BANACH ALGEBRAS VALUED λ – ADDITIVE MEASURES

Ion Chiţescu[1], Alice Dagmar Anghelescu[2], Ştefania Constantinescu[3] and Roxana - Maria Vîlcescu[4]

We extend the concept of λ–additivity for measures which take their values in commutative unital Banach algebras. Among other facts, an important idea is to see that (like in the positive case) such measures can be "representable" sometimes, i.e. can be obtained from additive measures via a suitable functional composition. We use this in order to obtain a non trivial example and an extension of Lyapounov's convexity theorem.

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1. Introduction

Classical measure theory deals with additive (or, which is more, with countably additive) measures. Recently, it was seen that more varied tools, other than additive measures, are necessary in order to describe and study a multitude of phenomena. These tools are the *(positive) generalized measures* which are montone and possibly non additive. Superadditive measures indicate a cooperative action or synergy between the measured items (sets), whilst subadditive measures indicate inhibitory effects, lack of cooperation or incompatibilty between the measured items (sets). Additive measures can express non interaction or indifference.

The history of non additive measures is short. It began with [8]. We think that the decisive step in the development of the theory of non additive measures was Dr. Eng. dissertation [17] from 1974 of the Japanese scholar M. Sugeno. In [17] basic facts concerning generalized measures (called there "fuzzy measures") and non linear integrals (called there "fuzzy integrals") were introduced. As asserted in the Introduction of [17], the main goal was to study uncertainties. One must add that, nowadays, many authors call the fuzzy measures and fuzzy integrals simply generalized measures and generalized integrals.

It is widely accepted that the most important generalized measures are the λ*–measures* (more precisely the λ*– additive measures*) introduced in the same work [17]. An important result primarily due to Z. Wang (see [20]) asserts that any λ–measure can be obtained from a classical measure via a canonical procedure – composition with a special increasing function (see also [21], [22], [2], [15]). We called the generalized measures which can be obtained in such a way *representable measures* (see [5]). M. Sugeno uses the term "distorted measures" for the representable measures on intervals (see [18] and [19]). Special types of λ– measure

[1]Professor, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: ionchitescu@yahoo.com
[2]Master Student, Faculty of Mathematics and Informatics, University of Bucharest, Romania, e-mail: alicee_dag@yahoo.com
[3]Teaching Assistant, Technical University of Civil Engineering of Bucharest, Romania, e-mail: stefania.constantinescu@utcb.ro
[4]PhD Student, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: roxam86@yahoo.com

were studied in [6] and [7]. Other types appear in the mathematical theory of evidence, created by G. Shafer (belief and possibility measures, see [16] and [21]). There are a lot of domains where generalized measures (in particular λ – measures) appear, see e.g. the joint monograph [12].

As it is well known, classical measure and integration theory was substantially generalized starting with the first half of the 20th century when vector measures and integrals were introduced and studied. The present paper is written following this line of thinking.

Namely, *we pass from positive valued λ–additive measures to vector valued λ–additive measures*, trying to extend the facts and methods from the positive case to the vector case. More precisely, we naturally extend the concept of λ–additivity to measures which take their values in Banach algebras.

A brief survey of the content follows.

After the present Introduction, the paper continues with a section containing necessary "Preliminary Facts". The main section ("Results") is divided into four subsections. The first one contains some basic algebraic computations. The second one establishes a representability – type correspondence between additive and λ–additive measures, whilst the third one establishes a similar correspondence between σ–additive and $\sigma - \lambda$–additive measures. In the fourth subsection we exhibit a non trivial example of $\sigma - \lambda$–additive vector measure and a logarithmic convexity result.

For classical measure theory, see [13]. For vector measures theory, see [9], [10] and [11]. For generalized measure theory, see [21] and [22]. For Banach algebras, see [1] and [4].

2. Preliminary Facts

We shall work with $\mathbb{C}$ = the field of complex numbers. Then $\mathbb{C} \supset \mathbb{R}_+ = [0, \infty) \supset \mathbb{N} = \{1, 2, \ldots, n, \ldots\}$. All sequences will be indexed with $\mathbb{N}$ and $(x_n)_n \subset A$ means that $x_n \in A$ for any $n \in \mathbb{N}$ (if A is a non empty set).

Let A, B, C be non empty sets and $A_1 \subset A, B_1 \subset B$ be non empty subsets. Assume that fhe functions $f : A_1 \to B$ and $g : B_1 \to C$ are such that $f(A_1) \subset B_1$. Then the (generalized) composition $g \circ f : A_1 \to C$ is defined via $(g \circ f)(x) \overset{def}{=} g(f(x))$ for any $x \in A_1$ (we shall use it in the sequel).

