SOME RESULTS OF FREDHOLM ALTERNATIVE TYPE FOR OPERATORS OF $\lambda J_\varphi - S$ FORM WITH APPLICATIONS

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În această lucrare sunt reluate şi generalizate unele rezultate de tip Alternativă Fredholm. În partea de început sunt prezentate câteva rezultate care pregătesc subiectul, dintre care unele sunt originale. Rezultatele de tip Alternativă Fredholm sunt aplicate în rezolvarea, deasemenea originală, a câtorva probleme, care implică p-*laplaceianul şi pseudo-laplaceianul.*

In this paper some results of the Fredholm Alternative type are discussed and generalized. At the beginning of the paper some results which prepare the subject, part of them being original, are presented. The results of the Fredholm Alternative type are applied in the solution, also original, for some problems involving the p-*Laplacian and the pseudo-Laplacian..*

Keywords: Fredholm alternative, strongly closed operator, strongly continuous operator, weak solution, *a*-homogeneous operator, *a*-quasi-homogeneous operator, duality map, (*K*, *L*, *a*)-map, Carathéodory function, Sobolev space, *p* - Laplacian, pseudo-Laplacian

1. Introduction

This paper starts its theoretical part with some propositions which have their origin in a theorem own to J. Nečas ([1], [2]). They are obtained based on this theorem, but also on others results own to the author from [3] and here. We apply these results to weak solutions of partial differential equations which are also discussed in [3], and together with others methods in [4], [5] and [6].

The aim of this work is to give some new results of the Fredholm Alternative type and, using them, to highlight new results in weak solutions for some types of partial differential equations involving the *p*-Laplacian and the pseudo-Laplacian.

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2. Preliminary notions and results

2.1. Preliminaries

Definitions. Let X, Y be real normed spaces and let $F: X \to Y$, $F_0 : X \to Y$.

F is *strongly closed* respectively *strongly continuous* if

$$x_n \xrightarrow{w} a \text{ and } F(x_n) \to \alpha \Rightarrow \alpha = F(a)$$

respectively

$$x_n \xrightarrow{w} a \Rightarrow F(x_n) \to F(a).$$

For instance, each linear compact operator between Banach spaces is strongly continuous ([7]).

Let a be a real strictly positive number.

F is a-*homogeneous* if

$$F(tu) = t^a F(u) \ \ \forall\, u \in X, \forall\, t \geq 0.$$

F is a-*quasi-homogeneous relative to* F_0, F_0 a-homogeneous, if

$$t_n \downarrow 0,\ u_n \xrightarrow{w} u_0 \text{ and } t_n^a\, F\left(\frac{u_n}{t_n}\right) \to \gamma \Rightarrow \gamma = F_0(u_0).$$

F is a-*strongly quasi-homogeneous relative to* F_0, if

$$t_n \downarrow 0,\ u_n \xrightarrow{w} u_0 \Rightarrow t_n^a\, F\left(\frac{u_n}{t_n}\right) \to F_0(u_0).$$

Proposition 1. *If* F *is* a-*homogeneous and strongly closed* (respectively *strongly continuous*), *then* F *is* a-*quasi-homogeneous* (respectively a-*strongly-homogeneous*) *relative to* F.

Proof. In both cases, we can remark that $t_n^a\, F\left(\frac{u_n}{t_n}\right) = F_0(u_0)$. □

Proposition 2. *If* F *is* a-*strongly quasi-homogeneous relative to* F_0 , *then* F_0 *is* a-*homogeneous and strongly continuous*.

