ON THE HILBERT DEPTH OF CERTAIN MONOMIAL IDEALS AND APPLICATIONS

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We study the Stanley depth and the Hilbert depth for I and S/I, where $I \subset S = K[x_1,\dots,x_n]$ is the intersection of monomial prime ideals with disjoint sets of variables. As an application, we obtain bounds for the Stanley depth of $I_{n,m}^t$ and $J_{n,m}^t$, where $I_{n,m}$ is the m-path ideal of the path graph of length n and $J_{n,m}$ is the the m-path ideal of the cycle graph of length n.

Keywords Monomial ideal, Stanley depth, Hilbert depth, Depth, Path ideal
MSC2020 05A18, 06A07, 13C15, 13P10, 13F20.

Introduction

Let K be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_iK[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1,\dots,x_n\}$ such that $m_iK[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\dots,r} |Z_i|$ and $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D})|\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vladoiu and Zheng show in [9] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [1], J. Apel restated a conjecture firstly given by Stanley in [13], namely that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see [8]. For a friendly introduction in the thematic of Stanley depth, we refer the reader [10].

Let M be a finitely generated graded S-module. The Hilbert depth of M, denoted by $\operatorname{hdepth}(M)$, is the maximal depth of a finitely generated graded S-module N with the same Hilbert series as M. In [6] we introduced a new method to compute the Hilbert depth of a quotient J/I of two squarefree monomial ideals $I \subset J \subset S$; see Section 1.

In Section 2 we consider the edge ideal of a complete bipartite graph, that is

$$I := (x_1,\dots,x_n) \cap (x_{n+1},\dots,x_{n+m}) \subset S := K[x_1,\dots,x_{n+m}],$$

and we study the Stanley depth and the Hilbert depth of I and S/I.

Assume $m \leq n$. In Proposition 2.**2** we show that

$$m \geq \operatorname{sdepth}(S/I) \geq \min\{m, \left\lceil \frac{n}{2} \right\rceil\}.$$

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Also, in Theorem 2.**6** we prove that

$$\operatorname{hdepth}(S/I) \leq \left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}} \right\rfloor.$$

In particular, we note that $\operatorname{hdepth}(S/I) < m$ if and only if $n \leq 2m-2$.

In Theorem 2.**9** we prove that

$$\operatorname{hdepth}(I) = \operatorname{sdepth}(I) = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil,$$

if n and m are not both even. Also, we prove that if $n=2s$ and $m=2t$ then

$$t+s \leq \operatorname{sdepth}(I) \leq \operatorname{hdepth}(S/I) = t+s+1.$$

In particular, we have $\operatorname{hdepth}(I) = \left\lfloor \frac{n+m+2}{2} \right\rfloor$ for any $n \geq m \geq 1$.

In Section 3 we consider a generalization of the ideal from the previous section, namely

$$I := I_{n_1,\ldots,n_r} := (x_1,\ldots,x_{n_1}) \cap (x_{n_1+1},\ldots,x_{n_1+n_2}) \cap \cdots \cap (x_{n_1+\cdots+n_{r-1}+1},\ldots,x_N) \subset S,$$

where $N = n_1+\cdots+n_r$ and $S = K[x_1,\ldots,x_N]$. In Theorem 3.**3** we prove that

$$\left\lfloor \frac{N+r}{2} \right\rfloor \geq \operatorname{hdepth}(I) \geq \operatorname{sdepth}(I) \geq \left\lceil \frac{n_1}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil.$$

Also, we conjecture that

$$\operatorname{hdepth}(I) = \left\lfloor \frac{N+r}{2} \right\rfloor.$$

This formula holds for $r=2$ and if $r \geq 3$ and at most one of the numbers $n_1,\ldots,n_r$ is even. In Proposition 3.**8** we characterize $\operatorname{hdepth}(S/I)$ and $\operatorname{hdepth}(I)$ in combinatorial terms. In Proposition 3.**10** we show that

$$\operatorname{hdepth}(S/I) \leq \min\{d \geq r \ : \ \binom{N-d+r-1}{r} < n_1 n_2 \cdots n_r\} - 1.$$

Based on Proposition 3.**13**, we conjecture that $\operatorname{hdepth}(S/I) \approx N - \left\lceil \sqrt[r]{r! n_1 n_2 \cdots n_r} \right\rceil$.

Let $n > m \geq 2$ and $t \geq 1$ be some integers. In Section 4 we apply the results from Section 3 in order to obtain sharper bounds for the Stanley depth of $I_{n,m}^t$ and $J_{n,m}^t$, where

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots, x_{n-m+1}\cdots x_n) \subset S := K[x_1,\ldots,x_n],$$

is the m-path ideal associated to path graph of length n and

$$J_{n,m} = I_{n,m} + (x_{n-m+2}\cdots x_n x_1, \ldots, x_n x_1 \cdots x_{m-1}) \subset S$$

is the m-path ideal associated to the cycle graph of length n.

In Theorem 4.**3** we show that

$$\operatorname{sdepth}(I_{n,m}^t) \leq \min\{n - \left\lceil \frac{t_0}{2} \right\rceil, n - \left\lfloor \frac{n-t_0+1}{m+1} \right\rfloor + 1\},$$

where $t_0 = \min\{t, n-m\}$. In Theorem 4.**5** we show that

$$\operatorname{sdepth}(J_{n,m}^t) \leq \left\lfloor \frac{n+d}{2} \right\rfloor,$$

for any $t \geq n-1$, where $d = \gcd(n,m)$.

1. **Preliminaries**

First, we fix some notations and we recall the main result of [6].

We denote $[n] := \{1, 2, \dots, n\}$ and $S := K[x_1, \dots, x_n]$.

For a subset $C \subset [n]$, we denote $x_C := \prod_{j\in C} x_j \in S$.

For two subsets $C \subset D \subset [n]$, we denote $[C, D] := \{A \subset [n] \ : \ C \subset A \subset D\}$, and we call it the *interval* bounded by C and D.

