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ON POSITIVE SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS INVOLVING THE *p(x)*-LAPLACIAN WITH MULTIPLE PARAMETERS

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In this article, we consider the system of differential equations

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)}[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)] & \text{in}\Omega \\ -\Delta_{p(x)} v = \lambda^{p(x)}[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)] & \text{in}\Omega \\ u = v = 0 \text{on} & \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ *is a bounded domain with* C^2 *boundary* $\partial\Omega$, $1 < p(x) \in C^1(\overline{\Omega})$ *is a function. The operator* $-\Delta_{p(x)} u = -\text{div}(|\nabla u|^{p(x)-2} \nabla u)$ *is called* $p(x)$*-Laplacian,* $\lambda, \lambda_1, \lambda_2, \mu_1$ *and* μ_2 *are positive parameters. We prove the existence of positive solutions when*

$$\lim_{u \to +\infty} \frac{f\left(M(g(u))^{\frac{1}{p^- -1}}\right)}{u^{p^- -1}} = 0, \quad \forall M > 0,$$

$$\lim_{u \to +\infty} \frac{h(u)}{u^{p^- -1}} = 0, \quad \lim_{u \to +\infty} \frac{\tau(u)}{u^{p^- -1}} = 0$$

via sub-supersolutions without assuming sign conditions on $f(0), g(0), h(0)$ *or* $\tau(0)$.

Keywords: Positive solutions; Sub-super solution; Variable exponent elliptic systems; $p(x)$-Laplacian problems.

1. Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [4, 16, 23]). Many results have been obtained on this kind of problems, for example [1, 4, 5, 7, 14, 23]. In [8, 9, 10],

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Fan et al. give the regularity of weak solutions for differential equations with variable exponent. On the existence of solutions for elliptic systems with variable exponent, we refer to [14, 18, 19, 21]. In this paper, we mainly consider the existence of positive weak solutions for the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}[\lambda_1 a(x)f(v)+\mu_1 c(x)h(u)] & \text{in}\Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)}[\lambda_2 b(x)g(u)+\mu_2 d(x)\tau(v)] & \text{in}\Omega, \\ u = v = 0 & \text{on}\partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $1 < p(x) \in C^1(\overline{\Omega})$ is a function. The operator $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x) \equiv p$ (a constant), (1) is the well-known p-Laplacian system. There are many papers on the existence of solutions for p-Laplacian elliptic systems, for example [3, 5, 13, 15]. Owing to the nonhomogeneity of $p(x)$-Laplacian problems are more complicated than those of p-Laplacian, many results and methods for p-Laplacian are invalid for $p(x)$-Laplacian; for example, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [12]), and maybe the first eigenvalue and the first eigenfunction of $p(x)$-Laplacian do not exist, but the fact that the first eigenvalue $\lambda_p > 0$ and the existence of the first eigenfunction are very important in the study of p-Laplacian problems. There are more difficulties in discussing the existence of solutions of variable exponent problems. In [13], the authors consider the existence of positive weak solutions for the following p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in}\Omega, \\ -\Delta_p v = \lambda g(u) & \text{in}\Omega, \\ u = v = 0 & \text{on}\partial\Omega. \end{cases} \tag{2}$$

The first eigenfunction is used to construct the subsolution of p-Laplacian problems successfully. On the condition that λ is large enough and

$$\lim_{u\to+\infty} \frac{f[M(g(u))^{\frac{1}{(p-1)}}]}{u^{p-1}} = 0 \text{for every} M > 0,$$

the authors give the existence of positive solutions for problem (2).

On the $p(x)$-Laplacian problems, maybe the first eigenvalue and the first eigenfunction of $p(x)$-Laplacian do not exist. Even if the first eigenfunction of $p(x)$-Laplacian exist, because of the nonhomogeneity of $p(x)$-Laplacian, the first eigenfunction cannot be used to construct the subsolution of $p(x)$-Laplacian problems. In [2, 18, 21], the authors studied the existence of solutions for the problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda f(v) & \text{in}\Omega, \\ -\Delta_{q(x)}v = \lambda g(u) & \text{in}\Omega, \\ u = v = 0 & \text{on}\partial\Omega, \end{cases} \tag{3}$$

when some symmetric conditions are imposed. In [20] Q.H. Zhang investigated the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} f(v) & \text{in}\Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)} g(u) & \text{in}\Omega, \\ u = v = 0 & \text{on}\partial\Omega, \end{cases} \tag{4}$$

without any symmetric conditions. Motivated by the ideas introduced in [2, 20], in this paper, we study the existence of positive solutions for system (1), where $p \in C^1(\overline{\Omega})$ is a function, $\lambda, \lambda_1, \lambda_2, \mu_1$ and μ_2 are positive parameters, a, b, c, d are continuous functions and $\Omega \subset \mathrm{R}^N$ is a bounded domain. It should be noticed that we do not assume the symmetric condition as in [2, 18, 21].

