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SOME NEW INTEGRAL INEQUALITIES VIA ϕ-PREINVEX FUNCTIONS

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In this paper, we derive a new integral identity involving differentiable function. Utilizing this identity as an auxiliary result we obtain several new Hermite-Hadamard type of integral inequalities. Special cases of the main results are also discussed in detail.

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1. Introduction and Preliminaries

The classical invex sets are defined as: A set $\mathcal{K}_\zeta$ is said to be invex, if there exists a bifunction $\zeta(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\mathfrak{u} + \mu\zeta(\mathfrak{v}, \mathfrak{u}) \in \mathcal{K}_\zeta,$$

where $\mathfrak{u}, \mathfrak{v} \in \mathcal{K}_\zeta$ and $\mu \in [0, 1]$.
It has been observed by [15] that the class of invex sets includes the class of classical convex function as special case, if $\zeta(\mathfrak{u}_2, \mathfrak{u}_1) = \mathfrak{u}_2 - \mathfrak{u}_1$.
Using invex sets as a domain of the function, Weir et al. [15] defined the concept of preinvex functions.

Definition 1.1 ([15])**.** *The function $\mathcal{F} : \mathcal{C}_\zeta \to \mathbb{R}$ is said to be preinvex with respect to bifunction $\zeta(.,.)$, if*

$$\mathcal{F}(\mathfrak{u}_1 + \varrho\zeta(\mathfrak{u}_2, \mathfrak{u}_1)) \leq (1 - \varrho)\mathcal{F}(\mathfrak{u}_1) + \varrho\mathcal{F}(\mathfrak{u}_2), \forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{C}_\zeta, \varrho \in [0, 1].$$

Mohan et al. [7] also presented some interesting properties of preinvex functions.
Varosanec [14] introduced the class of ϕ-convex function, which defined as:

Definition 1.2 ([14])**.** *Suppose $\Omega, \mathcal{J}$ be intervals in $\mathbb{R}$, (0, 1) $\subseteq \mathcal{J}$, and let $\phi : \mathcal{J} \to \mathbb{R}$ be a real function. We say that a real function $\mathcal{F} : \Omega \to \mathbb{R}$ is called ϕ-convex function, if*

$$\mathcal{F}((1 - \varrho)\mathfrak{u}_1 + \varrho\mathfrak{u}_2) \leq \phi(1 - \varrho)\mathcal{F}(\mathfrak{u}_1) + \phi(\varrho)\mathcal{F}(\mathfrak{u}_2), \forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{C}_\zeta, \varrho \in [0, 1].$$

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Remark 1.1 ([14])**.** *For different suitable choices of the function $\phi(.)$, the class of ϕ-convex functions reduce to some other classes of convex functions. For more details readers are referred to* [14].

Recently, Noor et al. [12] introduced the notion of ϕ-preinvex function and obtained several new refinements of Hermite-Hadamard's inequalities via ϕ-preinvex functions.

Definition 1.3 ([12])**.** *The function $\mathcal{F} : \mathcal{C}_\zeta \to \mathbb{R}$ is said to be ϕ-preinvex with respect to bifunction $\zeta(.,.)$, if*

$$\mathcal{F}(\mathfrak{u}_1 + \varrho\zeta(\mathfrak{u}_2, \mathfrak{u}_1)) \leq \phi(1-\varrho)\mathcal{F}(\mathfrak{u}_1) + \phi(\varrho)\mathcal{F}(\mathfrak{u}_2), \forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{C}_\zeta, \varrho \in [0,1].$$

Remark 1.2. *It is worth to mention here that if $\phi(\varrho) = \varrho, \varrho^s, \varrho^{-1}, \varrho^{-s}$ and $\phi(\varrho) = 1$, then we have the classes of s-preinvex functions of Brckner type, Q-preinvex functions, s-preinvex functions of Godunova-Levin type and P-preinvex functions respectively. For more information, see* [12].