Let $I \subset J \subset S$ be two square free monomial ideals. We let:

$$\mathrm{P}_{J/I} := \{C \subset [n] \ : \ x_C \in J \setminus I\} \subset 2^{[n]}.$$

A partition of $P_{J/I}$ is a decomposition:

$$\mathcal{P} : \ \mathrm{P}_{J/I} = \bigcup_{i=1}^{r} [C_i, D_i],$$

into disjoint intervals.

If $\mathcal{P}$ is a partition of $\mathrm{P}_{J/I}$, we let $\operatorname{sdepth}(\mathcal{P}) := \min_{i=1}^{r} |D_i|$. The Stanley depth of $P_{J/I}$ is

$$\operatorname{sdepth}(P_{J/I}) := \max\{\operatorname{sdepth}(\mathcal{P}) \ : \ \mathcal{P} \text{ is a partition of } \mathrm{P}_{J/I}\}.$$

Herzog, Vlădoiu and Zheng proved in [9] that:

$$\operatorname{sdepth}(J/I) = \operatorname{sdepth}(\mathrm{P}_{J/I}).$$

Let $\mathrm{P} := \mathrm{P}_{J/I}$, where $I \subset J \subset S$ are square-free monomial ideals. For any $0 \leq k \leq n$, we denote:

$$\mathrm{P}_k := \{A \in \mathrm{P} \ : \ |A| = k\} \text{ and } \alpha_k(J/I) = \alpha_k(\mathrm{P}) = |\,\mathrm{P}_k\,|.$$

For all $0 \leq d \leq n$ and $0 \leq k \leq d$, we consider the integers

$$\beta_k^d(J/I) := \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(J/I). \tag{1.1}$$

From (1.1) we can easily deduce that

$$\alpha_k(J/I) = \sum_{j=0}^{k} \binom{d-j}{k-j} \beta_k^d(J/I), \text{ for all } 0 \leq k \leq d. \tag{1.2}$$

Also, we have that

$$\beta_k^d(J/I) = \alpha_k(J/I) - \binom{d}{k}\beta_0^d(J/I) - \binom{d-1}{k-1}\beta_1^d(J/I) - \cdots - \binom{d-k+1}{1}\beta_{k-1}^d(J/I). \tag{1.3}$$

Theorem 1.1. *(*[6, Theorem 2.4]*) With the above notations, the* Hilbert depth *of J/I is*

$$\operatorname{hdepth}(J/I) := \max\{d \ : \ \beta_k^d(J/I) \geq 0 \textit{ for all } 0 \leq k \leq d\}.$$

As a basic property of the Hilbert depth, we state the following:

Proposition 1.2. *Let $I \subset J \subset S$ be two square-free monomial ideals. Then*

$$\operatorname{sdepth}(J/I) \leq \operatorname{hdepth}(J/I).$$

2. **Edge ideal of a complete bipartite graph**

Let n and m be two positive integers. We let $S = K[x_1, x_2, \ldots, x_{n+m}]$ and we consider the square free monomial ideal:

$$I := (x_1, \ldots, x_n) \cap (x_{n+1}, \ldots, x_{n+m}) \subset S.$$

Our aim is to study the Stanley depth and the Hilbert depth of I and S/I.

As usual, given a positive integer k, we denote $[k] := \{1, 2, \ldots, k\}$.

Remark 2.1. Let $K_{n,m} = (V, E)$ be the complete bipartite graph, that is $V = V' \cup V''$, where $V' = \{1, \ldots, n\}$, $V'' = \{n+1, \ldots, n+m\}$ and $E = \{\{i, j\} \ : \ i \in [n],\ j - n \in [m]\}$. Note that $I = (x_i x_{n+j} \ : \ i \in [n],\ j \in [m])$ is the edge ideal of $K_{n,m}$.

Also, we mention that $\operatorname{depth}(S/I) = 1$, which can be easily checked.

Proposition 2.2. *Let $n \geq m \geq 1$ be two integers. Then:*
(1) $m \geq \operatorname{sdepth}(S/I) \geq \min\{m, \lceil \frac{n}{2} \rceil\}$.
(2) $m + \lceil \frac{n}{2} \rceil \geq \operatorname{sdepth}(I) \geq \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$.
(3) *If $n \geq 2m - 1$ then* $\operatorname{sdepth}(S/I) = m$.

Proof. (1) Since $I = I'S \cap I''S$, where $I' = (x_1, \ldots, x_n) \subset S' = K[x_1, \ldots, x_n]$ and $I'' = (x_{n+1}, \ldots, x_{n+m}) \subset S'' = K[x_{n+1}, \ldots, x_{n+m}]$, from [3, Theorem 1.3(2)] it follows that

$$\operatorname{sdepth}(S/I'S) \geq \operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/I'S),\ \operatorname{sdepth}_{S''}(S''/I'') + \operatorname{sdepth}_{S'}(I')\}.$$

As $S/I'S \cong S''$, we have that $\operatorname{sdepth}(S/I'S) = m$.

Also, $S''/I'' \cong K$, so $\operatorname{sdepth}_{S''}(S''/I'') = 0$.

Finally, $\operatorname{sdepth}_{S'}(I') = \lceil \frac{n}{2} \rceil$, see [2, Theorem 2.2].

(2) Since $(I : x_{n+1}) = I'S$, from [12, Proposition 2], [2, Theorem 2.2] and [9, Lemma 3.6] we have

$$\operatorname{sdepth}(I) \leq \operatorname{sdepth}(I : x_{n+1}) = \operatorname{sdepth}(I'S) = m + \operatorname{sdepth}_{S'}(I') = m + \left\lceil \frac{n}{2} \right\rceil.$$

The other inequality follows from [11, Lemma 1.1] and [2, Theorem 2.2].