Proof. *First statement*. t > 0. Let u_0 be any fixed in X and $t_n \downarrow 0$, $u_n \xrightarrow{w} u_0$. Then $t_n^a\, F\left(\frac{u_n}{t_n}\right) \to F_0(u_0)$, hence $(tt_n)^a F\left(\frac{u_n}{t_n}\right) \to t^a F_0(u_0)$, but $(tt_n)^a F\left(\frac{tu_n}{tt_n}\right) \to F_0(tu_0)$ because $tu_n \xrightarrow{w} tu_0$, $t^a F_0 (u_0) = F_0 (t\, u_0)$. t > 0. We have to prove that $F_0 (0) = 0$. Take $t_n \downarrow 0$ and $(u_n)_{n\geq 1}$ with $u_n = 0\ \ \forall\, n$. Then $t_n^a\, F\left(\frac{u_n}{t_n}\right) \to F_0(0)$, but $F\left(\frac{u_n}{t_n}\right) = 0$ etc.

The second statement. From the hypothesis,

$$\lim_{t \to 0+} t^{a} F\left(\frac{u}{t}\right) = F_0(u) \quad \forall\, u \in X. \tag{1}$$

We assume *par absurdum* F_0 as being not strictly continuous. Then there exist u_n, $u_0 \in X$, $u_n \xrightarrow{w} u_0$ and

$$F_0(u_n) \not\to F_0(u_0). \tag{2}$$

$\varepsilon > 0$ being fixed any, from (2), $\exists$ a subsequence of (u_n), we use for it the same notation, such that

$$\|F_0(u_n) - F_0(u_0)\| \geq \varepsilon \quad \forall\, n \geq 1. \tag{3}$$

In addition, from (1), for each $n \in \mathbf{N}$ $\exists\, t_n$, $0 < t_n \leq \frac{1}{n}$, for which

$$\left\|F_0(u_n) - t_n^a F\left(\frac{u_n}{t_n}\right)\right\| \leq \frac{\varepsilon}{2}. \tag{4}$$

Then $\varepsilon \overset{(3)}{\leq} \|F_0(u_0) - F_0(u_n)\| \leq \left\|F_0(u_0) - t_n^a F\left(\frac{u_n}{t_n}\right)\right\| + \left\|t_n^a F\left(\frac{u_n}{t_n}\right) - F_0(u_n)\right\| \overset{(4)}{\leq}$ $\frac{\varepsilon}{2} + \left\|F_0(u_0) - t_n^a F\left(\frac{u_n}{t_n}\right)\right\|$, take the limit for $n \to \infty$, obtain $\varepsilon \leq \frac{\varepsilon}{2}$, which is a contradiction. □

Proposition 3. *The single-valued duality map* J_φ *is* a-*homogeneous iff* φ *is* a-*homogeneous*.

Proof. Necessary. $\forall\, u \neq 0$, $\forall\, t \geq 0$, $J_\varphi(tu) = \dfrac{\varphi(t\,\|u\|)}{\varphi(\|u\|)} J_\varphi u$ ([3], I, 1.1, 5°), $J_\varphi(tu) = t^{a} J_\varphi u$, and taking the norm,

$$\varphi(t\|u\|) = t^{a} \varphi(\|u\|);$$

one takes into account that $u \to \|u\|$ takes all its values in $\mathbf{R}_+$. *Sufficient* – the same formulae. □

For $p \in (1, +\infty)$, $-\Delta_p$ and $-\Delta_p^s$ are $(p-1)$-homogeneous maps on $W_0^{1,p}(\Omega)$ as it results from the following section.

2.2. Results for the operators $-\Delta_p$ and $-\Delta_p^s$

Ω designates an open set with the finite Lebesgue measure from $\mathbf{R}^N$, $N \geq 2$ and $p \in (1, +\infty)$. The norm on $W_0^{1,p}(\Omega)$ will be $u \to \|u\|_{1,p}$, where $\|u\|_{1,p} :=$

$\| \, |\nabla u| \|_{L^p}$ (= $\| \, |\nabla u| \, \|_{0,\,p}$)[2]. We designate the dual of ($W_0^{1,p}(\Omega)$, $\| \cdot \|_{1,\,p}$) with $W^{-1,p'}(\Omega)$, where p' is the conjugate with p exponent.

Consider the operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u). \tag{5}$$

This acts, according to [13],

$$\langle -\Delta_p u\,,\, v\rangle = \int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla v\,dx \quad \forall\; u, v \in W_0^{1,p}(\Omega). \tag{6}$$

For the followed aim, the most important is the next property of the p-Laplacian to be identical with a certain duality map.