For any non empty set T, let $\mathcal{P}(T)$ be the set of all subsets of T and let $\mathcal{C} \subset \mathcal{P}(T)$ be a ring of sets. Then, if X is a normed space or $X = \mathbb{R}_+$, a function $m : \mathcal{C} \to X$ having the property $m(\emptyset) = 0$ will be called a *generalized measure*. A generalized measure $m : \mathcal{C} \to X$ having the property that $m(A \cup B) = m(A) + m(B)$, whenever $A \cap B = \emptyset$, will be called an *additive measure*. It is equivalent for m to be *finitely additive*, i.e. $m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i)$, whenever $A_1, A_2, \ldots, A_n$ in $\mathcal{C}$ are mutually disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$). The generalized measure $m : \mathcal{C} \to X$ is called *σ–additive (countably additive)* in case $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ (convergence in X), whenever the sequence $(A_i)_i \subset \mathcal{C}$ is disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$. If m is σ–additive, then m is additive, the converse being false.

If $m : \mathcal{C} \to X$ is a generalized measure, we say that a set $A \in \mathcal{C}$ is an *atom of m* if $m(A) \neq 0$ and one has either $m(B) = 0$ or $m(B) = m(A)$ for any $\mathcal{C} \ni B \subset A$. In case there does not exist any atom for m, we say that m is *non atomic*.

Let X be a commutative and unital Banach algebra over $\mathbb{C}$. The norm of $x \in X$ is $\|x\|$ and the unit of X is u. The invertible elements of X form the set $G(X)$ (of course $u \in G(X)$).

For any $0 < M < \infty$, let

$$B_M = \left\{ x \in X \middle| \|x\| < M \right\}.$$

Then, we define

$$B = \left\{ x \in X \middle| x - u \in B_1 \right\} = \left\{ x \in X \middle| \|x - u\| < 1 \right\}$$

and notice that $B \subset G(X)$.

For any $x \in X$, the *spectrum of* x is the set

$$\mathrm{Sp}(x) = \left\{ \lambda \in \mathbb{C} \middle| \lambda u - x \notin G(X) \right\}$$

and we know that $\mathrm{Sp}(x)$ is a non empty compact subset of $\left\{ \lambda \in \mathbb{C} \middle| |\lambda| \leq \|x\| \right\}$. The exponential function $\exp : X \to X$ is defined via

$$\exp(x) = u + \sum_{n=1}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

(the series converges absolutely).

We have $\exp(x+y) = \exp(x)\exp(y)$ for any x, y in X. Also $\exp(X) \subset$ $\subset G(X)$ and $\exp(x)^{-1} = \exp(-x)$ for any $x \in X$.

Consider the open half plane $D = \left\{ z = a + ib \in \mathbb{C} \middle| a > 0 \right\}$. One knows that, for any $y \in X$ such that $\mathrm{Sp}(y) \subset D$, there exists $x \in X$ such that $y = \exp(x)$ (one uses the functional calculus in Banach algebras). In particular, one can prove that for any $y \in B$ one has $\mathrm{Sp}(y) \subset D$, hence we can find $x \in X$ such that $\exp(x) = y$. In this case, call $x = \log(y)$. We defined the analytical function $\log : B \to X$ and we have the formula (for y as above)

$$\log(y) = -\sum_{n=1}^{\infty} \frac{1}{n} (u - y)^n.$$

We retain the fact that, for any $y \in B$, one has $\exp(\log(y)) = y$, consequently $\exp(X) \supset B$.

It is natural to introduce the set

$$\Delta = \left\{ \log(y) \middle| y \in B \right\}.$$

Previous facts show that exp is injective on Δ. For any $y \in B$ one has $\exp(\log(y)) = y$, hence $\log(\exp(\log(y))) = \log y$, i.e. $\log(\exp(x)) = x$ for any $x = \log y \in \Delta$. This enables us to introduce the functions (use restrictions and corestrictions)

$$\mathrm{Exp} : \Delta \to B, \mathrm{Exp}(x) \stackrel{def}{=} \exp(x)$$

$$\mathrm{Log} : B \to \Delta, \mathrm{Log}(y) \stackrel{def}{=} \log(y)$$

and Log=Exp^{-1}, Exp=Log^{-1}.