(3) If $n \geq 2m - 1$ then $\lceil \frac{n}{2} \rceil \geq m$, hence the result follows from (1). □

Lemma 2.3. *Let $n \geq m \geq 1$ be two integers and $N := n + m$. We have that*
(1) $\alpha_k(I) = \begin{cases} 0, & 0 \leq k \leq 1 \\ \sum_{j=1}^{k-1} \binom{n}{j}\binom{m}{k-j}, & 2 \leq k \leq N \end{cases}$.
(2) $\alpha_k(I) = \binom{N}{k} - \binom{n}{k} - \binom{m}{k} + \delta_{k0}$*, for all* $0 \leq k \leq N$.
(3) $\alpha_k(S/I) = \binom{n}{k} + \binom{m}{k} - \delta_{k0}$*, for all* $0 \leq k \leq N$*, where* $\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$ *is the Kronecker symbol.*

Proof. (1) Since I is generated in degree 2, we have $\alpha_0(I) = \alpha_1(I) = 1$. Any squarefree monomial $u \in I$ with $\deg(u) = k \geq 2$ can be written as $u = v \cdot w$, where $v \in S' = K[x_1, \ldots, x_n]$ and $w \in S'' = K[x_{n+1}, \ldots, x_N]$ are squarefree monomials. Assume $\deg(v) = j$ with $1 \leq j \leq k-1$. Then $\deg(w) = k - j$. Since there are $\binom{n}{j}$ squarefree monomials of degree j in S' and $\binom{m}{k-j}$ squarefree monomials of degree $k - j$ in S'', we easily get the required conclusion.

(2) For $k \leq 1$ the identity can be easily checked. Assume $k \geq 2$. From (1) and the well known combinatorial formula $\sum\limits_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k} = \binom{N}{k}$, we get the required conclusion.

(3) It follows immediately from (2).

□

Lemma 2.4. *For any integers* $0 \leq k \leq d$ *and* $n \geq 0$ *we have that*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n}{j} = (-1)^k\binom{d-n}{k} = \binom{n-d+k-1}{k}.$$

Proof. Using the identity $(-1)^k\binom{x}{k} = \binom{-x+k-1}{k}$ and the Chu–Vandermonde summation, we get the required formula. □

Lemma 2.5. *Let* $n \geq m \geq 1$ *and* $0 \leq k \leq d \leq N := n+m$ *some integers. We have that*

(1) $\beta_k^d(S/I) = \binom{n-d+k-1}{k} + \binom{m-d+k-1}{k} + (-1)^{k+1}\binom{d}{k}$,

(2) $\beta_k^d(I) = \binom{N-d+k-1}{k} - \binom{n-d+k-1}{k} - \binom{m-d+k-1}{k} + (-1)^k\binom{d}{k}$.

Proof. (1) From (1.1), Lemma 2.**3**(3) and Lemma 2.**4** we have that

$$\beta_k^d(S/I) = \sum_{j=1}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n}{j} + \sum_{j=1}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{m}{j} - (-1)^k\binom{d}{k} =$$
$$= \binom{n-d+k-1}{k} + \binom{m-d+k-1}{k} + (-1)^{k+1}\binom{d}{k},$$

as required.

(2) The proof is similar, using (1.1), Lemma 2.**3**(2) and Lemma 2.**4**. □

Note that, if $n \geq 2m-1$ then, according to Proposition 2.**2**(3) and Proposition 1.**2** we have $\operatorname{hdepth}(S/I) \geq \operatorname{sdepth}(S/I) = m$. Also, $\operatorname{sdepth}(S/I) \leq m$, for any $n \geq m$.

Theorem 2.6. *Let* $n \geq m \geq 1$ *be two integers. Then*

$$\operatorname{sdepth}(S/I) \leq \operatorname{hdepth}(S/I) \leq \left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}}\right\rfloor.$$

In particular, if $n \leq 2m-2$ *then* $\operatorname{hdepth}(S/I) < m$.

Proof. The first inequality follows from Proposition 1.**2**. We consider the quadratic function

$$\varphi(t) = \frac{1}{2}t(t-1) - (n+m)t + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1).$$

Note that, according to Lemma 2.**5**(1), we have that

$$\beta_2^d(S/I) = \frac{1}{2}((n-d)(n-d+1) + (m-d)(m-d+1) - d(d-1)) =$$
$$= \frac{1}{2}d(d-1) - (n+m)d + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1) = \varphi(d).$$

The roots of $\varphi(t) = 0$ are $t_{1,2} = n+m+\frac{1}{2} \pm \sqrt{2mn+\frac{1}{4}}$ and therefore

$$\varphi(t) < 0 \text{ if and only if } t \in \left(n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}}, n+m+\frac{1}{2}+\sqrt{2mn+\frac{1}{4}}\right).$$

From the fact that $\beta_2^d(S/I) = \varphi(d)$ and the above, it follows that

$$\beta_2^d(S/I) < 0 \text{ for } \left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}}\right\rfloor + 1 \leq d \leq n+m.$$

From Theorem 1.**1**, we get $\operatorname{hdepth}(S/I) \leq \left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}} \right\rfloor$.

In order to prove the last part, we consider the function

$$\psi(x) = x+m+\frac{1}{2}-\sqrt{2mx+\frac{1}{4}},\ x\in[m,\infty).$$

Since $\frac{d\psi}{dx}(x)>0$, $m\leq n\leq 2m-2$ and $\psi(2m-1)=m$, it follows that

$$\left\lfloor 2m+\frac{1}{2}-\sqrt{2m^2+\frac{1}{4}} \right\rfloor \leq \lfloor \psi(n) \rfloor = \left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}} \right\rfloor < \psi(2m-1)=m,$$

as required. □

Theorem 2.7. *Let $n\geq m\geq 2$ be two integers. Then*

$$\operatorname{hdepth}(S/I) = \max\{d\leq n+m\ :\ \binom{d-n}{2\ell}+\binom{d-m}{2\ell}\geq\binom{d}{2\ell} \text{ for all } 1\leq\ell\leq\left\lfloor\frac{d}{2}\right\rfloor\}.$$

Moreover, $\operatorname{hdepth}(S/I) < m$ *if* $n\leq 2m-2$*. Also,* $m\leq\operatorname{hdepth}(S/I)\leq n-m+1$ *if* $n\geq 2m-1$.

