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ON STAR MOMENT SEQUENCES OF COMMUTING MULTIOPERATORS

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In această notă , generalizăm Definiția 1.2. din [5] în cazul $A=(A_1,\ldots,A_n) \in L(H)^n$, $n \geq 1$, este un multioperator comutativ pe un spațiu Hilbert H real și, de asemenea, în cazul în care $T=(T_1\ldots T_n) \in L(H)^n$, $n \geq 2$ este un multioperator comutativ definit pe un spațiu Hilbert complex H . În Secțiunea 2 a acestei note, introducem noțiunea de multioperator comuativ care admite şir de "star-momente reale" In cazul H spațiu Hilbert real si de multioperator comutativ care admite şir de "star-momente complexe" In cazul H spațiu Hilbert complex .În Secțiunea 3, generalizăm Teorema 2.2. din [5] şi dăm condiții necesare şi suficiente pentru ca un multioperator comutativ să admită şir de "star -momente reale" în cazul H spațiu real , respective şir de "star-momente complexe" în cazul H spațiu Hilbert complex. In Propoziția 3.2. a acestei note dăm condiții necesare pentru ca suportul măsurii de reprezentare a unui multioperator comutativ ca şir de "star-momente reale" , respectiv "star-momente complexe" să aibă suportul inclus în diferite mulțimi din $\boldsymbol{R}^n$ sau $\boldsymbol{C}^n$, mulțimi descrise prin condiții algebrice. In aceeaşi notă introducem noțiunea de multioperator comutativ care admite şir de " star L-momente" şi dăm o condiție suficientă pentru ca un multioperator comutativ să admită şir de "star L-momente complexe".

In this note we generalize the Definition 1.2. in [5] in case when $A=(A_1,\ldots,A_n) \in L(H)^n$, $n \geq 1$ is a commuting multioperator and H a real Hilbert space and also in case $T=(T_1\ldots T_n) \in L(H)^n$, $n \geq 2$ is a commuting multioperator on a complex Hilbert space H. We introduce in Section 2 of this note the notion of commuting multioperator that admits a star real moment sequence and the notion of commuting g multioperator that admits a complex moment sequence. We also generalize Th. 2.2. in [5] and give in Section 3 necessary and sufficient conditions for a commuting multioperator to admit a " star real moment sequence representation ,respectively a star complex moment sequence representation. In Proposition 3.2. of this note, we give a necessary condition such that the representation measure of a commuting multioperator as star moment sequence to have the support included in some special sets in R^n or C^n, particular sets described by algebraic conditions. We also introduce the notion of commuting multioperator that admit a star L complex-moment sequence and provide a

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sufficient condition such that a commuting multioperator to admit a star L- complex moment sequence.

Key words: star real and complex moment sequences ,commuting multioperator , linear functional, joint spectrum, Borel measure.

1. Introduction

Let H be a real Hilbert space and $L(H)$ the real algebra of all bounded linear operators on *H*. We denote with $A=(A_1,...,A_n) \in L(H)^n, n \geq 1$ a commuting multioperator on *H* that is $A_i A_j = A_j A_i$, $1 \leq i, j \leq n$,briefly a c.m. In case that *H* is a complex Hilbert space and $T=(T_1...T_n) \in L(H)^n, n \geq 2$ is a commuting multioperator, we denote with $T^*=(T_1^*,...,T_n^*) \in L(H)^n, n \geq 2$ the commuting adjoint of T. Let $t=(t_1,...,t_n) \in R^n$ denote the real variable in the Euclidian space and $z=(z_1,...,z_n) \in C^n$ the complex variable in the complex Euclidian space. If Let $m=(m_1,...,m_n) \in N^n$ is a multiindices, we have also $t^m=(t_1^{m_1}..t_n^{m_n}) \in \mathbf{R}$, $z^m=(z_1^{m_1}..z_n^{m_n}) \in \mathbf{C}$, $\bar{z}^m=(\bar{z}_1^{m_1}..\bar{z}_n^{m_n}) \in \mathbf{C}$, $T^m=T_1^{m_1} \circ ... \circ T_n^{m_n} \in L(H)$, $T^{*m}=T^{*m_1}_1 \circ ... \circ T^{*m_n}_n \in L(H)$.