For more details on recent research on this field, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 16, 17]

2. Auxiliary Results

In this section, we derive our new integral identity, which will play significant role in obtaining main results of next sections. Before we derive the result let us assume that $\Omega = [\mathfrak{a}_1, \mathfrak{a}_1 + \zeta(\mathfrak{a}_2, \mathfrak{a}_1)]$ and Ω° be the interior of Ω unless otherwise specified.

Lemma 2.1. *Suppose $\mathcal{F} : \Omega^\circ \to \mathbb{R}$ be a differentiable function. If $\mathcal{F}' \in \mathcal{L}_1[\Omega]$, $\varrho, \mu \in \mathbb{R}$ and $\mathfrak{n} \in \mathbb{N}^\star$, then*

$$\begin{aligned}
\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta) \quad := \quad & \psi(\mathfrak{n}; \varrho, \mu; \mathfrak{a}_1, \mathfrak{a}_1 + \zeta(\mathfrak{a}_2, \mathfrak{a}_1)) - \frac{1}{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)} \\
& \times \left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_1 + \frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)} \mathcal{F}(x)\mathrm{d}x + \int_{\mathfrak{a}_1 + \frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}^{\mathfrak{a}_1 + \zeta(\mathfrak{a}_2, \mathfrak{a}_1)} \mathcal{F}(x)\mathrm{d}x \right] \\
= \quad & \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2} \int_0^1 \left[(1-\varrho-t)\mathcal{F}'\left(\mathfrak{a}_1 + \frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right. \\
& \left. + (\mu - t)\mathcal{F}'\left(\mathfrak{a}_1 + \frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right] \mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
& \psi(\mathfrak{n}; \varrho, \mu; \mathfrak{a}_1, \mathfrak{a}_1 + \zeta(\mathfrak{a}_2, \mathfrak{a}_1)) \\
& = \frac{1}{\mathfrak{n}+1} \left[\varrho\mathcal{F}(\mathfrak{a}_1) + (1-\varrho)\mathcal{F}\left(\mathfrak{a}_1 + \frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right. \\
& \quad \left. + (1-\mu)\mathcal{F}\left(\mathfrak{a}_1 + \frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) + \mu\mathcal{F}(\mathfrak{a}_1 + \zeta(\mathfrak{a}_2, \mathfrak{a}_1)) \right].
\end{aligned}$$

Proof. Take

$$\begin{aligned} I &= \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\int_0^1\Bigg[(1-\varrho-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right) \\ &\quad +(\mu-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\Bigg]\mathrm{d}t \\ &= \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}(I_1+I_2). \end{aligned} \tag{2.1}$$

Let

$$\begin{aligned} I_1 &= \int_0^1\left[(1-\varrho-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right] \\ &= \frac{\mathfrak{n}+1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\Bigg[\varrho\mathcal{F}(\mathfrak{a}_1)+(1-\varrho)\mathcal{F}\left(\mathfrak{a}_1+\frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right) \\ &\quad -\int_0^1\mathcal{F}\left(\mathfrak{a}_1+\frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\mathrm{d}t\Bigg] \\ &= \frac{\mathfrak{n}+1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\Bigg[\varrho\mathcal{F}(\mathfrak{a}_1)+(1-\varrho)\mathcal{F}\left(\mathfrak{a}_1+\frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right) \\ &\quad -\frac{\mathfrak{n}+1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\int_{\mathfrak{a}_1}^{\mathfrak{a}_1+\frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_2)}{\mathfrak{n}+1}}\mathcal{F}(x)\mathrm{d}x\Bigg]. \end{aligned} \tag{2.2}$$

Now

$$\begin{aligned} I_2 &= \int_0^1(\mu-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right) \\ &= \frac{\mathfrak{n}+1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\Bigg[(1-\mu)\mathcal{F}(\mathfrak{a}_1+\frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)) \\ &\quad +\mu\mathcal{F}\left(\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)-\frac{\mathfrak{n}+1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\int_{\mathfrak{a}_1+\frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}^{\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\mathcal{F}(x)\mathrm{d}x\Bigg]. \end{aligned} \tag{2.3}$$

