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MODIFIED ISHIKAWA ITERATION PROCESS FOR TWO NONEXPANSIVE SEMIGROUPS IN CAT(0) SPACESS

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In this paper, we introduce modified Ishikawa iteration process for two nonexpansive semigroups in CAT(0) spaces. Then, we prove strong and △-convergence theorems for such iterative process in CAT(0) spaces. Moreover, we present △-convergence theorems of Ishikawa iteration process for a family of nonexpansive mappings in CAT(0) spaces. The results obtained in this paper extend and improve some recent known results.

Keywords: Nonexpansive semigroup, Ishikawa iterative process, △-convergence, CAT(0) space, Common fixed point.

MSC2010: : 47H10, 47H09.

1. Introduction

A mapping T of X into itself is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for each $x, y \in X$. We denote by $F(T)$ the set of fixed point of T . Fixed point theory in CAT(0) spaces (the initials of term "CAT" are in honor of E. Cartan, A.D. Alexanderov and V.A. Toponogov) was first studied by Kirk [1]. Since then the fixed point theory for nonexpansive mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see e.g., [2-11] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in R - trees) can be applied to graph theory, biology, and computer science (see e.g., 12-17).

The Mann iterative process [18] is defined in a CAT(0) space by

$$x_{n+1} = a_n x_n \oplus (1 - a_n)Tx_n, \quad \forall n \geq 0$$

where $\{a_n\}$ is a sequence in [0,1]. The Ishikawa iterative sequence [19] is defined by

$$y_n = a_n x_n \oplus (1 - a_n)Tx_n, \quad \forall n \geq 0$$

$$x_{n+1} = a_n x_n \oplus (1 - a_n)Ty_n, \quad \forall n \geq 0$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in [0,1]. In [7], Dhompongsa and Panyanak obtained △-convergence theorems for the Mann and Ishikawa iterations for a nonexpansive single valued mappings in CAT(0) spaces.

A family $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ of mappings on a closed convex subset D of a CAT(0) space X is called a nonexpansive semigroup if it satisfies the following conditions:

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(i) $T(0)x = x$ for all $x \in D$;

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$ and $s \geq 0$;

(v) for all $x \in D$, $s \longrightarrow T(s)x$ is continuous.

We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup $\mathcal{T}$, i.e., $F(\mathcal{T}) = \{x \in D : T(s)x = x, \forall s \geq 0\}$. In [20], Suzuki introduced an iterative process $\{x_n\}$ for nonexpansive semigroup on D, where D is a compact and convex subset of a Banach space X defined by $x_1 \in D$ and

$$x_{n+1} = \lambda T(t_n)x_n + (1-\lambda)x_n$$

where $\lambda \in (0, 1)$ and $\{t_n\} \subset [0, \infty)$. Then the author proved that $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$. Recently Cho , Ciric and Wang [9] generalized the result of Suzuki for CAT(0) spaces. In this paper, we introduce modified Ishikawa iteration process for two nonexpansive semigroups in CAT(0) spaces, and then prove strong and $\triangle$-convergence theorems for such iterative process in CAT(0) spaces. Moreover, we present a convergence theorem for a sequence of nonexpansive mappings in CAT(0) spaces. Our results generalized some recent known results.

2. **Preliminaries**

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is called convex if D includes every geodesic segment joining any two points of itself.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x, y) \leq d_{R^2}(\overline{x}, \overline{y})$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \longrightarrow \infty} d(x_n, x).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a CAT(0) space, $A(\{x_n\}))$ consists of exactly one point [2]. A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$.

Lemma 2.1. *([5]). Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2. *([6]). If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3. *([7]). If $\{x_n\}$ is a bounded sequence in complete CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4. *([7]). Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have*

(i) $d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$,
(ii) $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$.

Lemma 2.5. *([7]). Let D be a nonempty closed convex subset of a complete CAT(0) space X, and $T : D \longrightarrow D$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in D such that $\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$ and $\triangle - \lim_{n\longrightarrow\infty} x_n = v$, then $v = Tv$.*

3. Main Result

In this section we present some strong and $\triangle$-convergent theorems of Ishikawa iterative process for two nonexpansive semigroups in a CAT(0) space.