2. Star real moment sequences

We generalize Definition 1.2. in [5].

Definition 2.1. Let H be an arbitrary real Hilbert space and $A=(A_1,...,A_n) \in L(H)^n, n \geq 1$ a commuting multioperator on H .The commuting multioperator A has a star real moment sequence supported on a compact set in $\mathbf{R}^n$ if there exists a compact set $\sigma(A) \subset R^n$, nonzero vectors $u, v \in H$ and a positive Borel measure μ on $\sigma(A)$ such that we have the representations:

$$< A^m u, v >_H = \int_{\sigma(A)} t^m d\mu(x) \quad ,\forall m \in N^n.$$

Remark. It is immediately that the commuting multioperators $A=(A_1,...,A_n) \in L(H)^n, n \geq 1$ that admit integral representations on compact sets in $\mathbf{R}^n$ with respect to Borel positive , operator-valued measures , admits also “star-real moment sequences”. From the class of commuting multioperators that admit integral representations on compact sets in $\mathbf{R}^n$ we mention the class of commuting self-adjoint multioperators. Commuting selfadjoint multioperators admit joint spectral measures which generate immediately the scalar positive

Borel representation measure as "star real-moment sequence". In this case, the vectors $u, v \in H$ are arbitrary and the compact set on which the scalar positive Borel measure is defined is the joint spectrum $\sigma(A)$ of the c.m. $A = (A_1, \ldots, A_n) \in L(H)^n, n \geq 1$.

Theorem 2.1. Let H be a real Hilbert space and $A = (A_1, \ldots, A_n) \in L(H)^n, n \geq 1$ a commuting multioperator. The following assertions are equivalent:

(i) The commuting multioperator $A \in L(H)^n, n \geq 1$ has a "star-real moment sequence" supported on the joint spectrum $\sigma(A)$ of A.

(ii) There exists nonzero vectors u and v in H such that $< u, v >_H \geq 0$ and

$$|< p(A)u, v >_H| \leq < u, v >_H \| p \|_\infty, \text{ for any } p \in \mathbf{R}[x_1, \ldots, x_n]$$

where $\| p \|_\infty := \sup_{x \in \sigma(A)} | p(x) |$.

Proof. (i) $\Rightarrow (ii)$. If we have (i) , for $m = (0, \ldots, 0) \in N^n$ we have $< A^0 u, v >_H = < Iu, v >_H = < u, v >_H = \int_{\sigma(A)} d\mu(x) = \mu(\sigma(A)) \geq 0.$ Let $p \in \mathbf{R}[x_1, \ldots, x_n] = \mathbf{R}[x]$ arbitrary ; we have $|< p(A)u, v >_H| = |\int_{\sigma(A)} p(x) d\mu(x)| \leq \mu(\sigma(A)) \| p \|_\infty = < u, v >_H \| p \|_\infty$,with $\| p \|_\infty := \sup_{x \in \sigma(A)} | p(x) |$; that is (ii).

