U.P.B. Sci. Bull., Series D, Vol. 77, Iss. 3, 2015 ISSN 1454-2358

MATHEMATHICAL MODEL FOR *RUNUP* PHENOMENON OF THE WAVES

Gabriel TATU[1], Alexandru DIMACHE[2], Iulian IANCU[3], Andreea COVEIANU[4]

A wave propagating towards the shore takes several shapes called, in scientific literature and in the order of their occurrence: ***spilling, plunging, collapsing, surging*** *and* ***runup;*** *in Romanian, there is a lack of specific terms describing this phenomenon. The last form, the runup, representing the final stage of propagation of the wave to the shore has a special importance for marine shores, especially for the sand beaches (having low slopes) because this is the phase when the beach erosion through wave action is produced. The paper proposes a mathematical model to calculate this phenomenon. Built on a scientific basis, unlike the former ones, this model can evaluate all the flow parameters and consequently its erosive potential.*

Key word: dike, surface wave, modelling, runup.

1. Introduction

In open sea, with some exceptions, the phenomenon of wave falls into the category of non-permanent movements without mass transport (the trajectories of the fluid particles are closed curves).

As the waves approach to the shore, they gradually acquire an increasingly accentuated character of a non-permanent movement **with** mass transport. The successive phases of this transformation are shown in Fig. 1.

[1] Professor, Hydraulics and Environmental Protection Department, Technical University of Civil Engineering Bucharest, Romania, e-mail: tatu.gabriel@yahoo.com.
[2] Reader, Hydraulics and Environmental Protection Department, Technical University of Civil Engineering Bucharest, Romania, e-mail: aldi@utcb.ro.
[3] Lecturer, Hydraulics and Environmental Protection Department, Technical University of Civil Engineering Bucharest, Romania, e-mail: iancuiulian@hidraulica.utcb.ro.
[4] PhD, Hydraulics and Environmental Protection Department, Technical University of Civil Engineering Bucharest, Romania, e-mail: andreea.coveianu@yahoo.com.

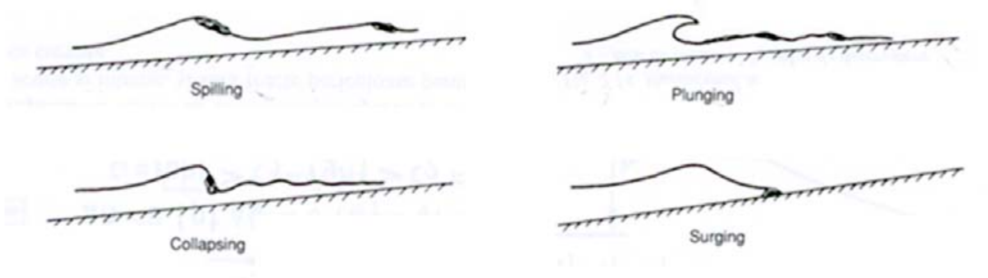

Fig. 1. Successive wave’s shape in the surf area

In Romanian language, there is a lack of specific terms describing this phenomenon but in English there are good classifications such as that in 1984 of U.S. Army Coastal Engineering Research Center [1]; all the sea surface where these phenomena occur is generically called *surf zone* (in free translation, foaming) and the successive wave’s shapes in this surf area are:

- **Spilling** is the phase where turbulence occurs at the crest of the wave, accompanied by foaming sliding on the front face of the wave; this produces a continuous but very gradual dissipation of wave energy which is producing a continuous and gradual decrease of its height;
- **Plunging** is the phase where the crest sharpens, bends and is "thrown" forward;
- **Collapsing** is the next phase after plunging and, just like it, has a much greater dissipative power; in this phase a new "generation" of waves is produced, more irregular, with a lower energy, propagating to the shore and taking more and more the shape of a traveling jump;
- **Surging** is the phase in which the traveling jump climbs up on the slope of the bank; this phase ends with the formation of a water blade relatively thin, which goes up to a certain height, i.e. away from the seashore (considered at the static level) and called ***runup*** (*figure 2*) .

The length of the water blade in the *runup* phase is important because it overlaps with the limit where all the various constructions or activities should be eliminated and, for sanded beaches, where the erosion occurs. In the scientific literature there are some computational models, but none of them (among those found by us) have been scientifically built, i.e. by applying the general laws of mechanics [1, 2, 3]. This paper proposes just such a model.

