_{\overline{\lambda}}$).*

Proof. Let $\overline{x} \in D$ be a type-I Pareto solution of the vector optimization problem (IVP) with multiple interval-valued objective function. Hence, by Definition 3.2, there does not exist other feasible solution x such that

$$f(x) \prec_{LU} f(\overline{x}).$$

Then, by the definition of the relation $\prec_{LU}$, it follows that there does not exist a feasible point x such that

$$(f^L(x) < f^L(\overline{x}) \wedge f^U(\mathrm{x}) \leqq f^U(\overline{x})) \text{ or } (f^L(\mathrm{x}) \leqq f^L(\overline{x}) \wedge f^U(x) < f^U(\overline{x})) \text{ or } (f^L(\mathrm{x}) < f^L(\overline{x}) \wedge f^U(\mathrm{x}) < f^U(\overline{x})). \tag{6}$$

Thus, the system of inequalities (6) and the feasibility of $\overline{x}$ yield that there is no a feasible solution x such that

$$\begin{matrix} f^L(x) - f^L(\overline{x}) < 0, \\ f^U(x) - f^U(\overline{x}) \leqq 0 \end{matrix} \quad \text{or} \quad \begin{matrix} f^U(x) - f^U(\overline{x}) < 0, \\ f^L(x) - f^L(\overline{x}) \leqq 0. \end{matrix} \tag{7}$$

By assumption, the objective functions $\mathrm{f}_{\mathrm{k}}, k \in K$, are convex interval-valued functions on C. Then, by Proposition 2.1, it follows that the functions f^L and f^U are convex on C (in the classical sense). Since the system of inequalities (7) has no a solution $x \in C$, by the theorem of the alternative (Theorem 2.1), there exist $\overline{\lambda}^L \in R^r$ and $\overline{\lambda}^U \in R^r$ with $\left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ such that, for all $x \in C$, the inequality

$$\sum_{k=1}^{r} \lambda_k^L\left(f_k^L(x) - f_k^L(\bar{x})\right) + \sum_{k=1}^{r} \lambda_k^U \left(f_k^U(x) - f_k^U(\bar{x})\right) \geqq 0$$

holds. This means that there exist $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ such that the inequality

$$\sum_{k=1}^{r} \lambda_k^L f_k^L(x) + \sum_{k=1}^{r} \lambda_k^U f_k^U(x) \geqq \sum_{k=1}^{r} \lambda_k^L f_k^L(\bar{x}) + \sum_{k=1}^{r} \lambda_k^U f_k^U(\bar{x})$$

holds for all $x \in C$. This means, by Definition 3.3, that $\overline{x}$ is an optimal solution of (WOP$_{\overline{\lambda}}$). This completes the proof of this theorem. ■

Theorem 3.4. *Let the objective functions f_k, $k \in K$, be convex on C. If $\overline{x} \in C$ is a weakly type-I Pareto solution of the vector optimization problem (IVP) with multiple interval-valued objective function, then there exists $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$ with $\left(\overline{\lambda}_k^L, \overline{\lambda}_k^U\right) \geq 0$ for each $k \in K$, where $\overline{\lambda}^L = \left(\overline{\lambda}_1^L, \dots, \overline{\lambda}_r^L\right)$ and $\overline{\lambda}^U = \left(\overline{\lambda}_1^U, \dots, \overline{\lambda}_r^U\right)$, such that $\overline{x}$ is an optimal solution of the weighting optimization problem ($WOP_{\overline{\lambda}}$).*

Proof. Let $\overline{x} \in D$ be a weakly type-I Pareto solution of (IVP). Hence, by Definition 3.1, there does not exist other feasible solution x such that

$$f_k(x) \prec_{\boldsymbol{LU}} f_k(\overline{x}), \ \ \forall k \in K.$$

Then, by the definition of the relation $\prec_{LU}$, it follows that there does not exist a feasible point x such that, for any $k \in K$,

$$(f_k^L(x) < f_k^L(\overline{x}) \wedge f_k^U(\mathrm{x}) \leqq f_k^U(\overline{x})) \text{ or } (f_k^L(\mathrm{x}) \leqq f_k^L(\overline{x}) \wedge f_k^U(x) < f_k^U(\overline{x})) \text{ or } (f_k^L(\mathrm{x}) < f_k^L(\overline{x}) \wedge f_k^U(\mathrm{x}) < f_k^U(\overline{x})). \tag{8}$$