Theorem 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ and $\mathcal{S} := \{S(t) : t \geq 0\}$ be two nonexpansive semigroups. Assume that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be sequence generated the following algorithm:*

$$y_n = (1 - a_n)x_n \oplus a_n T(t_n)x_n,$$
$$x_{n+1} = (1 - b_n)x_n \oplus b_n S(t_n)y_n,$$

where $\{a_n\}$,$\{b_n\}$ and $\{t_n\}$ satisfy the following conditions:

(i) $a_n, b_n \in [a, b] \subset (0, 1)$
(ii) $t_n > 0$, $\liminf_{n\longrightarrow\infty} t_n = 0$, $\limsup_{n\longrightarrow\infty} t_n > 0$, $\lim_{n\longrightarrow\infty} t_{n+1} - t_n = 0$.

Then for $t > 0$ we have $lim_{n\longrightarrow\infty} d(x_n, T(t)x_n) = lim_{n\longrightarrow\infty} d(x_n, S(t)x_n) = 0$ and $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$.

Proof. Let $p \in \mathcal{F}$, then by using Lemma 2.4 we have

$$\begin{aligned} d(y_n, p) &= d((1 - a_n)x_n \oplus a_n T(t_n)x_n, p) \\ &\leq (1 - a_n)d(x_n, p) + a_n d(T(t_n)x_n, p) \\ &\leq (1 - a_n)d(x_n, p) + a_n d(x_n, p) = d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1},p) &= d((1-b_n)x_n \oplus b_n S(t_n)y_n, p) \\ &\le (1-b_n)d(x_n,p) + b_n d(S(t_n)y_n,p) \\ &\le (1-b_n)d(x_n,p) + b_n d(y_n,p) = d(x_n,p). \end{aligned}$$

Hence $\lim_{n\longrightarrow\infty} d(x_n,p)$ exists, and therefore $\{x_n\}$ is bounded. Also by Lemma 2.4 we have

$$\begin{aligned} d(y_n,p)^2 &= d((1-a_n)x_n \oplus a_n T(t_n)x_n, p)^2 \\ &\le (1-a_n)d(x_n,p)^2 + a_n d(T(t_n)x_n,p)^2 - a_n(1-a_n)d(x_n,T(t_n)x_n)^2 \\ &\le (1-a_n)d(x_n,p)^2 + a_n d(x_n,p)^2 - a_n(1-a_n)d(x_n,T(t_n)x_n)^2 \\ &\le d(x_n,p)^2 - a_n(1-a_n)d(x_n,T(t_n)x_n)^2 \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1},p)^2 &= d((1-b_n)x_n \oplus b_n S(t_n)y_n, p)^2 \\ &\le (1-b_n)d(x_n,p)^2 + b_n d(S(t_n)y_n,p)^2 - b_n(1-b_n)d(x_n,S(t_n)y_n)^2 \\ &\le (1-b_n)d(x_n,p)^2 + b_n d(y_n,p)^2 - b_n(1-b_n)d(x_n,S(t_n)y_n)^2 \\ &\le d(x_n,p)^2 - b_n a_n(1-a_n)d(x_n,T(t_n)x_n)^2 - b_n(1-b_n)d(x_n,S(t_n)y_n)^2. \end{aligned}$$