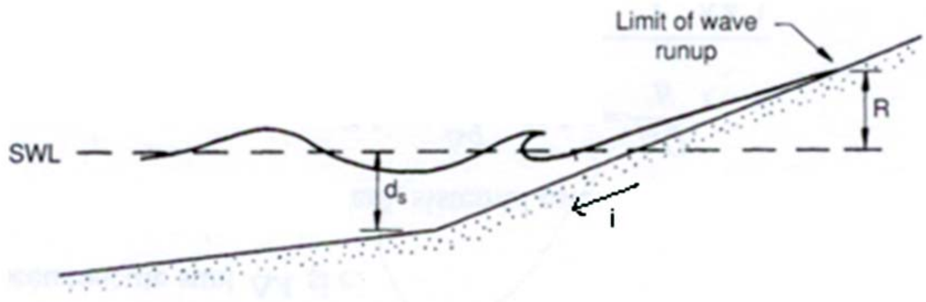

Fig. 2. The *runup* phase

2. The mathematical model

For the runup moment that occurs on a seaside with relatively small slope, the phenomenon can be schematized as in *figure 3*. The calculation scheme is based on the idea that, initially, at the beginning, the wave already took the form of a traveler jump, i.e. a certain volume of water *W* is heading towards the shore with a certain velocity V_0.

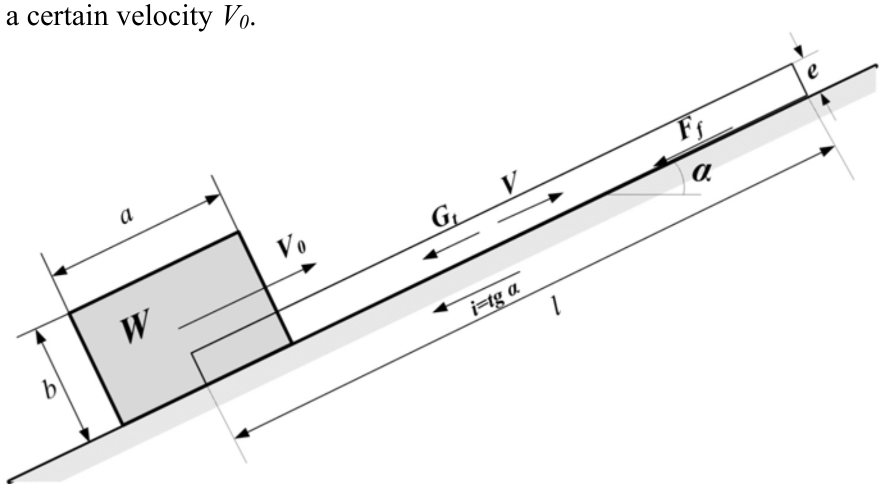

Fig. 3. The mathematical scheme for the calculation of the *runup* phase

Considering a plane problem, the initial volume *W* is supposed to have the shape of a parallelepiped with the dimensions *a, b* and *1* and, after climbing the counter slope *i*, after the time Δt, with the initial velocity V_0 it takes the shape of another, very thin, parallelepiped of size *l*, *e* and *1*. The average speed with which water ascends the slope, during the time interval Δt, is noted by *V*.

The volume W of the wave and its dimensions a and b are known, also the initial velocity V_0; the unknowns of the problem, needed to be found, are the dimensions of the *runup* blade l and e, the average speed V and the stopping time Δt (4 unknowns). The mass of the wave is $m = \rho \cdot W$ where ρ is the density of seawater.

From the geometry of the water bodies the following relations are resulting:

$$l = V \cdot \Delta t \tag{1}$$

$$W = l \cdot e \cdot 1 \tag{2}$$

Decreasing the velocity V_0 to zero along the climbing time Δt gives the relation:

$$V_0 = a \cdot \Delta t \text{ - cinematic equation} \tag{3}$$

where a is the deceleration due to the brake forces, i.e. the sum between the tangential component G_t of its own weight G and the friction force F_f:

$$a = \frac{G_t + F_f}{m} = \frac{G_t + F_f}{\rho \cdot W}$$

The tangential component of the weight is:

$$G_t = G \cdot \sin\alpha = m \cdot g \cdot \sin\alpha = \rho \cdot g \cdot W \cdot \sin\alpha = \gamma \cdot W \cdot \sin\alpha \ ,$$

$\gamma = \rho \cdot g$ being the specific gravity of sea water.