Proposition 4. *Let* Ψ: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\Psi(u) = \frac{1}{p}\,\| u \|_{1,p}^{p}.$$

Then Ψ *is Gâteaux differentiable on* $W_0^{1,p}(\Omega) \setminus \{0\}$ *and*

$$\Psi'(u) = -\Delta_p u = J_\varphi u \quad \forall\; u \in W_0^{1,p}(\Omega),$$

where $\varphi(t) = t^{p-1}$ ([8]).

Proof. As $\Psi(u) = \int_0^{\|u\|_{1,p}} \varphi(t)dt$, we have

$$J_\varphi u = \partial\Psi(u) \quad \forall\; u \in W_0^{1,p}(\Omega) \text{ ([3], I, 1.1)},$$

it remains to prove that Ψ is Gâteaux differentiable and

$$\Psi'(u) = -\Delta_p u \quad \forall\; u \in W_0^{1,p}(\Omega). \tag{7}$$

$\Psi = g \circ f$, where g: $L^p(\Omega) \to \mathbf{R}$, $g(u) = \frac{1}{p}\,\| u \|_{0,p}^{p}$, f: $W_0^{1,p}(\Omega) \to L^p(\Omega)$, $f(u) = |\nabla u|$.

g is of the C^1 Fréchet class on $L^p(\Omega) \setminus \{0\}$ ([3], VI, 1.1), f is Gâteaux differentiable on $W_0^{1,p}(\Omega) \setminus \{0\}$ and $f'(u)(h) = \dfrac{\nabla u\cdot\nabla h}{|\nabla u|}$ $\forall\; h \in W_0^{1,p}(\Omega)$ ([12]), we apply the generalized finite increments formula[3], $u \neq 0$ and $h \in W_0^{1,p}$ $\Rightarrow$

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[2] Let Ω be a nonempty open subset of $\mathbf{R}^N$. For $p \in [1, +\infty)$, $L^p(\Omega) := \{u : \Omega \to \mathbf{R}$, u measurable, $|u|^p$ Lebesgue integrable$\}$, $\| u \|_{L^p(\Omega)} := \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}$, $\|u\|_{0,\,p} := \| u \|_{L^p(\Omega)}$.

[3] Let X, Y be real normed spaces and $f : X \to Y$ Gâteaux differentiable, $F : Y \to \mathbf{R}$ of C^1 class Gâteaux. Then $g := F \circ f$ is Gâteaux differentiable and $g'(x) = F'(f(x)) \circ f'(x)$. This result is own to the author ([9], [3])

$$\Psi'(u)(h) = \int_\Omega |\nabla u|^{p-1} \frac{\nabla u \cdot \nabla h}{|\nabla u|} dx = \int |\nabla u|^{p-2} \nabla u \cdot \nabla h dx \overset{(6)}{=} \langle -\Delta_p \ u, \ h\rangle$$ and it remains only the case $u \neq 0$ to finish with (7).

But $\Psi'(0)(h) = \lim\limits_{t\to 0} \frac{1}{t}\Psi(th) = \lim\limits_{t\to 0} \frac{t^{p-1}}{p} \| h \|_{1,p}^{p} = 0 = \langle -\Delta_p 0, h\rangle$. □

Remarks. 1. The main important part of this proof is own to the author.

2. Ψ has even the C^1 class Fréchet on $W_0^{1,p}(\Omega)$ ([12]).

For the operator $-\Delta_p^s$, Ω also designates an open set with the finite Lebesgue measure from $\mathbf{R}^N$, $N \geq 2$ and $p \in (1, +\infty)$. The norm on $W_0^{1,p}(\Omega)$ will be now $u \to \|u\|_{1,p}$, where $\|u\|_{1,p} := \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^{p}\right)^{\frac{1}{p}}$. The dual of the space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is designated also by $W^{-1,p'}(\Omega)$, where p' is the conjugate with p exponent.

