A NOTE ON BIRDCAGE GROUPS

by Marius Tărnăuceanu[1]

In this note, we give a simpler proof of the classification theorem of birdcage groups. We also classify minimal non-birdcage groups.

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1. Introduction

A *birdcage group* is a finite group whose Hasse diagram is an unbranched graph. The starting point for our discussion is given by the paper [10], where these groups have been classified.

Theorem 1.1. *A birdcage group is either a cyclic group C_{p^n} of prime power order or a semidirect product group $C_p \rtimes C_q$ of two cyclic groups of prime orders (possibly $p = q$, in which case this is a direct product group $C_p \times C_q$).*

We first provide a simpler proof of this theorem. It is based on the classification of finite minimal non-cyclic groups, that is finite non-cyclic groups all of whose proper subgroups are cyclic.

Theorem A ([4]). *A finite group G is a minimal non-cyclic group if and only if it is isomorphic to one of the following groups:*

a) $C_p^2 = C_p \times C_p$, *where p is a prime;*
b) Q_8;
c) $\langle a, b \mid a^p = b^{q^m} = 1, b^{-1}ab = a^r\rangle$, *where p and q are distinct primes and $r \not\equiv 1 \pmod p$, $r^q \equiv 1 \pmod p$.*

We note that birdcage groups are particular planar groups, that is finite group with planar Hasse diagram. Such groups have been studied by Starr and Turner [8], Bohanon and Reid [1], and Schmidt [7]. We also note that a classification theorem of iterated birdcage groups, that is groups obtained by iterated application of group extensions to birdcage groups, cannot be done. For this, it suffices to observe that any finite solvable group is an iterated birdcage group since it has a composition series with cyclic factors of prime order.

Our second goal is to describe finite minimal non-birdcage groups, that is finite non-birdcage groups all of whose proper subgroups are birdcage groups.

Theorem 1.2. *A finite group G is a minimal non-birdcage group if and only if it is isomorphic to one of the following groups:*

a) C_p^3, *where p is a prime,*
b) He_p - *the Heisenberg group of order p^3, where p is a prime,*

[1]Professor, Alexandru Ioan Cuza University of Iaşi, Romania, e-mail: tarnauc@uaic.ro

c) $C_p^2 \rtimes C_q$, *where p and q are distinct primes,*
d) C_{p^2q}, C_{pqr}, *where p, q and r are distinct primes,*
e) $C_p \times C_{pq}$, *where p and q are distinct primes,*
f) Q_8, D_8, $C_p \times C_{p^2}$, $M(p^3)$, *where p is a prime and* $p \neq 2$ *for* $M(p^3)$,
g) $(C_q \rtimes C_p) \times C_r$, *where p, q and r are distinct primes,*
h) $C_q \rtimes C_{p^2}$, *where p and q are distinct primes,*
i) $C_q \rtimes C_p^2$, *where p and q are distinct primes,*
j) $C_{p^2} \rtimes C_q$, *where p and q are distinct primes.*

We remark that A_4 is a group of type c), $S_3 \times C_5$ is a group of type g), Dic_3 is a group of type h), and D_{20} and D_{18} are groups of types i) and j), respectively. Also, we remark that all groups in Theorem 1.2 are iterated birdcage groups and solvable.

For the proof of Theorem 1.2 we need to know the structure of finite groups with all non-trivial elements of prime order (CP_1-groups, in short) and the structure of finite groups containing a cyclic maximal subgroup. These are described in the next two theorems.

Theorem B ([3, 2]). *Let G be a* CP_1*-group. Then:*

a) *G is nilpotent if and only if G is a p-group of exponent p.*
b) *G is solvable and non-nilpotent if and only if G is a Frobenius group with kernel* $P \in Syl_p(G)$, *with P a p-group of exponent p and complement* $Q \in Syl_q(G)$, *with* $|Q| = q$. *Moreover, if* $|G| = p^n q$ *then G has a chief series* $G = G_0 > P = G_1 > G_2 > \cdots > G_k > G_{k+1} = 1$ *such that for every* $1 \leq i \leq k$ *one has* $G_i/G_{i+1} \leq Z(G/G_{i+1})$, *Q acts irreducibly on* G_i/G_{i+1} *and* $|G_i/G_{i+1}| = p^b$, *where b is the exponent of p* $(\text{mod } q)$.
c) *G is non-solvable if and only if* $G \cong A_5$.