Notice also that, for any $y_1, y_2, \ldots, y_n$ in B such that $y_1 y_2 \ldots y_n \in B$, one has

$$\mathrm{Log}(y_1 y_2 \ldots y_n) = \mathrm{Log}(y_1) + \mathrm{Log}(y_2) + \ldots + \mathrm{Log}(y_n).$$

For any $a \in B$ and any $t \in \mathbb{C}$, we shall introduce a^t as follows.
Let $b = \log(a)$. Then $a^t \stackrel{def}{=} \exp(tb)$.

A set $A \subset B$ will be called *logarithmically convex* if $x^{1-t} y^t \in A$, whenever x, y are in A and $t \in [0,1]$.

The most popular commutative unital algebra is

$$C(T) = \{ f : T \to \mathbb{C} | f \text{ is continuous} \}.$$

Here T is a compact Hausdorff space, $C(T)$ is equipped with the natural operations, the unit $u \in C(T)$ is the constant function equal to 1 everywhere and the norm of $f \in C(T)$ is $\|f\| = \sup \left\{ |f(t)| \middle| t \in T \right\}$. In case $T = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, with the discrete topology, we have $C(T) = \mathbb{C}^n$ (any norm on $\mathbb{C}^n$ gives the same topology, namely the topology of $C(T)$).

3. Results

Throughout this section, X will be a commutative unital Banach algebra.

3.1. Algebraic Considerations. Let λ and c be non null complex numbers. (c is deliberately not mentioned). First we define $h_\lambda : X \to X$ via

$$h_\lambda(x) = \frac{1}{\lambda}(\exp(c\lambda x) - u). \qquad (\mathbf{3}.1)$$

It is seen that, in case $x \in X$ is such that $c\lambda x \in \Delta$, one has $\|h_\lambda(x)\| < \frac{1}{|\lambda|}$, because $\exp(c\lambda x) = \exp(\log(y)) = y$ for some $y \in B$, hence $h_\lambda\left(\frac{1}{c\lambda}\Delta\right) \subset B_{1/|\lambda|}$.

Consequently, we have the injection (use that exp is injective on Δ)

$$H_\lambda : \frac{1}{c\lambda}\Delta \to B_{1/|\lambda|}, H_\lambda(x) = h_\lambda(x). \qquad (\mathbf{3}.2)$$

Now, we introduce the set

$$E(\lambda) = \left\{ u + \lambda y \middle| \|y\| < \frac{1}{|\lambda|} \right\} = \left\{ u + \lambda y \middle| y \in B_{1/|\lambda|} \right\},$$

hence $E(\lambda) \subset B$.

This enables us to define $\theta_\lambda : B_{1/|\lambda|} \to X$ via

$$\theta_\lambda(y) = \frac{1}{c\lambda}\log(u + \lambda y) \qquad (\mathbf{3}.3)$$

because $u + \lambda y \in E(\lambda)$.

In this case, one has $\log(u + \lambda y) \in \Delta$; it follows that $\theta_\lambda\left(B_{1/|\lambda|}\right) \subset$ $\subset \frac{1}{c\lambda}\Delta$ and we have the function

$$\Theta_\lambda : B_{1/|\lambda|} \to \frac{1}{c\lambda}\Delta, \Theta_\lambda(y) = \theta_\lambda(y). \qquad (\mathbf{3}.4)$$

Taking into account ($\mathbf{3}$.2) and ($\mathbf{3}$.4), we can see that $\Theta_\lambda, H_\lambda$ are bijections, $\Theta_\lambda = H_\lambda^{-1}$ and $H_\lambda = \Theta_\lambda^{-1}$.

3.2. Additivity and λ–Additivity in Correspondence. Let T be a non empty set and let $\mathcal{C} \subset \mathcal{P}(T)$ be a ring of sets. Let λ be a non null complex number.

Definition 3.1. *A generalized measure $\mu : \mathcal{C} \to X$ is called* λ–additive (satisfies the λ-rule) *if, for any A, B in $\mathcal{C}$ such that $A \cap B = \emptyset$ one has*

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B).$$

Remarks. *Of course, if we take $\lambda = 0$ into consideration, the 0–additive measures are exactly the (finitely) additive measure.*

Definition 3.2. *A generalized measure $\mu : \mathcal{C} \to X$ is called* finitely λ–additive *if, for any $A_1, A_2, \dots, A_n$ in $\mathcal{C}$ which are mutually disjoint, one has*

$$\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = \frac{1}{\lambda}\left(\prod_{i=1}^{n}(u + \lambda\mu(A_i)) - u\right)$$

(clearly, for $n = 2$, one has λ–additivity).