Consider now the operator $-\Delta_p^s: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$-\Delta_p^s u = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right).$$

This acts according to

$$\langle -\Delta_p^s u, h\rangle = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial h}{\partial x_i} dx, \ u, h \in W_0^{1,p}(\Omega) \qquad (8)$$

(Lions, [11]).

Proposition 5. *The function* Ψ: $\Psi(u) = \frac{1}{p}\|u\|_{1,p}^{p}$, $p \in (1, +\infty)$, *is Gâteaux differentiable on* $W_0^{1,p}(\Omega) \setminus \{0\}$ *and*

$$\Psi'(u) = -\Delta_p^s u = J_\varphi u, \ \varphi(t) := t^{p-1}.$$

Proof. We fix the index i, $1 \leq i \leq N$ and $g: W_0^{1,p}(\Omega) \to \mathbf{R}$, $g(u) = \left\|\frac{\partial u}{\partial x_i}\right\|_{0,p}^{p}$.

We have $g = F \circ f$, $f: W_0^{1,p}(\Omega) \to L^p(\Omega)$, $f(u) = \frac{\partial u}{\partial x_i}$, F: $L^p(\Omega) \to \mathbf{R}$, $F(v) = \|v\|_{0,p}^{p}$.

Because $f'(u)(h) = \frac{\partial h}{\partial x_i}$ and $F'(v)(h) = p \int_\Omega |v|^{p-2} vh dx$ ([14]), $g'(u) = F'(f(u)) \circ f'(u)$ (the announced finite increments formula).

$$g'(u)(h) = p \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial h}{\partial x_i}\,dx \tag{9}$$

and hence

$$\Psi'(u)(h) = \sum_{i=1}^{N}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{0,p}^{p}\right)'(h) \overset{(9),(8)}{=} \langle -\Delta_p^s u, h\rangle.$$

The rest of the proof is the same as that of the Proposition 4. □

Remark. The proof given here for this statement is own to the author.

Corollary. $u \to \|u\|_{1,\mathrm{p}}$, $p \in (1, +\infty)$, *is Gâteaux differentiable on* $W_0^{1,p}(\Omega) \setminus \{0\}$. *Consequently* $(W_0^{1,p}(\Omega), \|\cdot\|_{1,\mathrm{p}})$ *is smooth*.

Proof. By taking $\Phi(u) = \|u\|_{1,\mathrm{p}}$, we have

$$\Phi(u) = p^{\frac{1}{p}}(\Psi(u))^{\frac{1}{p}}.$$

We also apply the above generalized finite increments formula. For the second affirmation – see I, 1.2 from [3]. □

Remark. This last proof is own to the author.

3. Fredholm alternative

We continue in this section with the results completing the previous theory made in the subsection 2.1.

Proposition 6. *If the Banach space* X *is reflexive, smooth and with the* (H) *property*[4], *then any duality map* J_φ *on* X *is strongly closed*.

Proof. Let $u_n \xrightarrow{w} u_0$ and $J_\varphi u_n \to \gamma$. We have

$$\langle J_\varphi u_n - J_\varphi u_0, u_n - u_0\rangle \to 0,$$

but

$$\langle J_\varphi u_n - J_\varphi u_0, u_n - u_0\rangle \geq [\varphi(\|u_n\|) - \varphi(\|u_0\|)](\|u_n\| - \|u_0\|) \geq 0 \text{ ([3], I, 1.1)},$$

hence $\lim_{n\to 0} [\varphi(\|u_n\|) - \varphi(\|u_0\|)](\|u_n\| - \|u_0\|) = 0$, which implies $\|u_n\| \to \|u_0\|$ (see the proof of I, 4.16, [3]). X having the (H) property, it results $u_n \to u_0$, therefore

[4] A Banach space has the (*h*) property if

$$x_n \xrightarrow{w} x \text{ and } \|x_n\| \to \|x\| \Rightarrow x_n \to x.$$

A Banach space has the (H) property if it is strictly convex and also has the (*h*) property.