Proof. Let $q:=\max\{d\leq n+m\ :\ \binom{d-n}{2\ell}+\binom{d-m}{2\ell}\geq\binom{d}{2\ell}$ for all $1\leq\ell\leq\lfloor\frac{d}{2}\rfloor\}$.

From Lemma 2.**5**(1) and the identity $\binom{x}{k}=\binom{-x+k-1}{k}$ it follows that

$$\beta_{2\ell}^d(S/I)=\binom{d-n}{2\ell}+\binom{d-m}{2\ell}-\binom{d}{2\ell}. \tag{2.1}$$

Hence $\operatorname{hdepth}(S/I)\leq q$.

On the other hand, from the proof of Theorem 2.**6** and (2.1), it follows that

$$q\leq\left\lfloor n+m+\frac{1}{2}-\sqrt{2mn+\frac{1}{4}} \right\rfloor. \tag{2.2}$$

We consider two cases:

(i) $n\leq 2m-2$. From Theorem 2.**6**, it follows that $q<m$. From Lemma 2.**5**(2) and $0\leq k\leq q$ with k odd, we have

$$\beta_k^q(S/I)=\binom{n-q+k-1}{k}+\binom{m-q+k-1}{k}+\binom{q}{k}\geq\binom{n-m+k}{k}+1+\binom{q}{k}>0.$$

Since, by the definition of q, we have $\beta_k^q(S/I)\geq 0$ for all $0\leq k\leq q$ with k even, we conclude that $\operatorname{hdepth}(S/I)\geq q$. Hence $\operatorname{hdepth}(S/I)=q<m$, as required.

(ii) $n\geq 2m-1$. First, note that

$$m=\operatorname{sdepth}(S/I)\leq\operatorname{hdepth}(S/I)\leq q.$$

From (2.2) and the above it follows that

$$m\leq q\leq\left\lfloor n+m+\frac{1}{2}-\sqrt{2m(2m-1)+\frac{1}{4}}\right\rfloor=\left\lfloor n+m+\frac{1}{2}-(2m-\frac{1}{2})\right\rfloor=n-m+1.$$

From Lemma 2.**5**(2) and $0\leq k\leq q$ with k odd, we have

$$\beta_k^q(S/I)=\binom{n-q+k-1}{k}+\binom{q-m}{k}+\binom{q}{k}\geq\binom{m}{k}+0+\binom{m}{k}\geq 0.$$

Using the same argument as in the case (i), it follows that $\operatorname{hdepth}(S/I)=q$, as required.

Thus, the proof is complete. □

Lemma 2.8. *Let $n \geq m \geq 1$ be two integers. Then*

$$\operatorname{hdepth}(I) \leq \left\lfloor \frac{n+m+2}{2} \right\rfloor.$$

Proof. If $n+m=2$, that is $n=m=1$, then there is nothing to prove. Assume $n+m \geq 3$. From Lemma 2.**5**(2) and straightforward computations, it follows that

$$\beta_3^d(I) = \frac{nm(n+m-2d+2)}{2} < 0,$$

if and only if $d > \frac{n+m+2}{2}$. Hence, we get the required result. □

Theorem 2.9. *Let $n \geq m \geq 1$ be two integers.*

(1) *If n and m are not both even then we have that:*

$$\operatorname{sdepth}(I) = \operatorname{hdepth}(I) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil.$$

(2) *If $n = 2t$ and $m = 2s$ then we have that:*

$$t+s \leq \operatorname{sdepth}(I) \leq \operatorname{hdepth}(I) \leq t+s+1.$$

In both cases, we have $\operatorname{hdepth}(I) = \left\lfloor \frac{n+m+2}{2} \right\rfloor$.

Proof. (1) According to Proposition 2.**2**(2), we have that

$$\operatorname{sdepth}(I) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil. \tag{2.3}$$

On the other hand, according to Proposition 1.**2** and Lemma 2.**8**, we have that

$$\operatorname{sdepth}(I) \leq \operatorname{hdepth}(I) \leq \left\lfloor \frac{n+m}{2} \right\rfloor + 1. \tag{2.4}$$

Note that, if n and m are not both even, then

$$\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil = \left\lfloor \frac{n+m}{2} \right\rfloor + 1. \tag{2.5}$$

Hence, (1) follows from (2.3), (2.4) and (2.5).

(2) From (2.3) and (2.4) we have that

$$t+s \leq \operatorname{sdepth}(I) \leq \operatorname{hdepth}(I) \leq t+s+1.$$

On the other hand, for $0 \leq k \leq t+s+1$, from Lemma 2.**5**(2) we have

$$\beta_k^{t+s+1}(I) = \binom{t+s-2+k}{k} - \binom{t-s-2+k}{k} - \binom{s-t-2+k}{k} + (-1)^k \binom{t+s+1}{k}. \tag{2.6}$$

By direct computations, from (2.6) it follows that

$$\beta_0^{t+s+1}(I) = 0,\ \beta_1^{t+s+1}(I) = 0,\ \beta_2^{t+s+1}(I) = 4st \text{ and } \beta_3^{t+s+1}(I) = 0.$$

Also, by straightforward computations, we get

$$\beta_4^{t+s+1}(I) = \beta_5^{t+s+1}(I) = \frac{ts(2s^2+2t^2-1)}{3} > 0.$$

Now, assume $6 \leq k \leq t+s+1$. Without any loss of generality, we assume that $t = s+a$, where a is a nonnegative integer. In order to prove that $\beta := \beta_k^{t+s+1}(I) \geq 0$, we consider the following cases:

(i) k is even.