Using the values I_1 and I_2 from (2.2) and (2.3) in (2.1), we have

$$\begin{aligned} I &= \frac{1}{\mathfrak{n}+1}\Bigg[\varrho\mathcal{F}(\mathfrak{a}_1)+(1-\varrho)\mathcal{F}(\mathfrak{a}_1+\frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)) \\ &\quad +(1-\mu)\mathcal{F}(\mathfrak{a}_1+\frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1))+\mu\mathcal{F}(\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1))\Bigg] \\ &\quad -\frac{1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\Bigg[\int_{\mathfrak{a}_1}^{\mathfrak{a}_1+\frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\mathcal{F}(x)\mathrm{d}x+\int_{\mathfrak{a}_1+\frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}^{\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\mathcal{F}(x)\mathrm{d}x\Bigg]. \end{aligned}$$

This implies

$$\begin{aligned} &\psi(\mathfrak{n};\varrho,\mu;\mathfrak{a}_1,\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1))-\frac{1}{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)} \\ &\times\Bigg[\int_{\mathfrak{a}_1}^{\mathfrak{a}_1+\frac{1}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\mathcal{F}(x)\mathrm{d}x+\int_{\mathfrak{a}_1+\frac{\mathfrak{n}}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}^{\mathfrak{a}_1+\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}\mathcal{F}(x)\mathrm{d}x\Bigg] \\ &=\frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\int_0^1\Bigg[(1-\varrho-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right) \\ &\quad+(\mu-t)\mathcal{F}'\left(\mathfrak{a}_1+\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\Bigg]\mathrm{d}t. \end{aligned}$$

This completes the proof. □

3. **Main Results**

In this section, we derive our main results.

Theorem 3.1. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex function, then*

$$\begin{aligned} &|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\ &\leq\frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1))}{(\mathfrak{n}+1)^2}\left[\left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}}K_1^{\frac{1}{\mathfrak{q}}}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}}K_2^{\frac{1}{\mathfrak{q}}}\right], \end{aligned}$$

where

$$K_1=\int_0^1|1-\varrho-t|\left(\phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+\phi\left(\frac{1-t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right)\mathrm{d}t,$$

and

$$K_2=\int_0^1|\mu-t|\left(\phi\left(\frac{t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+\phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right)\mathrm{d}t.$$

Proof. Using Lemma 2.1 and the property of modulus, we have

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left[\overbrace{\int_0^1 |1-\varrho-t|\left|\mathcal{F}'\left(\mathfrak{a}_1+\frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right|\mathrm{d}t}^{I_1}\right. \\
&\qquad \left.+\overbrace{\int_0^1 |\mu-t|\left|\mathcal{F}'\left(\mathfrak{a}_1+\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right|\mathrm{d}t}^{I_2}\right] \\
&= \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_2)}{(\mathfrak{n}+1)^2}\left[I_1+I_2\right]. \qquad (3.1)
\end{aligned}$$

Using power mean's inequality and ϕ-preinvexity, we have

$$\begin{aligned}
I_1 &\leq \left(\int_0^1 |1-\varrho-t|\,\mathrm{d}t\right)^{1-\frac{1}{\mathfrak{q}}} \\
&\quad\times\left(\int_0^1 |1-\varrho-t|\left(\phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+\phi\left(\frac{1-t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right)\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}} \\
&= \left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}} K_1^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

Now

$$\begin{aligned}
I_2 &\leq \left(\int_0^1 |\mu-t|\,\mathrm{d}t\right)^{1-\frac{1}{\mathfrak{q}}} \\
&\quad\times\left(\int_0^1 |\mu-t|\left(\phi\left(\frac{t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+\phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right)|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right)\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}} \\
&= \left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}} K_2^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

Using the value of I_1 and I_2 in (3.1), we have the required result. □

Now we will discuss some cases for Theorem 3.1.