Conversely. *(ii)* $\Rightarrow (i)$. We assume that $|< p(A)u, v >_H| = |\int_{\sigma(A)} p(x) d\mu(x)| \leq \mu(\sigma(A)) \| p \|_\infty = < u, v >_H \| p \|_\infty$ for all $p \in \mathbf{R}[x]$. Let the **R**-linear functional $L : \mathbf{R}[x] \to \mathbf{R}$, $L(p) = < p(A)u, v >_H$; because $L(1) = < Iu, v >_H = < u, v >_H \geq 0$, it results that L is positive on $\mathbf{R}[x]$ and from (ii), L is bounded on $\mathbf{R}$[x] $\Leftrightarrow$ L is continuous on $\mathbf{R}[x]$. In these conditions, via Hahn-Banach theorem we can extend L on the space of continuous functions $C(\sigma(A))$ with preserving the linearity, the continuity and the norm, that is $\| L_{ext} \| = \| L \|$; we denote the extension on $C(\sigma(A))$ with L_{ext}. Because L_{ext} is linear, continuous, positive on $C(\sigma(A))$ and $\sigma(A)$ is compact, from Riesz representation theorem we have an unique positive Borel measure μ on $\sigma(A)$ such that $L_{ext}(p) = \int_{\sigma(A)} p(x) d\mu(x) = < p(A)u, v >_H$, $\forall p \in \mathbf{R}[x]$. If we consider $p(x) = x^n, n \in \mathbf{N}^n$, we obtain $< A^n u, v >_H = < \int_{\sigma(A)} x^n d\mu(x) >_H, \forall n \in \mathbf{N}^n$; the required representation in (i).

3. Star complex moment sequences

Let H a complex Hilbert space and $T = (T_1, ..., T_n) \in L(H)^n, n \geq 2$ a commuting multioperator. We generalized Definition 1.2. in [5] in case $n \geq 2$. That is:

Definition 3.2. The commuting multioperator $T = (T_1, ..., T_n) \in L(H)^n, n \geq 2$ admit a star complex moment sequence supported on a compact set in $\mathbf{C}^n$ if there exists a compact set $\sigma(T) \subset \mathbf{C}^n$, nonzero vectors $u, v \in H$ and a positive Borel measure μ on $\sigma(T)$ such that we have the representations $< T^{\bullet j} T^k u, v >_H = \int_{\sigma(T)} \overline{z}^j z^k d\mu(z), \forall j, k \in \mathbf{N}^n$.

Remark. It is immediately that the commuting multioperators that admit integral representations with respect with spectral measures , admit also star complex moment sequences. We mention in this case the class of commuting normals or subnormals.

The following theorem generalized the same Th.2.1. in [5]. The basic ideas of its proof is the same as in Theorem 2.1. in this note and also with Th.2.1. in [5]. It gives a necessary and sufficient condition on a commuting multioperator to admit star complex moment sequence.

Theorem 3.1. Let H be a complex Hilbert space and $T = (T_1, ..., T_n) \in L(H)^n, n \geq 2$,be a commuting multioperator. The following assertions are equivalent:

(i) The commuting multioperator $T \in L(H)^n, n \geq 2$ has a "star-complex moment sequence" supported on the joint spectrum $\sigma(T)$ of T.

(ii) There exists nonzero vectors u and v in H such that $< u, v >_H \geq 0$ and

$$|< p(T^*, T)u, v >_H| \leq < u, v >_H \| p \|_\infty, \text{ for any}$$

$p \in \mathrm{C}[\overline{z}_1, ..., \overline{z}_n, z_1, ..., z_n] = \mathrm{C}[\overline{z}, z]$,

where $\| p \|_\infty := \sup_{zx \in \sigma(T)} | p(\overline{z}, z) |, z = (z_1, ..., z_n), \overline{z} = (\overline{z}_1, ..., \overline{z_n})$.

The basic ideas of the proof of th.3.1 is quite similar with those from proof of th. 2.1. in this note and also with th.2.1. in [5]. The difference, from the proof in Th.2.1. Section 2 of this note , is that the moment functional is defined by $L(\bar{z}^j z^k) = < T^{*j}T^k u, v >_H$, $\forall j,k \in \mathbf{N}^n$ and is extended by **C**-linearity to $\mathbf{C}[\bar{z}, z]$**.** Because of similarity with the proof in Th.2.1. Section 2, we omit it.