(i.1) $a = 0$. From (2.6) and the fact that $s \geq 1$ it follows that

$$\beta = \binom{2s-2+k}{k} - \binom{k-2}{k} - \binom{k-2}{k} + \binom{2s+1}{k} \geq \binom{k}{k} + \binom{2s+1}{k} > 0.$$

(i.2) $a = 1$. From (2.6) and the fact that $s \geq 1$ it follows that

$$\beta = \binom{2s-1+k}{k} - \binom{k-1}{k} - \binom{k-3}{k} + \binom{2s+2}{k}\binom{k+1}{k} + \binom{2s+2}{k} > 0.$$

(i.3) $a \geq 2$ and $k \geq a+2$. From (2.6) we get

$$\beta = \binom{2s+a+k-2}{k} - \binom{a+k-2}{k} + \binom{2s+a+1}{k} > \binom{2s+a+1}{k} \geq 0.$$

(i.4) $a \geq 5$ and $k \leq a+1$. From (2.6), using $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$, we get

$$\beta = \binom{2s+a+k-2}{k} - \binom{a+k-2}{k} - \binom{a+1}{k} + \binom{2s+a+1}{k} =$$
$$= \binom{2s+a+k-2}{k} - \binom{a+k-2}{k} + \binom{2s+a+1}{k} - \binom{a+1}{k} > 0.$$

(ii) k is odd.

(ii.1) $a = 0$. From (2.6) and the fact that $k \geq 6$ it follows that

$$\beta = \binom{2s-2+k}{k} - \binom{k-2}{k} - \binom{k-2}{k} - \binom{2s+1}{k} \geq \binom{2s+4}{k} - \binom{2s+1}{k} > 0.$$

(ii.2) $a = 0$. From (2.6) and the fact that $k \geq 6$ it follows that

$$\beta = \binom{2s-1+k}{k} - \binom{k-1}{k} - \binom{k-3}{k} - \binom{2s+2}{k} \geq \binom{2s+5}{k} - \binom{2s+2}{k} > 0.$$

(ii.3) $a \geq 2$ and $k \geq a+2$. From (2.6) and the fact that $k \geq 6$ and $s \geq 1$ we get

$$\beta = \binom{2s+a+k-2}{k} - \binom{a+k-2}{k} - \binom{2s+a+1}{k} > \binom{2s+a+k-3}{k-1} -$$
$$- \binom{a+k-2}{k} \geq \binom{a+k-1}{k-1} - \binom{a+k-2}{k} = \frac{ak+k^2-k-a^2+a}{a(a-1)}\binom{a+k-2}{k} > 0.$$

(ii.4) $a \geq 5$ and $k \leq a+1$. From (2.6), using $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$, we get

$$\beta = \binom{2s+a+k-2}{k} - \binom{a+k-2}{k} + \binom{a+1}{k} - \binom{2s+a+1}{k} =$$
$$= \binom{2s+a+k-2}{k} - \binom{2s+a+1}{k} - \left(\binom{a+k-2}{k} - \binom{a+1}{k}\right) =$$
$$= \sum_{\ell=2s+a+1}^{2s+a+k-3} \binom{\ell}{k-1} - \sum_{\ell=a+1}^{a+k-3} \binom{\ell}{k-1} > 0.$$

Hence, $\operatorname{hdepth}(I) = s+t+1$ and the proof is complete. □

3. **A generalization**

Let $n_1, n_2, \ldots, n_r$ be some positive integers, $N = n_1 + \cdots + n_r$ and $S := K[x_1, \ldots, x_N]$. We consider the ideal

$$I := I_{n_1,n_2,\ldots,n_r} := (x_1, \ldots, x_{n_1}) \cap (x_{n_1+1}, \ldots, x_{n_1+n_2}) \cap \cdots \cap (x_{n_1+\cdots+n_{r-1}+1}, \ldots, x_N) \subset S,$$

which generalize the ideal I from the previous section.

Lemma 3.1. *With the above notations, we have that*

$$\alpha_k(I) = \begin{cases} 0, & k \leq r-1 \\ \sum\limits_{\substack{\ell_1,\ell_2,\ldots,\ell_r \geq 1 \\ \ell_1+\cdots+\ell_r=k}} \binom{n_1}{\ell_1}\binom{n_2}{\ell_2}\cdots\binom{n_r}{\ell_r}, & k \geq r \end{cases}.$$

Proof. Since I is generated by square free monomials of degree k, it is clear that $\alpha_k(I)=0$ for $k \leq r-1$. Assume $k \geq r$ and let $u \in I$ a square free monomial of degree k. It follows that $u = u_1 \cdots u_r$, where

$$1 \neq u_j \in I_j := (x_{n_1+\cdots+n_{j-1}+1}, \ldots, x_{n_1+\cdots+n_j}) \text{ for all } 1 \leq j \leq r.$$

Let $\ell_j = \deg(u_j) \geq 1$. Since there are $\binom{n_j}{\ell_j}$ squarefree monomials of degree ℓ_j in I_j, we get the required conclusion. □

Lemma 3.2. *With the above notations, we have that*

(1) $\alpha_k(I) = 0$ *for* $k \leq r-1$.
(2) $\alpha_r(I) = n_1 n_2 \cdots n_r$.
(3) $\alpha_{r+1}(I) = n_1 n_2 \cdots n_r \cdot \frac{n_1+\cdots+n_r-r}{2}$.
(4) $\alpha_k(I) = \binom{N}{k}$ *for* $k \geq N - \min\limits_{i=1}^{r} n_i + 1$.

Proof. (1), (2) and (3) follow immediately from Lemma 3.**1**. In order to prove (4), it is enough to observe that any squarefree monomial $u \in S$ of degree $k \geq N - \min\limits_{i=1}^{r} n_i + 1$, belongs to I. □

Theorem 3.3. *With the above notations, we have that:*

$$\left\lfloor \frac{N+r}{2} \right\rfloor \geq \operatorname{hdepth}(I) \geq \operatorname{sdepth}(I) \geq \left\lceil \frac{n_1}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil.$$

Proof. In order to prove the first inequality, let $d > \frac{n_1+\cdots+n_r+r}{2}$ be an integer. From Theorem 1.**1** and Lemma 3.**2**(1,2,3) it follows that

$$\beta_k^d(I) = 0 \text{ for } 0 \leq k \leq r-1,\ \beta_r^d(I) = \alpha_r(I) = n_1 n_2 \cdots n_r \text{ and}$$
$$\beta_{r+1}^d(I) = \alpha_{r+1}(I) - (d-r)\alpha_r(I) = n_1 n_2 \cdots n_r \left(\frac{n_1+\cdots+n_r+r}{2} - d \right) < 0.$$

Hence $\operatorname{hdepth}(I) \leq \frac{n_1+\cdots+n_r+r}{2}$.