(1) If $\phi(t) = 1$, then we have result for P-preinvex function.

Corollary 3.1. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is P-preinvex function, then*

$$\begin{aligned}&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)|\\ &\leq \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\varrho^2+\mu^2-(\varrho+\mu)+1\right)\left(|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right).\end{aligned}$$

(2) If $\phi(t)=t$, then we have result for preinvex function.

Corollary 3.2. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex function, then*

$$\begin{aligned}&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)|\\ &\leq \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left[\left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}}{6(\mathfrak{n}+1)}\left((3\mathfrak{n}+1)-(6\mathfrak{n}+3)\varrho\right.\right.\right.\\ &\left.\left.+6(\mathfrak{n}+1)\varrho^2-2\varrho^3\right)+\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{6(\mathfrak{n}+1)}\left(2-3\varrho+2\varrho^3\right)\right)^{\frac{1}{\mathfrak{q}}}\\ &+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}}{6(\mathfrak{n}+1)}\left(2-3\mu+2\mu^3\right)\right.\\ &\left.\left.+\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{6(\mathfrak{n}+1)}\left((3\mathfrak{n}+1)-(6\mathfrak{n}+3)\mu+6(\mathfrak{n}+1)\mu^2-2\mu^3\right)\right)^{\frac{1}{\mathfrak{q}}}\right].\end{aligned}$$

(3) If $\phi(t)=t^s$, then we have result for s-preinvex function of Breckner type.

Corollary 3.3. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is s-preinvex function of Breckner type, then*

$$\begin{aligned}&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)|\\ &\leq \frac{\zeta(\mathfrak{a}_2,(\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left[\left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}}\left[\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^s}\left(\frac{(s+2)(\varrho-1)\mathfrak{n}^{s+1}+2(1-\varrho+\mathfrak{n})^{s+2}}{(s+1)(s+2)}\right.\right.\right.\\ &\left.+\frac{(s+2)(\mathfrak{n}+1)^{s+1}\varrho-\mathfrak{n}^{s+2}-(\mathfrak{n}+1)^{s+2}}{(s+1)(s+2)}\right)\\ &\left.+\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^s}\left(\frac{2\varrho^{s+2}+(s+2)(1-\varrho)-1}{(s+1)(s+2)}\right)\right]^{\frac{1}{\mathfrak{q}}}\\ &+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}}\left[\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^s}\left(\frac{2\mu^{s+2}+(s+2)(1-\mu)-1}{(s+1)(s+2)}\right)\right.\\ &+\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^s}\left(\frac{(s+2)(\mu-1)\mathfrak{n}^{s+1}+(1-\mu+\mathfrak{n})^{s+2}}{(s+1)(s+2)}\right.\\ &\left.\left.\left.+\frac{(s+2)(\mathfrak{n}+1)^{s+1}\mu-\mathfrak{n}^{s+2}-(\mathfrak{n}+1)^{s+2}}{(s+1)(s+2)}\right)\right]^{\frac{1}{\mathfrak{q}}}.\end{aligned}$$

(4) If $\phi(t)=t^{-s}$, then we have result for s-preinvex function of Godunova-Levin type.