Theorem C ([5]). *A finite group G contains a cyclic maximal subgroup if and only if it is of one of the following types:*

a) $G = P \times K_1$, *where* K_1 *is an arbitrary finite cyclic group and P is a Sylow p-subgroup of G containing a cyclic maximal subgroup.*
b) $G = (K_2 \rtimes P) \times K_1$, *where* K_1 *is an arbitrary finite cyclic group which is a Hall subgroup of G, P is a Sylow p-subgroup of G containing a cyclic maximal subgroup,* $K_2 \rtimes P$ *is non-nilpotent, and the centralizer* $C_P(K_2)$ *is a maximal cyclic subgroup of P.*
c) $G = (P \rtimes K_2) \times K_1$, *where* K_1 *is an arbitrary finite cyclic group which is a Hall subgroup of G and* $G_1 = P \rtimes K_2$ *is a non-nilpotent group satisfying the following conditions: P is a SyIow p-subgroup of* G_1, $C_P(K_2) \supseteq \Phi(P)$, *and* $C_P(K_2)$ *is a cyclic invariant subgroup of* G_1 *such that* $K_2C_P(K_2)/C_P(K_2)$ *is a maximal subgroup of* $G_1/C_P(K_2)$.

The particular case of finite p-groups containing a cyclic maximal subgroup is exhaustively treated in Theorem 4.1 of Suzuki's monograph [9], volume II.

Theorem D. *A finite p-group G of order* p^n, $n \geq 3$, *contains a cyclic maximal subgroup if and only if one of the following conditions holds:*

a) *G is abelian of type* C_{p^n} *or* $C_p \times C_{p^{n-1}}$;
b) *G is non-abelian and isomorphic to the modular group*
   – $M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ y^{-1}xy = x^{p^{n-2}+1} \rangle$

   when p is odd, or to
   – $M(2^n)$ $(n \geq 4)$,
   – *the dihedral group*

$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle,$$

– *the generalized quaternion group*

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1,\ yxy^{-1} = x^{2^{n-1}-1} \rangle,$$

– *the quasi-dihedral group*

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{n-2}-1} \rangle \ (n \geq 4)$$

when $p = 2$.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [9]. For subgroup lattice concepts we refer the reader to [6].

2. Proofs of the main results

Proof of Theorem 1.1. Let G be a birdcage group. Then all maximal subgroups of G are cyclic of prime power order. It follows that G is either cyclic or a minimal non-cyclic group. In the first case we get $G \cong C_{p^n}$ or $G \cong C_{pq}$, where p, q are distinct primes and n is a positive integer. By using Theorem A, in the second case we get:

a) $G \cong C_p^2$ - is a birdcage group,

b) $G \cong Q_8$ - is not a birdcage group,

c) $G \cong \langle a, b \mid a^p = b^{q^m} = 1, b^{-1}ab = a^r \rangle$ - is a birdcage group if and only if $n = 1$ since it contains cyclic maximal subgroups of order pq^{n-1}; thus we have the birdcage group $C_p \rtimes C_q$.

Hence the birdcage groups are of type C_{p^n}, C_{pq}, C_p^2 or $C_p \rtimes C_q$, as desired. $\square$

Proof of Theorem 1.2. Let G be a minimal non-birdcage group. Then all maximal subgroups of G are of type C_{p^n}, C_{pq}, C_p^2 or $C_p \rtimes C_q$ by Theorem 1.1. We distinguish the following two cases:

Case 1. G has no cyclic maximal subgroups

It follows that the maximal subgroups of G are of type C_p^2 or $C_p \rtimes C_q$ and therefore G is a CP_1-group. Then Theorem B shows that:

a) If G is nilpotent, then it is a p-group of exponent p; we get
$$G \cong C_p^3 \text{ or } G \cong \mathrm{He}_p,$$
the groups a) and b) in Theorem 1.2.;

b) If G is solvable and non-nilpotent, then it is a nontrivial semidirect product of a p-group P of exponent p and a cyclic group Q; since P is a birdcage group, we obtain $P \cong C_p^2$, implying that
$$G \cong C_p^2 \rtimes C_q,$$
the group c) in Theorem 1.2.;

c) If G is non-solvable, then $G \cong A_5$, which is not a minimal non-birdcage group.