The inequality $\operatorname{hdepth}(I) \geq \operatorname{sdepth}(I)$ follows from Proposition 1.**2**, and the last inequality follows from [3, Corollary 1.9(1)] and [2, Theorem 2.2]. □

Based on our computer experiments, we propose the following conjecture:

Conjecture 3.4. *With the above notations, we have*

$$\operatorname{hdepth}(I) = \left\lfloor \frac{N+r}{2} \right\rfloor.$$

Note that, according to Theorem 2.**9**, Conjecture 3.**4** holds for $r=2$. Also, according to Theorem 3.**3**, Conjecture 3.**4** is true when at most one of the numbers $n_1, \ldots, n_r$ is even.

Proposition 3.5. *With the above notations, we have that*

$$N - \min_{i=1}^{r} n_i \geq \operatorname{hdepth}(S/I) \geq \operatorname{sdepth}(S/I) \geq \left\lceil \frac{n_1}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil - \min_{i=1}^{r} \left\lceil \frac{n_i}{2} \right\rceil.$$

Proof. From Lemma 3.**2**(4) it follows that

$$\alpha_k(S/I) = 0 \text{ for all } k \geq N - \max_{i=1}^{r} n_i + 1.$$

Hence, from [6, Lemma 1.3], we get the first inequality.

The second inequality follows from [3, Corollary 1.9] and [2, Theorem 2.2]. $\square$

Note that, Proposition 3.**5** reproves the fact that $\operatorname{hdepth}(S/I) \geq \operatorname{depth}(S/I) = r - 1$.

Lemma 3.6. *For any* $0 \leq k \leq N$*, we have that:*

1. $\alpha_k(I) = \sum_{J \subsetneq [n]} (-1)^{|J|} \binom{N - \sum_{i\in J} n_i}{k}$.
2. $\alpha_k(I) = \sum_{\emptyset \neq J \subsetneq [n]} (-1)^{|J|+1} \binom{N - \sum_{i\in J} n_i}{k}$.

Proof. (1) For all $1 \leq i \leq r$ we let:

$$A_i = \{(\ell_1, \ldots, \ell_d) \ : \ \ell_1 + \cdots + \ell_r = k, \ell_i = 0 \text{ and } \ell_j \geq 0 \text{ for } j \neq i\}.$$

Also, we consider the set:

$$A = \{(\ell_1, \ldots, \ell_d) \ : \ \ell_1 + \cdots + \ell_r = k \text{ and } \ell_i \geq 0 \text{ for all } 1 \leq i \leq r\}.$$

For any nonempty subset $J \subset [n]$, we let $A_J = \bigcup_{i\in J} A_i$. Also, we denote $A_\emptyset = A$. From Lemma 3.**1** it follows that

$$\alpha_k(I) = \sum_{(\ell_1,\ldots,\ell_r)\in A_\emptyset \setminus (\bigcup_{i=1}^r A_i)} \binom{n_1}{\ell_1} \cdots \binom{n_r}{\ell_r}. \tag{3.1}$$

Note that this equality holds also for $k < r$ as both terms are zero in this case. It is well known that

$$\sum_{(\ell_1,\ldots,\ell_r)\in A} \binom{n_1}{\ell_1} \cdots \binom{n_r}{\ell_r} = \binom{n_1 + \cdots + n_r}{k} = \binom{N}{k}.$$

Similarly, if $J \subset [n]$ then

$$\sum_{(\ell_1,\ldots,\ell_r)\in \bigcap_{i\in J} A_i} \binom{n_1}{\ell_1} \cdots \binom{n_r}{\ell_r} = \binom{N - \sum_{i\in J} n_i}{k}. \tag{3.2}$$

From (3.1) and (3.2), using the inclusion exclusion principle, we get the required conclusion.

(2) Follows from (1) and the fact that $\alpha(S/I) = \binom{N}{k} - \alpha(I)$. $\square$

Note that Lemma 3.**6** generalizes Lemma 2.**3**(2,3). From Lemma 3.**6** and Lemma 2.**4** we get the following generalization of Lemma 2.**5**:

Lemma 3.7. *For any* $0 \leq k \leq d \leq N$*, we have that:*

1. $\beta_k^d(I) = \sum\limits_{J \subsetneq [n]} (-1)^{|J|} \binom{N - \sum_{i\in J} n_i - d + k - 1}{k}$.
2. $\beta_k^d(S/I) = \sum\limits_{\emptyset \neq J \subsetneq [n]} (-1)^{|J|+1} \binom{N - \sum_{i\in J} n_i - d + k - 1}{k}$.

Proposition 3.8. *With the above notations, we have that:*

(1) $\operatorname{hdepth}(I) = \max\{d \ : \ \left\lceil \frac{n_1}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil \leq d \leq \left\lfloor \frac{N+r}{2} \right\rfloor$ *and*

$$\sum_{J \subsetneq [n]} (-1)^{|J|} \binom{N - \sum_{i\in J} n_i - d + k - 1}{k} \geq 0 \text{ for all } r \leq k \leq d\}.$$

(2) $\operatorname{hdepth}(S/I) = \max\{d \ : \ \left\lceil \frac{n_1}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil - \min_{i=1}^{r} \left\lceil \frac{n_i}{2} \right\rceil \leq d \leq N - \min_{i=1}^{r} n_i$ *and*

$$\sum_{\emptyset \neq J \subsetneq [n]} (-1)^{|J|+1} \binom{N - \sum_{i\in J} n_i - d + k - 1}{k} \geq 0 \text{ for all } r \leq k \leq d\}.$$

Proof. (1) It follows from Theorem 3.**3** and Lemma 3.**7**(1).