Thus we have

$$a^2(1-b)d(x_n,T(t_n)x_n)^2 \le b_n a_n(1-a_n)d(x_n,T(t_n)x_n)^2 \le d(x_n,p)^2 - d(x_{n+1},p)^2,$$

since $\lim_{n\longrightarrow\infty} d(x_n,p)$ exists, by taking Limit in above inequality we obtain that

$$\lim_{n\longrightarrow\infty} d(x_n,T(t_n)x_n) = 0.$$

By a similar argument we obtain

$$\lim_{n\longrightarrow\infty} d(x_n,S(t_n)y_n) = 0.$$

Also we have

$$d(x_n,y_n) = d(x_n,(1-a_n)x_n \oplus a_n T(t_n)x_n) \le a_n d(x_n,T(t_n)x_n) \longrightarrow 0 \quad n \longrightarrow \infty$$

and hence

$$\begin{aligned} d(x_n,S(t_n)x_n) &\le d(x_n,S(t_n)y_n) + d(S(t_n)y_n,S(t_n)x_n) \\ &\le d(x_n,S(t_n)y_n) + d(y_n,x_n) \longrightarrow 0 \quad n \longrightarrow \infty. \end{aligned}$$

Now we show that for a fixed $t > 0$

$$\lim_{n\longrightarrow\infty} d(x_n,T(t)x_n) = \lim_{n\longrightarrow\infty} d(x_n,S(t)x_n) = 0.$$

With the same proof, we only show that $\lim_{n\longrightarrow\infty} d(x_n,T(t)x_n) = 0$, without loss of generality, as in [21] we can assume that

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} \frac{d(x_n,T(t_n)x_n)}{t_n} = 0.$$

$$d(x_n, T(t)x_n) \leq \sum_{k=0}^{[\frac{t}{t_n}]-1} d(T((k+1)t_n)x_n, T(kt_n)x_n) + d(T([\frac{t}{t_n}]t_n)x_n, T(t)x_n)$$
$$\leq [\frac{t}{t_n}]d(T(t_n)x_n, x_n) + d(T(t - [\frac{t}{t_n}]t_n)x_n, x_n)$$
$$\leq \frac{t}{t_n}d(T(t_n)x_n, x_n) + max\{d(T(s)x_n, x_n) : 0 \leq s \leq t_n\},$$

Now by continuity of the mapping $t \longrightarrow T(t)x, x \in D$ and $\lim_{n\longrightarrow\infty} d(x_n, T(t_n)x_n) = 0$, we obtain that $\lim_{n\longrightarrow\infty} d(x_n, T(t)x_n) = 0$. □

Theorem 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ and $\mathcal{S} := \{S(t) : t \geq 0\}$ be two nonexpansive semigroups. Assume that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be sequence generated the following algorithm:*

$$y_n = (1 - a_n)x_n \oplus a_nT(t_n)x_n,$$
$$x_{n+1} = (1 - b_n)x_n \oplus b_nS(t_n)y_n,$$

where $\{a_n\}$,$\{b_n\}$ and $\{t_n\}$ satisfy the following conditions:

(i) $a_n, b_n \in [a, b] \subset (0, 1)$

(ii) $t_n > 0$, $\liminf_{n\longrightarrow\infty} t_n = 0$, $\limsup_{n\longrightarrow\infty} t_n > 0$, $\lim_{n\longrightarrow\infty} t_{n+1} - t_n = 0$.

Then the sequence $\{x_n\}$, $\triangle$-converges to an element of $\mathcal{F}$.

Proof. It follows from Theorem 3.1 that

$$\lim_{n\longrightarrow\infty} d(T(t)x_n, x_n) = \lim_{n\longrightarrow\infty} d(S(t)x_n, x_n) = 0$$

for each $t > 0$. Now we let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset \mathcal{F}$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_n v_n = v \in D$. We show that $v \in \mathcal{F}$, indeed

$$d(v_n, T(t)v) \leq d(v_n, T(t)v_n) + d(T(t)v_n, T(t)v)$$
$$\leq d(v_n, T(t)v_n) + d(v_n, v)$$

hence

$$\limsup_{n\longrightarrow\infty} d(v_n, T(t)v) \leq \limsup_{n\longrightarrow\infty} d(v_n, v).$$