The friction force has the expression: $F_f = \tau_0 . A = \tau_0 \cdot l \cdot 1$ where $\tau_0 = \gamma \cdot R \cdot J$ is the average tangential effort at the wall.

The hydraulic radius is $R = \frac{e \cdot 1}{1} = e$ and the hydraulic slope is:

$$J = \frac{\lambda}{4R} \cdot \frac{V^2}{2g} = \frac{\lambda}{4e} \cdot \frac{V^2}{2g}$$

The kinetic (initial) energy of the water mass $\frac{\rho \cdot W \cdot V_0^2}{2}$ is transformed, on one hand, in the potential energy of the weight force $G=\gamma W=\rho g W$, climbing vertically, with the height $\left(\frac{l}{2} \cdot \sin\alpha + \frac{e}{2} - \frac{b}{2}\right)$, and, on the other hand, in the mechanical work of the friction force F_f which is moving on the distance l; the following relation is obtained:

$$\frac{\rho \cdot W \cdot V_0^2}{2} = \rho . g \cdot W \cdot \left(\frac{l}{2} \cdot \sin\alpha + \frac{e}{2} - \frac{b}{2}\right) + F_f \cdot l \text{ - dynamic equation} \tag{4}$$

In the equations (3) and (4), the existing friction force F_f depends on the motion regime (laminar or turbulent).

In laminar regime

For a laminar flow regime the known formula is:

$$\lambda = \frac{64}{Re} \text{ cu } Re = \frac{V \cdot D}{\nu}$$

valid for filled circular pipes with diameter *D*.

Written with the hydraulic radius *R=D/4*, this relation becomes:

$$\lambda = \frac{64}{Re} = \frac{64\nu}{4.R.V} = \frac{16}{\mathrm{Re}} \ with\ Re = \frac{V \cdot R}{\nu}$$

(ν being the cinematic viscosity of the sea water).

Knowing that, in this case, the friction is produced only on a small part of the cross-section, respectively on the bottom of the bed, λ is smaller as well as the constant corresponding to the numerator of the expression; it was appreciated that constant to be equal to *4*, i.e. at *25%* (from *16*) as it is, approximately, proportionally smaller the wet perimeter as compared with the perimeter of a full pipeline.

As a result, the relationship was used as:

$$\lambda = \frac{C}{Re} \text{ with } C = 4 \text{ and } e = \frac{V \cdot R}{\nu} = \frac{V \cdot e}{\nu} \text{ respectively } \lambda = \frac{C \cdot \nu}{V \cdot e}$$

Under these conditions, the hydraulic gradient becomes:

$$J = \frac{C \cdot \nu}{V \cdot e} \cdot \frac{1}{4 \cdot e} \cdot \frac{V^2}{2 \cdot g} = \frac{C \cdot \nu}{8 \cdot g \cdot e^2} \cdot V$$

and the friction force,

$$F_f = l \cdot \gamma \cdot e \cdot \frac{C \cdot \nu}{8 \cdot g \cdot e^2} \cdot V = \gamma \cdot \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e} \cdot V$$

In turbulent regime

For the turbulent flow regime, the known formulas are:

$$\lambda = \frac{8g}{C^2};\ C = \frac{1}{n} R^{1/6};\ C^2 = \frac{e^{1/3}}{n^2};$$

where *n* - friction coefficient, resulting:

$$\lambda = \frac{8 \cdot g \cdot n^2}{e^{1/3}} \text{ and } J = \frac{8 \cdot g \cdot n^2}{e^{1/3}} \cdot \frac{1}{4 \cdot e} \cdot \frac{V^2}{2 \cdot g} = \frac{n^2}{e^{4/3}} \cdot V^2$$

As a result, the friction force was calculated with the formula:

$$F_f = l \cdot \gamma \cdot e \cdot \frac{n^2}{e^{4/3}} \cdot V^2 = \gamma \cdot \frac{n^2 \cdot l}{e^{1/3}} \cdot V^2$$

Uniting these relationships, the following system of equations was obtained.