(1) It follows from Proposition 3.**5** and Lemma 3.**7**(2). □

Lemma 3.9. *Let $d \geq r$. We have that*

$$\beta_r^d(S/I) = \binom{N-d+r-1}{r} - n_1 n_2 \cdots n_r.$$

Proof. From Lemma 3.**2** it follows that

$$\alpha_k(S/I) = \binom{N}{k} \text{ for } k \leq r-1, \ \alpha_r(S/I) = \binom{N}{r} - n_1 n_2 \cdots n_r.$$

Hence, the required result follows from (1.1) and Lemma 2.**4**. □

Proposition 3.10. *With the above notations, we have that:*

$$\operatorname{hdepth}(S/I) \leq \min\{d \geq r \ : \ \binom{N-d+r-1}{r} < n_1 n_2 \cdots n_r\} - 1.$$

Proof. First of all, note that, according to (1.3), we have

$$\beta_0^N(S/I) = 1 \text{ and } \beta_k^N(S/I) = 0 \text{ for all } 1 \leq k \leq r-1.$$

Moreover, according to Lemma 3.**9**, (1.1) and (1.3), we have

$$\beta_r^N(S/I) = \binom{N-N+r-1}{r} - n_1 n_2 \cdots n_r = -n_1 n_2 \cdots n_r < 0.$$

Therefore, we have that $q := \min\{d \geq r \ : \ \binom{N-d+r-1}{r} < n_1 n_2 \cdots n_r\}$ is well defined and $q \leq N$. Now, it is enough to notice that, from above, $\beta_r^q(S/I) < 0$ and thus $\operatorname{hdepth}(S/I) \leq q - 1$, as required. □

Lemma 3.11. *We have that*

$$\binom{N-d+r-1}{r} \geq n_1 n_2 \cdots n_r \text{ for all } d \leq N - \left\lceil \sqrt[r]{r! n_1 n_2 \cdots n_r} \right\rceil.$$

Proof. We assume that $d = \lfloor aN \rfloor$, where $a \in (0,1)$. Then

$$\binom{N-d+r-1}{r} = \frac{(N-d+r-1)(N-d+r-2)\cdots(N-d)}{r!} \geq$$

$$\geq \frac{(N-aN+r-1)(N-aN+r-2)\cdots(N-aN)}{r!} \geq \frac{N^r(1-a)^r}{r!}. \tag{3.3}$$

On the other hand

$$\frac{N^r(1-a)^r}{r!} \geq n_1 n_2 \cdots n_r \Leftrightarrow (1-a)^r \geq \frac{r! n_1 n_2 \cdots n_r}{r! N^k} \Leftrightarrow a \leq 1 - \frac{\sqrt[r]{r! n_1 n_2 \cdots n_r}}{N} \tag{3.4}$$

The conclusion follows from (3.3) and (3.4). □

Remark 3.12. Note that, from the inequality of means, we have

$$\sqrt[r]{r!n_1n_2\cdots n_r} \leq \frac{N\sqrt[r]{r!}}{r},$$

with equality for $n_1 = n_2 = \cdots = n_r = n = \frac{N}{r}$. Therefore, from Lemma 3.**11**, we have that

$$\binom{N-d+r-1}{r} \geq n^r \text{ for all } d \leq N\left(1-\sqrt[r]{\frac{r!}{r^r}}\right).$$

Proposition 3.13. *With the above notations, we have that*

$$\beta_r^d(S/I) \geq 0 \textit{ for all } d \leq N - \left\lceil \sqrt[r]{r!n_1n_2\cdots n_r} \right\rceil.$$

Proof. It follows from Lemma 3.**9** and Lemma 3.**11**. □

Proposition 3.**13** allows us to conjecture that $\operatorname{hdepth}(S/I) \approx N - \lceil \sqrt[r]{r!n_1n_2\cdots n_r} \rceil$.

4. Applications

The m-path ideal of a path graph

Let $n \geq m \geq 1$ be two integers and

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots, x_{n-m+1}\cdots x_n) \subset S = K[x_1,\ldots,x_n],$$

be the m-path ideal associated to the path graph of length n. Let $t \geq 1$. We define:

$$\varphi(n,m,t) := \begin{cases} n-t+2-\left\lfloor \frac{n-t+2}{m+1} \right\rfloor - \left\lceil \frac{n-t+2}{m+1} \right\rceil, & t \leq n+1-m \\ m-1, & t > n+1-m \end{cases}.$$

According to [4, Theorem 2.6] we have that $\operatorname{sdepth}(S/I_{n,m}^t) \geq \operatorname{depth}(S/I_{n,m}^t) = \varphi(n,m,t)$. Assume that $t \leq n-m$ and let $S_{t+m} := K[x_1,\ldots,x_{m+t}]$. We consider the ideal

$$U_{m,t} = (x_{i_1}\cdots x_{i_m} \ :\ i_j \equiv j (\operatorname{mod} m), 1 \leq j \leq m) \subset S_{t+m}.$$

By Euclidean division, we write $t+m = am+b$, where $1 \leq b \leq m$. According to the proof of [4, Lemma 2.4], we have that

$$U_{m,t} = V_{m,1,a+1}\cap\cdots\cap V_{m,b,a+1}\cap V_{m,b+1,a}\cap\cdots\cap V_{m,m,a}, \text{ where } V_{m,j,k} = (x_j, x_{j+m},\ldots,x_{j+(k-1)m}). \quad (4.1)$$

Proposition 4.1. *We have that:* $\operatorname{sdepth}(U_{m,t}) \leq \operatorname{hdepth}(U_{m,t}) \leq m + \left\lfloor \frac{t}{2} \right\rfloor$.