Corollary 3.4. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is s-preinvex function of Godunova-levin type, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\
&\le \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left[\left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}}\right. \\
&\times\left[\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^{-s}}\left(\frac{(2-s)(\varrho-1)\mathfrak{n}^{1-s}+2(1-\varrho+\mathfrak{n})^{2-s}}{(1-s)(2-s)}\right.\right. \\
&+\left.\frac{(2-s)(\mathfrak{n}+1)^{1-s}\varrho-\mathfrak{n}^{2-s}-(\mathfrak{n}+1)^{2-s}}{(1-s)(2-s)}\right) \\
&+\left.\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^{-s}}\left(\frac{2\varrho^{2-s}+(2-s)(1-\varrho)-1}{(1-s)(2-s)}\right)\right]^{\frac{1}{\mathfrak{q}}} \\
&+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}}\left[\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^{-s}}\left(\frac{2\mu^{2-s}+(2-s)(1-\mu)-1}{(1-s)(2-s)}\right)\right. \\
&+\frac{|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^{-s}}\left(\frac{(2-s)(\mu-1)\mathfrak{n}^{1-s}+(1-\mu+\mathfrak{n})^{2-s}}{(1-s)(2-s)}\right. \\
&+\left.\left.\left.\frac{(2-s)(\mathfrak{n}+1)^{1-s}\mu-\mathfrak{n}^{2-s}-(\mathfrak{n}+1)^{2-s}}{(1-s)(2-s)}\right)\right]^{\frac{1}{\mathfrak{q}}}\right].
\end{aligned}$$

(5) If $\phi(t)=t(1-t)$, then we have result for tgs-preinvex function.

Corollary 3.5. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is tgs-preinvex function, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\
&\le \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}+|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(\mathfrak{n}+1)^2}\right)^{\frac{1}{\mathfrak{q}}}\left[\left(\frac{1}{2}-\varrho+\varrho^2\right)^{1-\frac{1}{\mathfrak{q}}}\right. \\
&\times\left\{\left(\frac{(1-\varrho)^2\left(1+4\mathfrak{n}+2\mathfrak{n}\varrho-\varrho^2\right)}{6}+\frac{6\mathfrak{n}\varrho+2\varrho-4\mathfrak{n}-1}{12}\right)\right\}^{\frac{1}{\mathfrak{q}}} \\
&+\left.\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{\mathfrak{q}}}\left\{\left(\frac{\mu^3\left(1-\mu+\mathfrak{n}\right)}{3}+\frac{6\mathfrak{n}\mu+2\mu-4\mathfrak{n}-1}{12}\right)\right\}^{\frac{1}{\mathfrak{q}}}\right].
\end{aligned}$$

Theorem 3.2. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex function, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\
&\le \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\left[\left((\varrho^{\mathfrak{p}+1})+(1-\varrho)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}K_1+\left(\mu^{\mathfrak{p}+1}+(1-\mu)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}K_2\right],
\end{aligned}$$

where

$$K_1=\left(|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}\int_0^1\phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right)\mathrm{d}t+|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\int_0^1\phi\left(\frac{1-t}{\mathfrak{n}+1}\right)\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}},$$

and

$$K_2 = \left(|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{t}{\mathfrak{n}+1}\right) \mathrm{d}t + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right) \mathrm{d}t \right)^{\frac{1}{\mathfrak{q}}}.$$

Proof. Using Lemma 2.1 and the property of modulus, we have

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2} \left[\int_0^1 |1-\varrho-t| \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right| \mathrm{d}t \right. \\
&\quad \left. + \int_0^1 |\mu - t| \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right| \mathrm{d}t \right] \\
&= \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2} [I_1 + I_2].
\end{aligned} \tag{3.2}$$

Using Holder's inequality, we have

$$\begin{aligned}
I_1 &= \int_0^1 |1-\varrho-t| \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right| \mathrm{d}t \\
&\leq \left(\int_0^1 |1-\varrho-t|^{\mathfrak{p}} \mathrm{d}t \right)^{\frac{1}{\mathfrak{p}}} \left(\int_0^1 \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right|^{\mathfrak{q}} \mathrm{d}t \right)^{\frac{1}{\mathfrak{q}}},
\end{aligned} \tag{3.3}$$

where $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Also

$$\int_0^1 |1-\varrho-t|^{\mathfrak{p}} \mathrm{d}t = \frac{(1-\varrho)^{\mathfrak{p}+1} + (\varrho)^{\mathfrak{p}+1}}{\mathfrak{p}+1}. \tag{3.4}$$