(X reflexive smooth $\Rightarrow J_\varphi$ demi-continuous, [3], I, 4.2) $J_\varphi u_n \xrightarrow{w} J_\varphi u_0$ and so $J_\varphi u_0 = \gamma$. □

Corollary. *If the Banach space* X *is reflexive*, *smooth and with the* (H) *property*, *any duality map on* X J_φ a-*homogeneous is* a-*quasi-homogeneous relative to* J_φ.

Proof. Combine the last proposition with Proposition 1. □

Passing to the main proposition of this paper, the conditions are slightly weakened as in the original result.

For this reason, we give the following

Definitions. The map $f : X \to Y$, X and Y normed spaces, is *regular surjective*, if it is surjective and $\forall R > 0 \ \exists r > 0$ such that

$$\|f(x)\| \le R \Rightarrow \|x\| \le r.$$

$T : X \to Y$, X and Y normed spaces, is a (K, L, a) map, where $K > 0$, $L > 0$, $a > 0$, if

$$K\|x\|^a \le \|Tx\| \le L\|x\|^a \quad \forall x \in X.$$

Proposition 7 (*Fredholm Alternative*). *Let* X, Y *real normed spaces*, T: X → Y (K, L, a)-*bijection* a-*homogeneous*, *odd and with continuous inverse*, S: X → Y a-*homogeneous odd compact operator*. *Then*, *for any* $\lambda \neq 0$, $\lambda T - S$ *is regular surjective iff* λ *is not an eigenvalue for the couple* (T, S) ([2], [3]).

Proof. Necessary. Let, *par absurdum*, be $x_0 \neq 0$ from X such that

$$\lambda T(x_0) - S(x_0) = 0. \tag{10}$$

We multiply (10) by t^a,

$$\lambda T(tx_0) - S(tx_0) = 0 \tag{11}$$

and as $\lim_{t\to+\infty} \|tx_0\| = +\infty$, (11) imposes (*par absurdum*!) the conclusion $\lambda T - S$ is not regular surjective, a contradiction.

Sufficient. Firstly we prove that

$$\rho := \inf_{\|x\|=1} \|\lambda Tx - S(x)\| > 0. \tag{12}$$

Assume *par absurdum*

$$\rho = 0. \tag{13}$$

With (13) we obtain a sequence $(x_n)_{n\in\mathbf{N}}$, $x_n \in X$, (14) $\|x_n\| = 1$ and

$$\lim_{n\to\infty} [\lambda T(x_n) - S(x_n)] = 0. \tag{15}$$

The sequence (x_n) being bounded, $(S(x_n))_{n\in\mathbf{N}}$ has a convergent in Y subsequence, let $\gamma = \lim_{n\to\infty} S(x_{k_n})$. But T is surjective and $\lambda \neq 0$, so $\exists\, x_0 \in X$ such that $\lambda T(x_0) = \gamma$ and then, from (15),

$$\lim_{n\to\infty} \lambda T(x_{k_n}) = \lambda T(x_0). \tag{16}$$

From (16) it results (T has a continuous inverse)

$$\lim_{n\to\infty} x_{k_n} = x_0. \tag{17}$$

(17) imposes on one hand $\|x_0\| \overset{(14)}{=} 1$ and on the other hand $\lim_{n\to\infty} S(x_{k_n}) = S(x_0)$, which combined with (15) and (16) gives $\lambda T(x_0) - S(x_0) = 0$, a contradiction and hence (12).

So being, from (12)

$$\left\|\lambda T\left(\frac{x}{\|x\|}\right) - S\left(\frac{x}{\|x\|}\right)\right\| \geq \rho \quad \forall\, x \in X \setminus \{0\},$$

so

$$\rho\|x\|^{a} \leq \|\lambda T(x) - S(x)\| \quad \forall\, x \in X \setminus \{0\}. \tag{18}$$

From (18)

$$\lim_{\|x\|\to+\infty} \|\lambda T(x) - S(x)\| = +\infty, \tag{19}$$

from which we conclude that $\lambda T - S$ is surjective ([3], II, 1.1).