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [7, 17]. If $\Omega \subset \mathrm{R}^N$ is an open domain, write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\},$$

$$h^+ = \sup_{x\in\Omega} h(x), \quad h^- = \inf_{x\in\Omega} h(x) \quad \text{for any } h \in C_+(\Omega).$$

Throughout the paper, we will assume that:

- $\Omega \subset \mathrm{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$;
- $p \in C^1(\overline{\Omega})$ and $1 < p^- \le p^+$;
- $h, \tau \in C^1([0,\infty))$ are nonnegative, nondecreasing functions such that
  $$\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = 0, \quad \lim_{u\to+\infty} \frac{\tau(u)}{u^{p^- -1}} = 0.;$$
- $f, g \in C^1([0,\infty))$ are nondecreasing functions, $\lim_{u\to+\infty} f(u) = +\infty$, $\lim_{u\to+\infty} g(u) = +\infty$, and

$$\lim_{u\to+\infty}\frac{f(M(g(u))^{\frac{1}{p^- -1}})}{u^{p^- -1}}=0,\quad \forall M>0.;$$

- $a,b,c,d:\overline{\Omega}\to(0,\infty)$ are continuous functions such that

$a_1=\min_{x\in\overline{\Omega}}a(x)$, $b_1=\min_{x\in\overline{\Omega}}b(x)$, $c_1=\min_{x\in\overline{\Omega}}c(x)$, $d_1=\min_{x\in\overline{\Omega}}d(x)$,
$a_2=\max_{x\in\overline{\Omega}}a(x)$ and $b_2=\max_{x\in\overline{\Omega}}b(x)$, $c_2=\max_{x\in\overline{\Omega}}c(x)$, $d_2=\max_{x\in\overline{\Omega}}d(x)$.

Denote

$$L^{p(x)}(\Omega)=\left\{u \mid u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\},$$

and $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ becomes a Banach space, we call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega),|.|_{p(x)})$ is a separable, reflexive and uniform convex Banach space (see [7, Theorems 1.10, 1.14]). The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}:|\nabla u|\in L^{p(x)}\}$, and it is equipped with the norm

$$\|u\|=|u|_{p(x)}+|\nabla u|_{p(x)},\quad \forall\, u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach space (see [7, Theorem 2.1]). We define

$$\langle L(u),v\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx,\quad \forall\, u,v\in W_0^{1,p(x)}(\Omega),$$

then $L:W_0^{1,p(x)}(\Omega)\to(W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded and strictly monotone operator, and it is a homeomorphism (see [11, Theorem 3.1].

If $u,v\in W_0^{1,p(x)}(\Omega),(u,v)$ is called a weak solution of (1) if it satisfies

$$\begin{cases}\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla q dx=\int_\Omega\lambda^{p(x)}[\lambda_1 f(v)+\mu_1 h(u)]q dx, & \forall\, q\in W_0^{1,p(x)}(\Omega),\\ \int_\Omega|\nabla v|^{p(x)-2}\nabla v\nabla q dx=\int_\Omega\lambda^{p(x)}[\lambda_2 g(u)+\mu_2\tau(v)]q dx, & \forall\, q\in W_0^{1,p(x)}(\Omega).\end{cases}$$

Define $A:W^{1,p(x)}(\Omega)\to(W_0^{1,p(x)}(\Omega))^*$ as

$$\langle Au,\varphi\rangle=\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi+j(x,u)\varphi)dx,\quad \forall u,\varphi\in W^{1,p(x)}(\Omega),$$

where $j(x,u)$ is continuous on $\overline{\Omega}\times\mathrm{R}$, and $j(x,\cdot)$ is increasing. It is easy to check that A is a continuous bounded mapping. Copying the proof of [22], we have the following lemma.

Lemma 1.1 (Comparison principle) *Let* $u, v \in W^{1,p(x)}(\Omega)$ *satisfy* $Au - Av \geq 0$ *in* $(W_0^{1,p(x)}(\Omega))^*$, $\varphi(x) = \min\{u(x) - v(x), 0\}$. *If* $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ *(i.e.* $u \geq v$ *on* $\partial\Omega$ *), then* $u \geq v$ *a.e. in* Ω.