<u>In laminar regime (with *C=4*)</u>

The kinematic equation is:

$$a = \frac{\gamma \cdot W \cdot \sin\alpha + \gamma \cdot \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e} V}{\rho \cdot W} = g \cdot \left(\sin\alpha + \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e \cdot W} \cdot V \right)$$

$$V_0 = g \cdot \Delta t \cdot \left(\sin\alpha + \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e \cdot W} \cdot V \right) = g \cdot \frac{l}{V} \cdot \left(\sin\alpha + \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e \cdot W} \cdot V \right)$$

Or:

$$V_0 = g \cdot \left(\frac{l}{V} \cdot \sin\alpha + \frac{C \cdot \nu \cdot l^3}{8 \cdot g \cdot e \cdot W^2} \right); \qquad \left(e = \frac{W}{l} \right)$$

The dynamic equation is:

$$\frac{\rho \cdot W \cdot V_0^2}{2} = \gamma \cdot W \cdot \left(\frac{l}{2} \cdot \sin\alpha + \frac{e}{2} - \frac{b}{2} \right) + \gamma \cdot \frac{C \cdot \nu \cdot l}{8 \cdot g \cdot e} \cdot V \cdot l \text{ with } e = \frac{W}{l}$$

Resulting:

$$V_0^2 = g \cdot \left[\left(l \cdot \sin\alpha + e - b \right) + \frac{C \cdot \nu \cdot l^2}{4 \cdot g \cdot e \cdot W} \cdot V \right] \text{ or}$$

$$V_0 = \sqrt{g \cdot \left(l \cdot \sin\alpha + e - b + K_1 \cdot \frac{l^2 \cdot V}{e} \right)} \text{ with } K_1 = \frac{C \cdot \nu}{4 \cdot g \cdot W} \text{ and } e = \frac{W}{l} \tag{b}$$

In order to solve the system iteratively together with the dynamic relationship, the kinematic equation was put on the form:

$$\mathrm{V} = \frac{l \cdot \sin\alpha}{\frac{V_0}{g} - K_2 \cdot l^3} \text{ with } K_2 = \frac{C \cdot \nu}{8 \cdot g \cdot W^2} = \frac{K_1}{2 \cdot W} \tag{a}$$

The system of equations (a) and (b) has two unknowns, *l* and *V*, and it is resolved by attempts as follows: values for *l* are given in equation (a), resulting *V* and then, from the equation (b), V_0 is calculated; successive attempts are made until the known value of V_0 is reached.

In turbulent regime (here *C* represents the *Chezy* coefficient)

λ => known; λ is calculated using the known formula $\lambda = \dfrac{8g}{C^2}; C = \dfrac{1}{n} R^{1/6}; \mathrm{R} = \mathrm{e};$ (*n*, roughness coefficient), choosing for *e* an approximate value that will be reviewed later.

$$J = \frac{\lambda}{4 \cdot R} \cdot \frac{V^2}{2 \cdot g} = \frac{\lambda}{4 \cdot e} \cdot \frac{V^2}{2 \cdot g};\ l = v \cdot \Delta t;\ W = l \cdot e \cdot 1$$

$$F_f = \gamma \cdot R \cdot J \cdot l \cdot 1 = \gamma \cdot e \cdot \frac{\lambda \cdot l}{4 \cdot e} \cdot \frac{V^2}{2 \cdot g} = \gamma \cdot \frac{\lambda \cdot V^2 \cdot l}{8 \cdot g}$$

The kinematic equation is:

$$a = \frac{\gamma \cdot W \cdot \sin\alpha + \gamma \cdot \dfrac{\lambda \cdot V^2}{8 \cdot g} \cdot l}{\rho \cdot W} = g \cdot \left(\sin\alpha + \frac{\lambda \cdot V^2 \cdot l}{8 \cdot g \cdot W} \right)$$

$$V_0 = a \cdot \Delta t = g \cdot \Delta t \cdot \left(\sin\alpha + \frac{\lambda \cdot V^2 \cdot l}{8 \cdot g \cdot W} \right) \text{ or}$$

$$V_0 = g \cdot \frac{l}{V} \cdot \left(\sin\alpha + \frac{\lambda \cdot V^2 \cdot l}{8 \cdot g \cdot W} \right) \text{, respectively}$$

$$(a) \quad V_0 = g \cdot l \cdot \left(\frac{\sin\alpha}{V} + \frac{\lambda \cdot V \cdot l}{8 \cdot g \cdot W} \right)$$