Since $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex functions, we have

$$\begin{aligned}
&\int_0^1 \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2, \mathfrak{a}_1)\right) \right|^{\mathfrak{q}} \mathrm{d}t \\
&\leq \int_0^1 \phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \mathrm{d}t + \int_0^1 \phi\left(\frac{1-t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \mathrm{d}t.
\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we have

$$\begin{aligned}
I_1 &\leq \left(\frac{(1-\varrho)^{\mathfrak{p}+1} + (\varrho)^{\mathfrak{p}+1}}{\mathfrak{p}+1} \right)^{\frac{1}{\mathfrak{p}}} \\
&\quad \times \left(\int_0^1 \phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \mathrm{d}t + \int_0^1 \phi\left(\frac{1-t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \mathrm{d}t \right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} I_2 &= \int_0^1 |\mu - t| \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right| \mathrm{d}t \\ &\leq \left(\int_0^1 |\mu - t|^{\mathfrak{p}} \mathrm{d}t\right)^{\frac{1}{\mathfrak{p}}} \left(\int_0^1 \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right|^{\mathfrak{q}} \mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}, \quad (3.7) \end{aligned}$$

where $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Also

$$\int_0^1 |\mu - t|^{\mathfrak{p}} \mathrm{d}t = \frac{(1-\mu)^{\mathfrak{p}+1} + (\mu)^{\mathfrak{p}+1}}{\mathfrak{p}+1}. \quad (3.8)$$

Since $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex functions, we have

$$\begin{aligned} \int_0^1 \left| \mathcal{F}'\left(\mathfrak{a}_1 + \frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\zeta(\mathfrak{a}_2,\mathfrak{a}_1)\right)\right|^{\mathfrak{q}} \mathrm{d}t &\leq \int_0^1 \phi\left(\frac{t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \mathrm{d}t \\ &\quad + \int_0^1 \phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \mathrm{d}t. (3.9) \end{aligned}$$

Using (3.8) and (3.9) in (3.7), we have

$$\begin{aligned} I_2 \leq& \left(\frac{(1-\mu)^{\mathfrak{p}+1} + (\mu)^{\mathfrak{p}+1}}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} \left(\int_0^1 \phi\left(\frac{t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \mathrm{d}t\right. \\ &\left. + \int_0^1 \phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right) |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}. \quad (3.10) \end{aligned}$$

Now using (3.6) and (3.10) in (3.2), we have

$$\begin{aligned} &|\Delta(\mathfrak{a}_1,\mathfrak{a}_2;\varrho,\mu;\mathfrak{n};\zeta)| \\ &\leq \frac{\zeta(\mathfrak{a}_2,\mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} \\ &\quad \times\left[\left(\varrho^{\mathfrak{p}+1} + (1-\varrho)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} K_1 + \left(\mu^{\mathfrak{p}+1} + (1-\mu)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} K_2\right], \quad (3.11) \end{aligned}$$

where

$$K_1 = \left(|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{\mathfrak{n}+t}{\mathfrak{n}+1}\right) \mathrm{d}t + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{1-t}{\mathfrak{n}+1}\right) \mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}},$$

and

$$K_2 = \left(|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{t}{\mathfrak{n}+1}\right) \mathrm{d}t + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}} \int_0^1 \phi\left(\frac{\mathfrak{n}+1-t}{\mathfrak{n}+1}\right) \mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}.$$

This completes the proof. □

Now we will discuss some special cases for Theorem 3.2.

(1) If $\phi(t) = t$, then we have a result for preinvex function.

Corollary 3.6. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is preinvex function, then we have*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\Bigg[\left(\left(\varrho^{\mathfrak{p}+1} + (1-\varrho)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\right. \\
&\quad \times \left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}(2\mathfrak{n}+1) + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{2(\mathfrak{n}+1)}\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad + \left(\mu^{\mathfrak{p}+1} + (1-\mu)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} + (2\mathfrak{n}+1)|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{2(\mathfrak{n}+1)}\right)^{\frac{1}{\mathfrak{q}}}\Bigg].
\end{aligned}$$

(2) If $\phi(t) = t^s$, then we have a result for s-preinvex function of Breckner type.