In various contexts, we can have a better localization of the support of the star real or complex moment sequence. We preserve the same notation as those in proof of Th.2.1.

Section 2 . For a better localization of the support of the representing measure of a "star-moment sequence" , we have the following :

Proposition 3.2. Let $T = (T_1, \ldots, T_n) \in L(\mathcal{H})^n, n \geq 1$,be a commuting multioperator which admits a star complex moment sequence defined on the joint spectrum $\sigma(T) \subset \mathbf{C}^n$ and

μ the Borel positive defined representation measure of T as a star-complex moment sequence.

Assume that there exists $r \in \mathbf{C}_R[\bar{z}, z]$ (that is $r(\bar{z}, z) \in \mathbf{R}$ $\forall z \in \mathbf{C}$) with the property

that $L_{ext}(rp) \geq 0$ for all $p \in \mathbf{C}[\bar{z}, z]$ for which $p(\bar{z}, z) \geq 0$ when $z \in \sigma(T)$. In this case

$\operatorname{supp} \mu \subset \{z \in \sigma(T), r(\bar{z}, z) \geq 0\}$.

If also $L_{ext}(rp) = 0$ for all $p \in \mathbf{C}[\bar{z}, z]$ with $p(\bar{z}, z) \geq 0$ when $z \in \sigma(T)$ and some

$r \in \mathbf{C}_R[\bar{z}, z]$ **,** then $\operatorname{supp} \mu \subset \{z \in \sigma(T), r(\bar{z}, z) = 0\}$.

Proof. (a) Let $L_{ext} : C(\sigma(T)) \to C$ the extended moment functional from Th.3.1., $L_{ext}(\bar{z}^j z^k) = < T^{*j}T^k u, v >_H = \int_{\sigma(T)} \bar{z}^j z^k d\mu(z)$ when $j,k \in \mathbf{N}^n$**.** Because of (a) we have

$L_{ext}(p(\bar{z}, z) r(\bar{z}, z)) = \int_{\sigma(T)} p(\bar{z}, z) r(\bar{z}, z) d\mu(z) \geq 0$ for all $p \in \mathbf{C}[\bar{z}, z]$ with $p(\bar{z}, z) \geq 0$ on $\sigma(T)$.

It follows ,from Weierstrass approximation theorem , that we have $L_{ext}(f(z) r(\bar{z}, z)) \geq 0$ for all continuous positive functions $f \in \mathbf{C}_R(\sigma(T))$. With an easy measure theoretic argument, it follows that $\mu(B) = 0$ when $B \subset \{z \in \sigma(T), r(\bar{z}, z) < 0\}$. From the above argument it follows that $\operatorname{supp} \mu \subset \{z \in \sigma(T), r(\bar{z}, z) \geq 0\}$; that is (a).

(b) If $L_{ext}(rp)=0$ for some $r\in \mathbf{C}_R[\bar{z},z]$ and all $p\in \mathrm{C}[\bar{z},z]$ with $p(\bar{z},z)\geq 0$ on $\sigma(T)$, then we have $L_{ext}(rf)=0$ for all real positive valued continuous functions on $\sigma(T)$; i.e. $f\in C(\sigma(T)), f(z)\geq 0$. Again, by a simple measure theoretic argument, we have supp $\mu\subset\{z\in\sigma(T),\quad r(\bar{z},z)=0\}$.