The dynamic equation is:

$$\frac{\rho \cdot W \cdot V_0^2}{2} = \gamma \cdot \frac{W}{2} \cdot \left(l \cdot \sin\alpha + e - b \right) + \gamma \cdot \frac{\lambda \cdot V^2 \cdot l^2}{8 \cdot g}$$

$$\text{Or } V_0^2 = g \cdot \left(l \cdot \sin\alpha + \frac{W}{l} - b \right) + \frac{\lambda \cdot V^2 \cdot l^2}{4 \cdot W} \text{, respectively}$$

$$(b) \quad V_0 = \sqrt{g \cdot \left(l \cdot \sin\alpha + \frac{W}{l} - b \right) + \frac{\lambda \cdot V^2 \cdot l^2}{4 \cdot W}}$$

The system of equations (a) and (b), having two unknown parameters, *l* and *V*, is resolved by successive attempts.

Values are given for *V* in equation (a) and *l* is resulting as solution of the equation of second degree:

$$\left(\frac{\lambda \cdot V}{8 \cdot W}\right) \cdot l^2 - \left(\frac{g \cdot \sin\alpha}{V}\right) \cdot l - V_0 = 0$$

after rearranging the equation (a), respectively

$$l = \frac{\sqrt{\left(\frac{g \cdot \sin\alpha}{V}\right)^2 + \frac{1}{2}\frac{\lambda \cdot V}{W}V_0} - \frac{g \cdot \sin\alpha}{V}}{\frac{1}{4}\frac{\lambda \cdot V}{W}}$$

Then, from the equation (b), V_0 is calculated and successive attempts are made for V until the known value of V_0 is reached. With the final values, the average thickness of the *runup* blade $e = \frac{W}{l}$ is calculated, and λ is recalculated with the new value of the hydraulic radius $R = e$.

3. Comparative calculations

To validate the model, we proceeded to perform calculations compared to other models in the scientific literature. In the studied literature, we didn't find deterministic models such as the one shown above, constructed on a scientific base, i.e. based on the general laws of mechanics.

We found only non-deterministic models such as that presented in the book of Sorensen (2006) [1] who found, first of all, by applying the π theorem of the dimensional analysis, the following dependence:

$$\frac{R}{H} = f\left(i, \frac{H}{gT^2}, \frac{d_s}{H}\right)$$

where H is the height of the wave, T is its period, i is the counter-slope of the bank and R and d_s are defined in *figure 2*.

Based on this objectionable dependency, because it does not contain the celerity (or wavelength), the theory is then filled up with a number of additional parameters determined experimentally and graphs that allow the calculation of the lifting R of the *runup* blade.

From this book we took the calculus example *2.9-1*, with the basic data $d_s=4$ *m; T=10 s; H=2 m; i=1:10=10 %*, grassy slope, resulting *R=1.5 m … 1.7 m*, i.e., at a slope of *10 %*, a length of the *runup* blade *l=15 m … 17 m*.

With the same basic data, the present model and procedures have been applied.

The initial velocity was appreciated as being equal with $V_o=5\ m/s$, starting from the information that the depth at the base of the slope is $d_s = 4\ m$ and then the celerity would be $c = \sqrt{gh} = \sqrt{10 \cdot 4} = 6.3\, m/s$ (applying then a reduction due to the overall dissipation in the *surf* area).

Applying the proposed calculation method for turbulent regime with a roughness coefficient appreciated at the value $n=0.015$ and taking for the b size, the height of the volume W for the incident jump, a value equal with the height of the wave, i;e. $b=H=2\ m$. For the second dimension, the length of volume W of the incident jump, two values, $a=2\ m$ and $a=1\ m$ were considered.

The results are presented in *table 1*. They show, first of all, a good concordance with the example above regarding the length l of the *runup* blade.