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω. Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\varepsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) < \varepsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a constant $\delta \in (0,1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, and $|\nabla d(x)| \equiv 1$.

Denote

$$v_1(x) = \begin{cases} \gamma d(x), & d(x) < \delta, \\ \gamma\delta + \int_\delta^{d(x)} \gamma (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, & \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, & 2\delta \leq d(x). \end{cases}$$

$$v_2(x) = \begin{cases} \gamma d(x), & d(x) < \delta, \\ \gamma\delta + \int_\delta^{d(x)} \gamma (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_2 b_1 + \mu_2 d_1)^{\frac{2}{p^- - 1}} dt, & \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma (\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_2 b_1 + \mu_2 d_1)^{\frac{2}{p^- - 1}} dt, & 2\delta \leq d(x). \end{cases}$$

Obviously, $0 \leq v_1(x), v_2(x) \in C^1(\overline{\Omega})$. Consider

$$\begin{cases} -\Delta_{p(x)} w(x) = \eta & \text{in}\,\Omega, \\ w = 0 & \text{on}\,\partial\Omega. \end{cases} \tag{5}$$

Lemma 1.2 (see[6]) *If the positive parameter* η *is large enough and* w *is the unique solution of (5), then we have*

- For any $\theta \in (0,1)$ there exists a positive constant C_1 such that

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \leq \max_{x \in \overline{\Omega}} w(x);$$

- There exists a positive constant C_2 such that $\max_{x \in \overline{\Omega}} w(x) \leq C_2 \eta^{\frac{1}{p^- - 1}}$.

2. Main result

In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

Theorem 2.1 *If the conditions* (H_1)-(H_5) *are satisfied then problem (1) has a positive solution when* λ *is large enough.*

Proof. We shall establish Theorem 2.1 by constructing a positive subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (1), such that $\Phi_1 \le z_1$ and $\Phi_2 \le z_2$. That is (Φ_1, Φ_2) and (z_1, z_2) satisfy

$$\begin{cases} \int_\Omega |\nabla \Phi_1|^{p(x)-2} \nabla \Phi_1 \cdot \nabla q dx \le \int_\Omega \lambda^{p(x)} [\lambda_1 a(x) f(\Phi_2) + \mu_1 c(x) h(\Phi_1)] q dx, \\ \int_\Omega |\nabla \Phi_2|^{p(x)-2} \nabla \Phi_2 \cdot \nabla q dx \le \int_\Omega \lambda^{p(x)} [\lambda_2 b(x) g(\Phi_1) + \mu_2 d(x) \tau(\Phi_2)] q dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla q dx \ge \int_\Omega \lambda^{p(x)} [\lambda_1 a(x) f(\Phi_2) + \mu_1 c(x) h(\Phi_1)] q dx, \\ \int_\Omega |\nabla z_2|^{p(x)-2} \nabla z_2 \cdot \nabla q dx \ge \int_\Omega \lambda^{p(x)} [\lambda_2 b(x) g(\Phi_1) + \mu_2 d(x) \tau(\Phi_2)] q dx, \end{cases}$$

for all $q \in W_0^{1,p(x)}(\Omega)$ with $q \ge 0$. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [6]), then (1) has a positive solution.

Step 1. We will construct a subsolution of (1). Let $\sigma \in (0, \delta)$ be small enough. Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, \quad \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \le d(x). \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_2 b_1 + \mu_2 d_1)^{\frac{2}{p^- - 1}} dt, \quad \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_2 b_1 + \mu_2 d_1)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \le d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\overline{\Omega})$. Denote

$$\alpha = \min\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)|+1)},1\},$$
$$\zeta = \min\{\lambda_1 a_1 f(0)+\mu_1 c_1 h(0), \lambda_2 b_1 g(0)+\mu_2 d_1 \tau(0), -1\}.$$

By some simple computations we can obtain

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\Delta d}{k}\right], \quad d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^- -1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d+\Delta d\right]\right\} \\ \times(ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}(\lambda_1 a_1+\mu_1 c_1), \quad \sigma<d(x)<2\delta, \\ 0, \quad 2\delta<d(x). \end{cases}$$

$$-\Delta_{p(x)}\phi_2 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\Delta d}{k}\right], \quad d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^- -1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d+\Delta d\right]\right\} \\ \times(ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}(\lambda_2 b_1+\mu_2 d_1), \quad \sigma<d(x)<2\delta, \\ 0, \quad 2\delta<d(x). \end{cases}$$