By uniqueness of the asymptotic center we obtain that $T(t)v = v$ for all $t > 0$ and hence $v \in F(\mathcal{T})$. Similarly we obtain $v \in F(\mathcal{S})$ and hence $v \in \mathcal{F}$. By Theorem 3.1 the limit $\lim_{n\longrightarrow\infty} d(x_n, v)$ exists . Hence by Lemma 2.3, $u = v \in \mathcal{F}$. This shows that $W_w(x_n) \subset \mathcal{F}$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset \mathcal{F}$ and $d(x_n, v)$ converges, by Lemma 2.3 we have $x = u$. □

Theorem 3.3. *Let D be a nonempty compact convex subset of a complete CAT(0) space X. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ and $\mathcal{S} := \{S(t) : t \geq 0\}$ be two nonexpansive semigroups. Assume that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be sequence generated the following algorithm:*

$$y_n = (1 - a_n)x_n \oplus a_nT(t_n)x_n,$$

$$x_{n+1} = (1-b_n)x_n \oplus b_n S(t_n) y_n,$$

where $\{a_n\}$,$\{b_n\}$ *and* $\{t_n\}$ *satisfy the following conditions:*

(i) $a_n, b_n \in [a,b] \subset (0,1)$

(ii) $t_n > 0$, $\liminf_{n\longrightarrow\infty} t_n = 0$, $\limsup_{n\longrightarrow\infty} t_n > 0$, $\lim_{n\longrightarrow\infty} t_{n+1} - t_n = 0$.

Then the sequence $\{x_n\}$*, converges strongly to an element of* $\mathcal{F}$*.*

Proof. It follows from Theorem 3.1 that

$$\lim_{n\longrightarrow\infty} d(T(t)x_n, x_n) = \lim_{n\longrightarrow\infty} d(S(t)x_n, x_n) = 0$$

for each $t > 0$. By compactness of D, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \longrightarrow w$. We shall show that $w \in \mathcal{F}$, indeed for all $t > 0$ we have

$$\begin{aligned} d(w, T(t)w) &\leq d(w, x_{n_i}) + d(x_{n_i}, T(t)x_{n_i}) + d(T(t)x_{n_i}, T(t)w) \\ &\leq 2d(w, x_{n_i}) + d(x_{n_i}, T(t)x_{n_i}) \longrightarrow 0 \quad n \longrightarrow \infty, \end{aligned}$$

hence we have $w = T(t)w$, i.e., $w \in F(\mathcal{T})$. Similarly we have

$$\begin{aligned} d(w, S(t)w) &\leq d(w, x_{n_i}) + d(x_{n_i}, S(t)x_{n_i}) + d(S(t)x_{n_i}, S(t)w) \\ &\leq 2d(w, x_{n_i}) + d(x_{n_i}, S(t)x_{n_i}) \longrightarrow 0 \quad n \longrightarrow \infty, \end{aligned}$$

thus $w \in F(\mathcal{S})$, and hence $w \in \mathcal{F}$. Since $\lim_{n\longrightarrow\infty} d(x_n, w)$ exists we obtain the result. □

Definition 3.4. Let D be a nonempty closed convex subset of a CAT(0) space X and $T_n : D \longrightarrow D$, where $n \in \mathbb{N}$. Then the family $\{T_n\}$ is called uniformly asymptotically regular on D, if for all $i \in \mathbb{N}$ and any bounded subset K of D we have

$$\lim_{n\longrightarrow\infty} \sup_{x\in K} d(T_i(T_n x), T_n x) = 0$$

Theorem 3.5. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_n : D \longrightarrow D$ be uniformly asymptotically regular and nonexpansive mappings such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be sequence generated the following algorithm:*

$$y_n = (1-a_n)x_n \oplus a_n T_n x_n,$$
$$x_{n+1} = (1-b_n)x_n \oplus b_n T_n y_n,$$

where $a_n, b_n \in [a,b] \subset (0,1)$. Then the sequence $\{x_n\}$, $\triangle$-converges to an element of $\mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. Then by Lemma 2.4 we have

$$\begin{aligned} d(y_n, p) &\leq d((1-a_n)x_n \oplus a_n T_n x_n, p) \\ &\leq (1-a_n)d(x_n, p) + a_n d(T_n x_n, p) \\ &\leq (1-a_n)d(x_n, p) + a_n d(x_n, p) = d(x_n, p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &\leq d((1-b_n)x_n \oplus b_n T_n y_n, p) \\ &\leq (1-b_n)d(x_n, p) + b_n d(T_n y_n, p) \\ &\leq (1-b_n)d(x_n, p) + b_n d(y_n, p) \leq d(x_n, p). \end{aligned}$$