This surjectivity is regular. Indeed, assuming *par absurdum* the contrary, we obtain $R > 0$ such that $\forall\, n \in \mathbf{N}\ \exists\, x_n' \in X$, $\|x_n'\| > n$ and

$$\|\lambda T(x_n') - S(x_n')\| \leq R. \tag{20}$$

But as

$\lim_{n\to\infty} \|x_n'\| = +\infty$, we have $\lim_{n\to\infty} \|\lambda T(x_n') - S(x_n')\| \overset{(19)}{=} +\infty$ and we obtain a contradiction with (20). □

And now, from the "surjectivity lemma" (Proposition 7, Fredholm Alternative), we can quickly obtain,

Proposition 8. *Let* X *be a real reflexive Banach space, smooth and with the* (H) *property, which is compact embedded in the real Banach space* Z, *and* N: Z → Z* *an* a-*homogeneous odd demi-continuous operator. Then the operator* $\lambda J_\varphi - N$, J_φ *the duality map on* X *with* $\varphi(t) = t^a$, $\lambda \neq 0$, *is regular surjective iff* λ *is not an eigenvalue for the couple* (J_φ, N).

Explanation. In the expressions $\lambda J_\varphi - N$ and (J_φ, N), N is, in fact, i' o N o i, i: $X \to Z$ linear compact injection, i': $Z^* \to X^*$ the transposed of i.

Proof. We apply Proposition 7 with $T := J_\varphi$, $S := i'$ o N o i, correctly as J_φ is (K, L, a) with $K = L = 1$, bijection with continuous inverse ([3], I, 4.14), odd and S is odd, a-homogeneous and compact (see the proof of III, 1.2, [3]).

Remark. The last proposition is due to the author.

4. Application

4.1 Application to the *p*-Laplacian and pseudo-Laplacian

We take now in the Proposition 8 $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, where $p \in (1, +\infty)$ and Ω is an open bounded of the C^1 class set from $\mathbf{R}^N$, $N \geq 2$ (so $J_\varphi = -\Delta_p$, $\varphi(t) = t^{p-1}$, Proposition 4), $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, N: $L^p(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $Nu = |u|^{p-2}u$.

$W_0^{1,p}(\Omega)$ is uniform convex ([3], IV, 4.5), and hence also reflexive ([3], I, 3.2) with the property (H), with its norm Gâteaux differentiable ([3], IV, above 4.5) and hence smooth ([3], I, 1.2). It is compact embedded in $L^p(\Omega)$ ([3], IV, 3.6).

Concerning N, it is the duality map on L^p relative to the weight $t \to t^{p-1}$ (see in the following Proposition 10), consequently N is a homeomorphism of L^p on $L^{p'}$ ([3], IV, 1.2), odd and $(p-1)$-homogeneous.

So being, we apply Proposition 8 in order to obtain

Proposition 9. *Let* p *be from* $(1, +\infty)$ *and* $\lambda \neq 0$. *If*

$$\lambda(-\Delta_p u) = |u|^{p-2}u \tag{21}$$

has not a nonzero solution in $W_0^{1,p}(\Omega)$, *then for any* h *from* $W^{-1,p'}(\Omega)$, *the equation*

$$\lambda(-\Delta_p u) = |u|^{p-2}u + h \tag{22}$$

has solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$[5].

[5] All the terms from the first relationship from (21) are considered as elements of $W^{-1,p'}(\Omega)$.

Explanation. The term $|u|^{p-2}u$ from (21) and (22) is considered to be its image through a compact embedding of $L^{p'}(\Omega)$ in $W^{-1,p'}(\Omega)$ (use the Schauder theorem).

Remark. This result is own to the author.