Corollary 3.7. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is s-preinvex function of Breckner type, then we have*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\Bigg[\left(\varrho^{\mathfrak{p}+1} + (1-\varrho)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} \\
&\quad \times \left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}((\mathfrak{n}+1)^{1+s} - \mathfrak{n}^{1+s}) + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(1+s)(\mathfrak{n}+1)^s}\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad + \left(\mu^{\mathfrak{p}+1} + (1-\mu)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} + ((\mathfrak{n}+1)^{1+s} - \mathfrak{n}^{1+s})|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(1+s)(\mathfrak{n}+1)^s}\right)^{\frac{1}{\mathfrak{q}}}\Bigg].
\end{aligned}$$

(3) If $\phi(t) = t^{-s}$, then we have a result for s-preinvex function of Godunova-Levin type.

Corollary 3.8. *Under the assumptions of Lemma 2.1 if $|\mathcal{F}'|^{\mathfrak{q}}$ is s-preinvex function of Godunova-Levin type, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\Bigg[\left(\varrho^{\mathfrak{p}+1}) + (1-\varrho)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} \\
&\quad \times \left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}}((\mathfrak{n}+1)^{1-s} - \mathfrak{n}^{1-s}) + |\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(1-s)(\mathfrak{n}+1)^{-s}}\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad + \left(\mu^{\mathfrak{p}+1} + (1-\mu)^{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\left(\frac{|\mathcal{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} + ((\mathfrak{n}+1)^{1-s} - \mathfrak{n}^{1-s})|\mathcal{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}}{(1-s)(\mathfrak{n}+1)^{-s}}\right)^{\frac{1}{\mathfrak{q}}}\Bigg].
\end{aligned}$$

(4) If $\phi(t) = t(1-t)$, then we have a result for tgs-preinvex function.

Corollary 3.9. *Under the assumptions of Lemma 2.1 if $|\mathfrak{F}'|^{\mathfrak{q}}$ is tgs-preinvex function, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\left(\frac{\varrho^{\mathfrak{p}+1}) + (1-\varrho)^{\mathfrak{p}+1}}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} + \left(\frac{\mu^{\mathfrak{p}+1}) + (1-\mu)^{\mathfrak{p}+1}}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\right) \\
&\quad \times \left(\frac{(3\mathfrak{n}+1)(|\mathfrak{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} + |\mathfrak{F}'(\mathfrak{a}_2)|^{\mathfrak{q}})}{6(\mathfrak{n}+1)^2}\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

(5) If $\phi(t) = 1$ then we have a result for P-preinvex function.

Corollary 3.10. *Under the assumptions of Lemma 2.1 if $|\mathfrak{F}'|^{\mathfrak{q}}$ is P-preinvex function, then*

$$\begin{aligned}
&|\Delta(\mathfrak{a}_1, \mathfrak{a}_2; \varrho, \mu; \mathfrak{n}; \zeta)| \\
&\leq \frac{\zeta(\mathfrak{a}_2, \mathfrak{a}_1)}{(\mathfrak{n}+1)^2}\left(\left(\frac{\varrho^{\mathfrak{p}+1} + (1-\varrho)^{\mathfrak{p}+1}}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} + \left(\frac{\mu^{\mathfrak{p}+1}) + (1-\mu)^{\mathfrak{p}+1}}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\right) \\
&\quad \times \left(|\mathfrak{F}'(\mathfrak{a}_1)|^{\mathfrak{q}} + |\mathfrak{F}'(\mathfrak{a}_2)|^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

Conclusion

A new generalized integral identity involving differentiable function was obtained. Using this identity we obtained several new integral inequalities of Hermite-Hadamard type. New special cases are also discussed in detail. It is expected that the ideas and techniques of the results obtained will inspire interested readers.

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