Thus, the system of inequalities (8) and the feasibility of $\overline{x}$ yield that there is no a feasible solution x such that, for any $k \in K$,

$$\begin{matrix} f_k^L(x) - f_k^L(\overline{x}) < 0, \\ f_k^U(x) - f_k^U(\overline{x}) \leqq 0 \end{matrix} \quad \text{or} \quad \begin{matrix} f_k^U(x) - f_k^U(\overline{x}) < 0, \\ f_k^L(x) - f_k^L(\overline{x}) \leqq 0. \end{matrix} \tag{9}$$

By assumption, the objective functions $f_k, k \in K$, are convex on C. Then, by Proposition 2.1, it follows that the functions f^L and f^U are convex on C (in the classical sense). Since the system of inequalities (9) has no a solution $x \in C$, by the theorem of the alternative (Theorem 2.1), for each $k \in K$, there exists $\overline{\lambda}_k = \left(\overline{\lambda}_k^L, \overline{\lambda}_k^U\right) \geq 0$ such that the inequality

$$\sum_{k=1}^{r} \bar{\lambda}_k^L f_k^L(x) + \sum_{k=1}^{r} \bar{\lambda}_k^U f_k^U(x) \geqq \sum_{k=1}^{r} \bar{\lambda}_k^L f_k^L(\bar{x}) + \sum_{k=1}^{r} \bar{\lambda}_k^U f_k^U(\bar{x})$$

holds for all $x \in C$. This means, by Definition 3.3, that $\overline{x}$ is an optimal solution of (WOP$_{\overline{\lambda}}$). This completes the proof of this theorem. ■

In order to illustrate the results established in the paper, we consider an example of a convex vector optimization problem with multiple interval-valued objective function which we solve by using the weighting method.

Example 3.1. Consider the following convex vector optimization problem with the multiple interval-valued objective function:

$$f(x) = \left(\left[\frac{1}{2}, 1\right](x_2^1 + x_2^2) + \left[\frac{1}{2}, 1\right](x_1 - x_2) + \left[\frac{1}{2}, 1\right],\right.$$
$$\left.[1,1](x_2^1 + x_2^2) + \left[-1, -\frac{1}{2}\right](x_1 - x_2) + \left[-1, -\frac{1}{2}\right]\right) \to min \quad (IVP1)$$
$$C = \{(x_1, x_2) \in R^2 : 0 \leqq x_1 \leqq 1 \ \wedge \ 0 \leqq x_2 \leqq 1\}.$$

Note that $\overline{x} = (0,0)$ is a feasible solution of the considered vector optimization problem (IVP1) with multiple interval-valued objective function. Note that all objective functions of the considered interval-valued multiobjective programming problem (IVP1) are convex. By Definition 3.2, it follows that $\overline{x}$ is a type-I Pareto solution of (IVP1). It can be shown that the Karush-Kuhn-Tucker necessary optimality conditions (see, for example, [8], [15]) are satisfied at $\overline{x} = (0,0)$ with Lagrange multipliers associated to the objective functions which are as follows $\bar{\lambda}_1^L = \frac{1}{4}$, $\bar{\lambda}_1^U = \frac{1}{4}$, $\bar{\lambda}_2^L = \frac{1}{4}$, $\bar{\lambda}_2^U = \frac{1}{4}$. Then, we use the weighting method for solving (IVP1). Therefore, we construct its associated weighting optimization problem (WOP1$_{\overline{\lambda}}$) as follows

$$\frac{7}{8}(x_2^1 + x_2^2) \to min \qquad (\text{WOP1}_{\overline{\lambda}})$$
$$x \in C,$$

where $\overline{\lambda} = \left(\overline{\lambda}^L, \overline{\lambda}^U\right) \geq 0$, $\overline{\lambda}^L = (\bar{\lambda}_1^L, \bar{\lambda}_2^L) = \left(\frac{1}{4}, \frac{1}{4}\right) > 0$, $\overline{\lambda}^U = (\bar{\lambda}_1^U, \bar{\lambda}_2^U) = \left(\frac{1}{4}, \frac{1}{4}\right) > 0$. Since all hypotheses of Theorem 3.3 are satisfied, therefore, $\overline{x} = (0,0)$ is a minimizer of (WOP1$_{\overline{\lambda}}$). Conversely, let $\overline{x} = (0,0)$ be a minimizer of the weighting optimization problem (WOP1$_{\overline{\lambda}}$). Moreover, all functions constituting (IVP1) are convex. Since $\overline{\lambda}^L = (\bar{\lambda}_1^L, \bar{\lambda}_2^L) = \left(\frac{1}{4}, \frac{1}{4}\right) > 0$, $\overline{\lambda}^U = (\bar{\lambda}_1^U, \bar{\lambda}_2^U) = \left(\frac{1}{4}, \frac{1}{4}\right) > 0$, by Theorem 3.2, it follows that $\overline{x} = (0,0)$ is also a type-I Pareto solution of (IVP1).

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