Table 1

The results calculated in turbulent regime table 1

a (m)	*b (m)*	*V (m/s)*	*l (m)*	*e (m)*
2	2	6.14	18.62	0.215
1	2	5.78	14.26	0.140

Secondly, however, the present model, unlike the model existing in the literature, allows the assessment of the speed V of the water climbing up the slope, resulting in average value of *6 m/s.* It means that the maximum speed is around 12 m/s, which for sandy beaches, creates the conditions for sand bubbling, lifting it in suspension state and driving it easily into the sea, in the next phase when the *runup* blade returns by gravity drainage.

Thus, it is demonstrated through scientific calculations what can be seen with the naked eye, on the Romanian Black Sea coast, namely that the waves that come to shore can only have an action of erosion, driving continuously the sand into the sea.

Table 2

The results calculated in laminar regime

a (m)	*b (m)*	*V (m/s)*	*l (m)*	*e (m)*	*Re*
2	2	8.81	44.27	0.091	796462
1	2	8.89	44.58	0.045	398898

In *table 2* are shown the results of calculation in the hypothesis of laminar regime, which obviously cannot be applied, given the high values of Reynolds numbers; and, obviously, as seen, the values of the *runup* length l do not fit at all with those from the literature.

3. Conclusions

In a research domain like this (the free surface waves in open sea), where random and chaos seem to govern and only probabilistic methods have been applied in the past, this paper proves that the main laws of the mechanics, the old laws of mass and energy conservation might be still very useful, with very successful results in the mathematical modeling of the physical phenomena.

The combination of the classical laws of mechanics with the particular ones of hydraulics conducted to a trusting mathematical model, validated by the probabilistic existing ones but giving more complete and detailed results than them.

In our opinion, using an approach like this for the mathematical modeling might be extended also for studying the other phases of waves' evolution (spilling, collapsing and plunging) and the scientific basis of such a research will also help to a better understanding of those phenomena themselves.

R E F E R E N C E S

[1] *Sorensen Robert,* Basic Coastal Engineering, Department of Civil and Environmental Engineering, Lehigh University, Bethlehem, Pennsylvania, 2006

[2] *Holthuijsen Leo H,* Waves in Oceanic and Coastal Waters, Delft University of Technology and UNESCO-IHE, 2007

[3] *Davidson-Arnott Robin,* An Introduction to Coastal Processes and Geomorphology, CAMBRIDGE UNIVERSITY PRESS, 2010.

[4] *Nwogu, O., Demirbilek, Z.* Infragravity wave motions and runup over shallow fringing reefs, J. Waterw. Port Coastal, Ocean Eng., 136, 295–305, 2010.

[5] *Senechal, N., Coco, G., Bryan, K. R., Holman, R.*, Wave runup during extreme storm conditions. Journal of Geophysical Research, 116(C7), 2011.

[6] *Johnson, B.D., Kobayashi, N., Gravens, M.B.* Cross-Shore numerical model CSHORE for waves, currents, sediment transport and beach profile evolution. ERDC/CHL Technical Report, in press,U.S. Army Engineer R&D Center, Vicksburg, MS, 2011.

[7] *Dutykh, D, Katsaounis, T., Mitsotakis, D.,* Finite Volume Schemes for Dispersive Wave Propagation and Runup. J Comput. Phys., Vol 230, pp 3035-3061, 2011.

[8] *Liu, X., Xia, J. (C.), Wright, G., Mahoney, R., & Arnold, L.* High watermark determination based on the wave runup height distribution and spatialcontinuity. Coasts and Ports 2011: Diverse and Developing: Proceedings of the20th Australasian Coastal and Ocean Engineering Conference and the 13thAustralasian Port and Harbour Conference (pp. 442-427). Perth, WA, 2011.

[9] *Pietropaolo, J., N. Kobayashi, and J.A. Melby.* Wave runup on dikes and beaches. Proc. 33rd ICCE, Santander, Spain, 2012.

[10] *Soldini, L., Antuono, M., Brocchini, M.* Numerical Modeling of the Influence of the Beach Profile on Wave Run-Up, Journal of Waterway, Port, Coastal and Ocean Engineering, Vol. 139, No. 1, pp. 61-71, January/February 2013.

[11] *de la Peña, J. M., Sánchez-González, J. F., Díaz-Sánchez, R.* Wave Runup in a Sand Bed Physical Model. Journal of Waterway, Port, Coastal and Ocean Engineering, Vol. 140, No. 4, July/August 2014.