Hence we have $d(x_{n+1},p) \le d(x_n,p)$, this shows that the limit $\lim_{n\longrightarrow\infty} d(x_n,p)$ exists. Applying Lemma 2.4 we have

$$\begin{aligned} d(y_n,p)^2 &\le d((1-a_n)x_n \oplus a_nT_nx_n,p)^2 \\ &\le (1-a_n)d(x_n,p)^2 + a_nd(T_nx_n,p)^2 - a_n(1-a_n)d(x_n,T_nx_n)^2 \\ &\le (1-a_n)d(x_n,p) + a_nd(x_n,p) - a_n(1-a_n)d(x_n,T_nx_n)^2 \\ &= d(x_n,p) - a_n(1-a_n)d(x_n,T_nx_n)^2 \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1},p)^2 &\le d((1-b_n)x_n \oplus b_nT_ny_n,p)^2 \\ &\le (1-b_n)d(x_n,p)^2 + b_nd(T_ny_n,p)^2 - b_n(1-b_n)d(x_n,T_ny_n)^2 \\ &\le (1-b_n)d(x_n,p)^2 + b_nd(y_n,p)^2 - b_n(1-b_n)d(x_n,T_ny_n)^2 \\ &\le d(x_n,p)^2 - b_n(1-b_n)d(x_n,T_ny_n)^2 - b_na_n(1-a_n)d(x_n,T_nx_n)^2. \end{aligned}$$

So we have

$$\begin{aligned} \sum_{n=1}^{\infty} a^2(1-b)d(x_n,T_nx_n)^2 &\le \sum_{n=1}^{\infty} a_nb_n(1-a_n)d(x_n,T_nx_n)^2 \\ &\le \sum_{n=1}^{\infty} d(x_n,p)^2 - d(x_{n+1},p)^2 < d(x_1,p)^2 < \infty \end{aligned}$$

this implies that

$$\lim_{n\longrightarrow\infty} d(x_n,T_nx_n) = 0.$$

Similarly we obtain

$$\lim_{n\longrightarrow\infty} d(x_n,T_ny_n) = 0.$$

Also we have $d(x_{n+1},x_n) \le b_nd(x_n,T_ny_n) \longrightarrow 0$ and thus

$$d(x_{n+1},T_nx_n) \le d(x_{n+1},x_n) + d(x_n,T_nx_n) \longrightarrow 0.$$

Now, by our assumption, for each $i \in \mathbb{N}$ we have

$$\begin{aligned} d(x_{n+1},T_ix_{n+1}) &\le d(x_{n+1},T_nx_n) + d(T_nx_n,T_i(T_nx_n)) + d(T_i(T_nx_n),T_ix_{n+1}) \\ &\le 2\,d(x_{n+1},T_nx_n) + \sup_{u\in\{x_n\}} d(T_nu,T_i(T_nu)) \longrightarrow 0 \quad n \longrightarrow \infty. \end{aligned}$$

We let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset \mathcal{F}$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_n v_n = v \in D$. By Lemma 2.5 we have $v \in \mathcal{F}$, and hence the limit $\lim_{n\longrightarrow\infty} d(x_n,v)$ exists . Hence by Lemma 2.3, $u = v \in \mathcal{F}$. This shows that $W_w(x_n) \subset \mathcal{F}$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset \mathcal{F}$ and $d(x_n,v)$ converges, by Lemma 2.3 we have $x = u$. □

Remark: Theorem 3.5, improves and extends Theorem 3.4 in [9], indeed, we remove an extra condition in Theorem 3.4 in [9].

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