From (H_3) and (H_4), there exists a positive constant $M>1$ such that

$$f(M-1)\geq 1, h(M-1)\geq 1, g(M-1)\geq 1, \gamma(M-1)\geq 1.$$

Let $\sigma=\frac{1}{k}\ln M$. Then

$$\sigma k=\ln M. \tag{6}$$

If k is sufficiently large from 6, we have

$$-\Delta_{p(x)}\phi_1 \leq -k^{p(x)}\alpha, \quad d(x)<\sigma. \tag{7}$$

Let $-\lambda\zeta = k\alpha$. Then

$$k^{p(x)}\alpha \geq \lambda^{p(x)}\zeta.$$

From (7), we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda^{p(x)}(\lambda_1 a_1 f(0) + \mu_1 c_1 h(0)) \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)), \quad d(x) < \sigma. \tag{8}$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, there exists a positive constant C_3 such that

$$\begin{aligned}
-\Delta_{p(x)}\phi_1 \quad &\leq (ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}(\lambda_1 a_1 + \mu_1 c_1) \\
&\quad \times \left|\left\{\frac{2(p(x)-1)}{(2\delta-\sigma)(p^- -1)} - \left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p \nabla d + \Delta d\right]\right\}\right| \\
&\leq C_3 (ke^{k\sigma})^{p(x)-1}(\lambda_1 a_1 + \mu_1 c_1)\ln k, \quad \sigma < d(x) < 2\delta.
\end{aligned}$$

If k is sufficiently large, since $-\lambda\zeta = k\alpha$, we have

$$(\lambda_1 a_1 + \mu_1 c_1)C_3(ke^{k\sigma})^{p(x)-1}\ln k = (\lambda_1 a_1 + \mu_1 c_1)C_3(kM)^{p(x)-1}\ln k \leq \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1).$$

Then

$$-\Delta_{p(x)}\phi_1 \leq \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1), \quad \sigma < d(x) < 2\delta.$$

Since $\phi_1(x), \phi_2(x) \geq 0$ and h, f are monotone, when λ is large enough we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)), \quad \sigma < d(x) < 2\delta. \tag{9}$$

Obviously

$$-\Delta_{p(x)}\phi_1 = 0 \leq \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1) \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)), \quad 2\delta < d(x). \tag{10}$$

Combining (8), (9) and (10), we can conclude that

$$-\Delta_{p(x)}\phi_1 \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)), \quad \text{a.e.on}\,\Omega. \tag{11}$$

Similarly

$$-\Delta_{p(x)}\phi_2 \leq \lambda^{p(x)}(\lambda_2 b(x) g(\phi_1) + \mu_2 d(x) \tau(\phi_2)), \quad \text{a.e.on}\,\Omega. \tag{12}$$

From (11) and (12), we can see that (ϕ_1, ϕ_2) is a subsolution of problem (1).

Step 2. We will construct a supersolution of problem (1). We consider

$$\begin{cases} -\Delta_{p(x)} z_1 = \lambda^{p^+} \mu(\lambda_1 a_2 + \mu_1 c_2) & \text{in}\Omega, \\ -\Delta_{p(x)} z_2 = \lambda^{p^+} (\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)) & \text{in}\Omega, \\ z_1 = z_2 = 0 & \text{on}\partial\Omega, \end{cases}$$

where $\beta = \beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu) = \max_{x\in\bar{\Omega}} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution of problem (1).

For $q \in W_0^{1,p(x)}(\Omega)$ with $q \geq 0$, it is easy to see that

$$\begin{aligned} &\int_\Omega |\nabla z_2|^{p(x)-2} \nabla z_2 \cdot \nabla q dx \\ &= \int_\Omega \lambda^{p^+} (\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)) q dx \qquad (13) \\ &\geq \int_\Omega \lambda^{p^+} \mu_2 d(x) g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)) q dx + \int_\Omega \lambda^{p^+} \lambda_2 b(x) g(z_1) q dx. \end{aligned}$$

By (H_5) for μ large enough, using Lemma 1.2, we have

$$g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)) \geq \tau([\lambda^{p^+} (\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu))]^{\frac{1}{p^- -1}}) \geq \tau(z_2). \qquad (14)$$

Hence,

$$\int_\Omega |\nabla z_2|^{p(x)-2} \nabla z_2 \cdot \nabla q dx \geq \int_\Omega \lambda^{p^+} \mu_2 d(x) \tau(z_2) q dx + \int_\Omega \lambda^{p^+} \lambda_2 b(x) g(z_1) q dx. \qquad (15)$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla q dx = \int_\Omega \lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2) \mu q dx.$$