4. Star *L*- moment sequences

Definition 4. 3. Let $T=(T_1,...,T_n)\in L(H)^n, n\geq 1$,be a commuting multioperator on H , a complex Hilbert space. The c.m. T has a "star L-moment sequence" supported on a compact set $K\subset \mathbf{C}^n$ if there exist the nonzero vectors $u,v\in H$, a positive Borel measure μ on K and a measurable bounded function $h:K\to \mathbf{R}$ with $0\leq h\leq L$ for a constant $L>0,$ such that the following representations hold :

$$<T^{*j}T^k u,v>_H=\int_{\sigma(T)}\bar{z}^j z^k h(z)d\mu(z) \text{ for all } j,k\in \mathbf{N}^n.$$

Proposition 4.1. Let $T=(T_1,...,T_n)\in L(H)^n, n\geq 1$ and $S=(S_1,...,S_n)\in L(H)^n, n\geq 1$, be commuting multioperators on H a complex Hilbert space such the joint spectrums

$\sigma(T)=\sigma(S)=K$. Let the following assumptions be :

There exists $u,v\in H$ such that $|<p(T^*,T)u,v>_H|\leq|<p(S^*,S)u,v>_H|\leq <u,v>\| p\|_\infty, \forall p\in \mathbf{C}[\overline{z,z}]$.

There exists a measurable function $h:K\to \mathbf{R}$ such the following representations

$$<T^{*j}T^k u,v>_H=\int_{\sigma(T)}\bar{z}^j z^k h(z)d\mu(z) \text{ for all } j,k\in \mathbf{N}^n \text{ and}$$

$$<S^{*j}S^k u,v>_H=\int_{\sigma(T)}\bar{z}^j z^k d\mu(z) \text{ for all } j,k\in \mathbf{N}^n \text{ hold.}$$

In this case, the implication (i)$\Rightarrow$(ii) is true.

Proof. If (i) is true, then from Th.3.1. Section 3. there exists a positive Borel measure μ on the joint spectrums $\sigma(T)=\sigma(S)=K\subset \mathbf{C}^n$ such that we have $<T^{*j}T^k u,v>_H=\int_{\sigma(T)}\bar{z}^j z^k d\lambda(z)$ for all $j,k\in \mathbf{N}^n$ and

$< S^{*j}S^k u, v >_H = \int_{\sigma(S)} \bar{z}^j z^k d\mu(z)$ for all $j,k \in \mathbf{N}^n$. We shall prove, as in paper [3], that the measure $d\lambda$ is absolutely continuous with respect to the measure $d\mu$.

Indeed, because of (i) and because the obtained representation as "star-complex moment sequences", we have $0 \le \int_{\sigma(T)} p(\bar{z},z)d\lambda(z) \le \int_{\sigma(S)} p(\bar{z},z)d\mu(z)$ for all $p \in \mathbf{C}[\bar{z},z]$ with

$p(\bar{z},z) \ge 0$ on *K*. Let $E \subset K$ be a measurable set with $\mu(E)=0$ and χ_E the characteristic function of E. We approximate χ_E pointwise and monotonically by a sequence $\{\varphi_m\}_{m\in N} \in C_R^+(K)$. For this sequence ,because of previous inequalities, we have

$$0 \le \int_K \varphi_m(z)d\lambda(z) \le \int_K \varphi_m(z)d\mu(z) \le \int_K \overline{\lim}_m \varphi_m d\mu(z) = \int_K \chi_E(z)d\mu(z) = \mu(E\cap K).$$

Hence, if $\mu(K\cap E)=0$, it results that also $\lambda(K\cap E)=0$. In this case there exists $h \in \mathrm{L}^1(K,\mu)$ such that $d\mu = hd\lambda$. Because $\int_K h\varphi d\mu \ge 0$ for any $\varphi \in C_R^+(K)$, we have $h \ge 0$ and because

$\int_K h\varphi d\mu \le \int_K \varphi d\mu$, we have $h \le 1$. That is , $0 \le h \le 1$ and we have the required representations

$< T^{*j}T^k u, v >_H = \int_{\sigma(T)} \bar{z}^j z^k h(z)d\mu(z)$ for all $j,k \in \mathbf{N}^n$ and

$< S^{*j}S^k u, v >_H = \int_{\sigma(T)} \bar{z}^j z^k d\mu(z)$ for all $j,k \in \mathbf{N}^n$. Q.E.D.

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R E F E R E N C E S

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