Proposition 10. *The duality map on* $L^p(\Omega)$, $p \in (1, +\infty)$, *with the weight* $\varphi(t) = t^{p-1}$ *is*

$$J_\varphi u = |u|^{p-1} \operatorname{sgn} u,\ u \in L^p(\Omega),$$

i.e.

$$\langle J_\varphi u, h\rangle = \int_\Omega |u|^{p-1} (\operatorname{sgn} u) h\, dx \quad \forall\ h \in L^p(\Omega).$$

Proof. Let Ψ: $\Psi(u) = \frac{1}{p}\|u\|_{0,p}^p$. As $\Psi(u) = \int_0^{\|u\|_{0,p}} \varphi(t)\, dt$, we have (see [3], I, 1.1, 3°)

$$J_\varphi(u) = \partial\Psi(u).$$

But $\Psi'(u)(h) = \int_\Omega |u|^{p-1} (\operatorname{sgn} u) h\, dx \quad \forall\ h$ from $L^p(\Omega)$ ([12]), and then the conclusion. □

Remark. The proof of this statement is own to the author.

Proposition 11. *In the statement of the Proposition* 9, *if* $p \in [2, +\infty)$, *then* $-\Delta_p$ *can be replaced by* $-\Delta_p^s$.

Proof. We take in the Proposition 8 $(X, \|\cdot\|) = (W_0^{1,p}(\Omega), |\!|\!|\cdot|\!|\!|_{1,p})$ (see at the beginning), $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, $N : L^p(\Omega) \to L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1$, $Nu = |u|^{p-2}u$, and we take into account IV, 5.7 from [3] and the Corollary of Proposition 5 (the compact embedding of $W_0^{1,p}$ in L^p is given by the equivalence of the norms $|\!|\!|\cdot|\!|\!|_{1,p}$ and $\|\cdot\|_{1,p}$[6]). □

4.2. Another application for *p*-Laplacian

Here we take in the Proposition 8 $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, Ω an open bounded of the C^1 class set in $\mathbf{R}^N$, $N \geq 2$, $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, N: L^p

[6] $|\!|\!|\cdot|\!|\!|_{1,p}$ is equivalent with the norm $|u|_{1,p} := \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$: $\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)} \leq N\ |\!|\!|u|\!|\!|_{1,p} \leq N\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$. But $|\cdot|_{1,p}$ is equivalent with $\|\cdot\|_{1,p}$, as $\|u\|_{1,p} \leq |u|_{1,p} \leq N\|u\|_{1,p}$.

$(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'} = 1$, $Nu = N_f u$[7], $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function which verifies

1° $|f(x, s)| \leq c_1 |s|^{p-1} + \beta(x)$ $\forall s \in \mathbf{R}$, $\forall x \in \Omega \setminus A$, $\mu(A) - 0$, where $c_1 \geq 0$, $\beta \in L^{p'}$;

2° f is odd and $(p - 1)$-homogeneous in the second variable. Then N_f is odd, $(p - 1)$-homogeneous and continuous ([3], IV, 5.13). Apply Proposition 8 (see also the previous section) and we obtain the following

Proposition 12. *Let* p *be from* $(1, +\infty)$ *and* $\lambda \neq 0$. *If*

$$\lambda(-\Delta_p u) = N_f u$$

has no nonzero solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$, *then*, *for any* h *from* $W^{-1,p'}(\Omega)$, *the equation*

$$\lambda(-\Delta_p u) = f(\cdot, u(\cdot)) + h, x \in \Omega$$

has a solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Remark. This statement own to the author can be compared with Proposition 10 from [5].

6. Conclusions

An original solution for some problems of partial differential equations is given in this paper starting from some propositions of the Fredholm Alternative type.

The introductive results concerning the *p*-Laplacian and the pseudo-Laplacian are improved by the author.

The main result the "Alternative Fredholm" is given with weakened condition. Another result of this type (Proposition 8) is own to the author.

Other novel results are developed in order to characterize weak solutions involving the *p*-Laplacian and the pseudo-Laplacian.

We can compare these results for the *p*-Laplacian and for the pseudo-Laplacian with the characterizations of the solutions for the problems of partial differential equations involving the *p*-Laplacian and for the pseudo-Laplacian from [4], [5] and [3].

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