Proof. According to Theorem 3.**3**, we have that $\operatorname{hdepth}(U_{m,t}) \leq \left\lfloor \frac{m+t+m}{2} \right\rfloor = m + \left\lfloor \frac{t}{2} \right\rfloor$. Now, apply Proposition 1.**2**. □

Lemma 4.2. *Let $I \subset S$ be a proper monomial ideal and $u \in S \setminus I$ a monomial. Then*

(1) $\operatorname{sdepth}(S/(I:u)) \geq \operatorname{sdepth}(S/I)$. *(*[3, Proposition 2.7(2)]*)*
(2) $\operatorname{sdepth}(I:u) \geq \operatorname{sdepth}(I)$. *(*[12, Proposition 2]*)*

Theorem 4.3. *Let $n \geq m \geq 1$ and $t \geq 1$. Let $t_0 := \min\{t, n-m\}$. We have that*

$$\operatorname{sdepth}(I_{n,m}^t) \leq \min\{n - \left\lceil \frac{t_0}{2} \right\rceil, n - \left\lfloor \frac{n-t_0+1}{m+1} \right\rfloor + 1\}.$$

Proof. If $t \geq n-m+1$, then $t_0 = n-m$ and, according to [4, Lemma 2.1], we have that $I_{n,m}^{t_0} = I_{n,m}^t : (x_{n-m+1}\cdots x_n)^{t-t_0}$. Therefore, from Lemma 4.**2**(2) it follows that $\operatorname{sdepth}(I_{n,m}^t) \leq \operatorname{sdepth}(I_{n,m}^{t_0})$. Hence, we can assume that $t \leq n-m$ and $t_0 = t$.

By Euclidean division, we write $n-t+1 = q(m+1)+r$, where $0 \leq r \leq m$. According to [4, Lemma 2.5], there exists a monomial $w \in S$ such that:

$$(I_{n,m}^t : w) = \begin{cases} U_{m,t} + P_{m,t,q}, & r < m \\ U_{m,t} + P_{m,t,q} + (x_{n-m+1}\cdots x_n), & r = m \end{cases}, \quad (4.2)$$

where $P_{m,t,q} \subset K[x_{t+m+1},\ldots,x_{t+q(m+1)-1}]$ is a prime monomial ideal of height $2(q-1)$. If $r < m$ then, from (4.2) and [3, Theorem 1.3] it follows that $\operatorname{sdepth}(I_{n,m}^t : w) \leq \min\{\operatorname{sdepth}(U_{m,t}S), \operatorname{sdepth}(P_{m,t,q}S)\}$. From Proposition 4.**1**, [9, Lemma 3.6] and [2, Theorem 2.2] we deduce that $\operatorname{sdepth}(I_{n,m}^t : w) \leq \min\{n - \left\lceil \frac{t}{2} \right\rceil, n-q+1\}$. In the case $r = m$, we obtain the same inequality. Therefore, the required conclusion follows from Lemma 4.**2**(2) and the fact that $q = \left\lfloor \frac{n-t+1}{m+1} \right\rfloor$. □

The m-path ideal of a cycle graph

Let $n > m \geq 2$ be two integer and

$$J_{n,m} = I_{n,m} + (x_{n-m+2}\cdots x_n x_1, \ldots, x_n x_1 \cdots x_{m-1}) \subset S = K[x_1,\ldots,x_n],$$

the m-path ideal associated to the cycle graph of length n.

Let $d := \gcd(n,m)$. We consider the ideal

$$U'_{n,d} = (x_1, x_{d+1}, \cdots, x_{d(r-1)+1}) \cap (x_2, x_{d+2}, \cdots, x_{d(r-1)+2}) \cap \cdots \cap (x_d, x_{2d}, \ldots, x_{rd}), \quad (4.3)$$

where $r := \frac{n}{d}$. Note that $U'_{n,1} = \mathbf{m} = (x_1,\ldots,x_n)$.

Proposition 4.4. *We have that:* $\operatorname{sdepth}(U'_{n,d}) \leq \operatorname{hdepth}(U'_{n,d}) \leq \left\lfloor \frac{n+d}{2} \right\rfloor$.

Proof. According to Theorem 3.**3**, we have that $\operatorname{hdepth}(U'_{n,d}) \leq \left\lfloor \frac{n+d}{2} \right\rfloor$. Now, apply Proposition 1.**2**. □

Let $t_0 := t_0(n,m)$ be the maximal integer such that $t_0 \leq n-1$ and there exists a positive integer α such that $mt_0 = \alpha n + d$. Let $t \geq t_0$ be an integer.

Theorem 4.5. *Let $n > m \geq 2$ and $t \geq t_0$. We have that* $\operatorname{sdepth}(J_{n,m}^t) \leq \left\lfloor \frac{n+d}{2} \right\rfloor$.

Proof. By [5, Lemma 2.2], there exists a monomial $w_t \in S$ such that $(J_{n,m}^t : w_t) = U'_{n,d}$. The conclusion follows from Lemma 4.**2**(2) and Proposition 4.**4**. □

5. **Conclusion**

Let n, m be two positive integers and $I = (x_1,\ldots,x_n) \cap (x_{n+1},\ldots,x_{n+m})$ be the ideal of $S = K[x_1,\ldots,x_{n+m}]$. We proved that $\operatorname{hdepth}(I) = \left\lfloor \frac{n+m+2}{2} \right\rfloor$. Also, we proved that $\operatorname{hdepth}(S/I) \leq \left\lfloor n+m+\frac{1}{2} - \sqrt{2mn+\frac{1}{4}} \right\rfloor$.

More generally, let $n_1, n_2, \ldots, n_r$ be some positive integers, $N = n_1 + \cdots + n_r$, $S := K[x_1,\ldots,x_N]$ and $I = (x_1,\ldots,x_{n_1}) \cap \cdots \cap (x_{n_1+\cdots+n_{r-1}+1},\ldots,x_N) \subset S$. We proved that $\operatorname{hdepth}(I) \leq \left\lfloor \frac{N+r}{2} \right\rfloor$ and we conjectured that we have equality.

Aknowledgments

We gratefully acknowledge the use of the computer algebra system Cocoa ([7]) for our experiments. The second author, Mircea Cimpoeaş, was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2021-1633, within PNCDI III.

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