Case 2. G has cyclic maximal subgroups

It follows that G is one of the groups in Theorem C.

Assume first that $G = P \times K_1$, where P and K_1 are as in item a) of Theorem C, and let $|P| = p^m$. Since P is a birdcage group, we have $m \leq 3$. For $m = 0$, we get $K_1 \in \{C_{q^2r}, C_{qrs}\}$, where q, r and s are distinct primes. For $m \geq 1$, we infer that G has a cyclic maximal subgroup of order $p^{m-1}|K_1|$. Since this must be of type C_{p^n} or C_{pq}, we have the next possibilities:

- $m = 1$, that is $P = C_p$, and $K_1 \in \{C_{q^2}, C_{qr}\}$,
- $m = 2$, that is $P = C_{p^2}$ or $P = C_p^2$, and $K_1 = C_q$,
- $m = 3$, that is $P \in \{Q_8, D_8, C_p \times C_{p^2}, M(p^3)\}$, and $K_1 = 1$,

where $q, r \neq p$ are distinct primes and $p \neq 2$ for $P = M(p^3)$. These lead to the groups d), e) and f) in Theorem 1.2.

Assume now that $G = (K_2 \rtimes P) \times K_1$, where P, K_1 and K_2 are as in item b) of Theorem C.

If $K_1 \neq 1$, then G has a proper non-nilpotent subgroup which is isomorphic to $K_2 \rtimes P$. Since this is contained in a maximal subgroup of G, it must be of type $C_q \rtimes C_p$ and we get
$$G \cong (C_q \rtimes C_p) \times C_r,$$
the group g) in Theorem 1.2 (again, p, q and r are distinct primes).

If $K_1 = 1$, then $G = K_2 \rtimes P$ is non-nilpotent. From Theorem C, we also know that P has a cyclic maximal subgroup P_1. Then K_2P_1 is a maximal subgroup of G and so either $K_2P_1 \cong C_{pq}$ or $K_2P_1 \cong C_q \rtimes C_p$. These lead to
$$G \cong C_q \rtimes C_{p^2} \text{ or } G \cong C_q \rtimes C_p^2,$$
i.e. to the groups h) and i) in Theorem 1.2.

Finally, assume that $G = (P \rtimes K_2) \times K_1$ with P, K_1 and K_2 as in item c) of Theorem C.

If $K_1 \neq 1$, then G has a proper non-nilpotent subgroup isomorphic to $P \rtimes K_2$. Again, this must be of type $C_p \rtimes C_q$ and we get
$$G \cong (C_p \rtimes C_q) \times C_r,$$
a group of type g) in Theorem 1.2.

If $K_1 = 1$, then $G = P \rtimes K_2$ is non-nilpotent and P is a Sylow p-subgroup of G. Since P is contained in a maximal subgroup of G, we infer that it must be of type C_p^2 or C_{p^n}. If $P \cong C_p^2$, it follows that $q = |K_2|$ is a prime and we obtain

$$G \cong C_p^2 \rtimes C_q,$$

a group of type c) in Theorem 1.2. If $P \cong C_{p^n}$, it follows that $P_1 = \Phi(P)$ is the unique subgroup of order p^{n-1} in P, implying that $C_P(K_2) = P_1$. Then P_1K_2 is a maximal subgroup of G of order $p^{n-1}|K_2|$, and we find that $n = 2$ and $q = |K_2|$ is a prime. Consequently,

$$G \cong C_{p^2} \rtimes C_q \text{ or } G \cong C_p^2 \rtimes C_q,$$

the groups j) and c) in Theorem 1.2.

This completes the proof. □

3. Conclusions and further research

All previous results show that the study of groups whose Hasse diagram satisfies a certain property is an interesting aspect of subgroup lattice theory. Clearly, it can successfully be extended to other significant lattices/posets associated to a group. This will surely constitute the subject of some further research.

Finally, we formulate an open problem related to the above results.

Open problem. Study the finite groups whose Hasse diagram of normal subgroups is an unbranched graph[1].

We note that this class of groups includes groups whose normal subgroup lattices are chains and, in particular, simple groups.

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[1]A suitable name for these groups would be *normal birdcage groups*.

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