By $(H_3),(H_4)$, when μ is sufficiently large, using Lemma 1.2, we have

$$\begin{aligned} &(\lambda_1 a_2 + \mu_1 c_2)\mu \\ &\geq \frac{1}{\lambda^{p^+}} [\frac{1}{C_2} \beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)]^{p^- -1} \\ &\geq \mu_1 h(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu)) + \lambda_1 f(C_2[\lambda^{p^+} (\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+} (\lambda_1 a_2 + \mu_1 c_2)\mu))]^{\frac{1}{p^- -1}}). \end{aligned}$$

Then

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla q dx \geq \int_\Omega \lambda^{p^+} \lambda_1 a(x) f(z_2) q dx + \int_\Omega \lambda^{p^+} \mu_1 c(x) h(z_1) q dx. \qquad (16)$$

According to (15) and (16), we can conclude that (z_1, z_2) is a supersolution for (1).

It only remains to prove that $\phi_1 \le z_1$ and $\phi_2 \le z_2$.

In the definition of $v_1(x)$, let

$$\gamma = \frac{2}{\delta}(\max_{x\in\overline{\Omega}} \phi_1(x) + \max_{x\in\overline{\Omega}} |\nabla \phi_1(x)|).$$

We claim that

$$\phi_1(x) \le v_1(x), \quad \forall x \in \Omega. \tag{17}$$

From the definition of v_1, it is easy to see that

$$\phi_1(x) \le 2\max_{x\in\overline{\Omega}} \phi_1(x) \le v_1(x), \text{when} d(x) = \delta$$

and

$$\phi_1(x) \le 2\max_{x\in\overline{\Omega}} \phi_1(x) \le v_1(x), \text{when} d(x) \ge \delta.$$

It only remains to prove that

$$\phi_1(x) \le v_1(x), \text{when} d(x) < \delta.$$

Since $v_1 - \phi_1 \in C^1(\overline{\partial\Omega_\delta})$, there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1(x_0) - \phi_1(x_0) = \min_{x_0\in\overline{\partial\Omega_\delta}} [v_1(x) - \phi_1(x)].$$

If $v_1(x_0) - \phi_1(x_0) < 0,$ it is easy to see that $0 < d(x) < \delta,$ and then

$$\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0.$$

From the definition of v_1, we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}(\max_{x\in\overline{\Omega}} \phi_1(x) + \max_{x\in\overline{\Omega}} |\nabla \phi_1(x)|) > |\nabla \phi_1(x_0)|.$$

It is a contradiction to $\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0$. Thus (17) is valid.

Obviously, there exists a positive constant C_3 such that

$$\gamma \le C_3 \lambda.$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 1.2, then there exists a positive constant C_4 such that

$$-\Delta_{p(x)} v_1(x) \le C_* \gamma^{p(x)-1+\theta} \le C_4 \lambda^{p(x)-1+\theta}, \quad \text{a.e.in} \Omega, \text{where} \theta \in (0,1).$$

When $\eta \ge \lambda^{p^+}$ is large enough, we have

$$-\Delta_{p(x)} v_1(x) \le \eta.$$

According to the comparison principle, we have

$$v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{18}$$

From (17) and (18), when $\eta \ge \lambda^{p^+}$ and $\lambda \ge 1$ is sufficiently large, we have

$$\phi_1(x) \le v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{19}$$

According to the comparison principle, when μ is large enough, we have

$$v_1(x) \le w(x) \le z_1(x), \quad \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (19), it is easy to see that

$$\phi_1(x) \le v_1(x) \le w(x) \le z_1(x), \quad \forall x \in \Omega.$$

When $\mu \ge 1$ and λ is large enough, from Lemma 1.2 we can see that $\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu)$ is large enough then

$$\lambda^{p^+}(\lambda_2 b_2 + \mu_2 d_2)h(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu))$$

is large enough. Similarly, we have $\phi_2 \le z_2$. This completes the proof.

3. Conclusions

The study of differential equations involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of last century. In this paper, we have proved that problem (1) has at least one positive solution provided that the parameter λ is large enough. The approach is based on the sub-supersolution arguments. This improves or complements the previous results [2, 18, 20, 21] in the sense that we consider problem (1) with weights $a(x), b(x), c(x), d(x)$ and the fact that we do not assume any symmetric conditions on the nonlinearities.

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