Actually, Definitions **3**.1 and **3**.2 are equivalent.

The first result shows how to pass from additivity to λ–additivity.

Theorem 3.1. *Let $m : \mathcal{C} \to X$ be an additive measure.*
*Then $\mu = h_\lambda \circ m : \mathcal{C} \to X$ is a λ–additive measure, for any $0 \neq c \in \mathbb{C}$. Using (**3**.1), recall that, for any $A \in \mathcal{C}$, one has*

$$\mu(A) = \frac{1}{\lambda}\left(\exp(c\lambda m(A)) - u\right).$$

Proof. Let $A, B \in \mathcal{C}, A \cap B = \emptyset$. Then

$$\mu(A \cup B) = \frac{1}{\lambda}\left(\exp(c\lambda m(A \cup B)) - u\right) = \frac{1}{\lambda}\left(\exp(c\lambda(m(A) + m(B))) - u\right) =$$

$$= \frac{1}{\lambda}\left(\exp(c\lambda m(A))\exp(c\lambda m(B) - u)\right).$$

At the same time:

$$\mu(A) + \mu(B) + \lambda\mu(A)\mu(B) = \frac{1}{\lambda}\left(\exp(c\lambda m(A)) - u\right) + \frac{1}{\lambda}\left(\exp(c\lambda m(B)) - u\right) +$$

$$+\frac{1}{\lambda}\left(\exp(c\lambda m(A)) - u\right)\left(\exp(c\lambda m(B)) - u\right) =$$

$$= \frac{1}{\lambda}\left(\exp(c\lambda m(A))\left(\exp(c\lambda m(B)) - u\right)\right) \text{ a.s.o.}$$

□

The next result will be in the opposite sense, namely we shall see how to pass from λ–additivity to additivity. A certain complication appears, because, for x, y in B on must have $xy \in B$ in order to be able to compute $\log(xy)$, as we shall see. For positive measures, this complication does not appear.

So, let $x = u + a, y = u + b$ in B (hence $\|a\| < 1, \|b\| < 1$) and let us see under which conditions one has $xy = u + a + b + ab \in B$, i.e. $\|a + b + ab\| < 1$. A sufficient condition would be $\|a\| + \|b\| + \|a\|\|b\| < 1$. Considering some $t > 0$ such that $\|a\| < t$ and $\|b\| < t$, it will be sufficient to have $t + t + t^2 \leq 1$, i.e. $0 < t \leq \sqrt{2} - 1$. Conclusion: if $x = u + a, y = u + b$, with $\|a\| < \sqrt{2} - 1, \|b\| < \sqrt{2} - 1$, then x, y and xy are in B.

Theorem 3.2. *Let $\mu : \mathcal{C} \to X$ be a λ–additive measure which is bounded and assume that*

$$M \stackrel{def}{=} \sup\left\{\|\mu(A)\| \,\middle|\, A \in \mathcal{C}\right\} < \frac{\sqrt{2} - 1}{|\lambda|}.$$

*Then $B_M \subset B_{1/|\lambda|}$ and the measure $m = \theta_\lambda \circ \mu : \mathcal{C} \to X$ (generalized composition) is additive, for any $0 \neq c \in \mathbb{C}$. Using (**3**.3), recall that, for any $A \in \mathcal{C}$, one has*

$$m(A) = \frac{1}{c\lambda}\log(u + \lambda\mu(A)).$$

Proof. We have $B_M \subset B_{1/|\lambda|}$, because $M < \frac{\sqrt{2}-1}{|\lambda|} < \frac{1}{|\lambda|}$.

Consequently, for any $A \in \mathcal{C}$, one has $\|\mu(A)\| \leq \frac{1}{|\lambda|}$ and $\|\lambda\mu(A)\| <$
$< \sqrt{2} - 1 < 1$, hence $u + \lambda\mu(A) \in B$. It follows that we can compute
$m(A) = (\theta_\lambda \circ \mu)(A) = \theta_\lambda(\mu)(A)) = \frac{1}{c\lambda}\log\left(u + \lambda\mu(A)\right) \in \frac{1}{c\lambda}\Delta$.

To prove the additivity of m, take A', B' in $\mathcal{C}$ such that $A' \cap B' = \emptyset$. Then

$$m(A' \cup B') = \frac{1}{c\lambda}\log\left(u + \lambda\mu(A' \cup B')\right) =$$

$$= \frac{1}{c\lambda}\log\left(u + \lambda\mu(A') + \lambda\mu(B') + \lambda^2\lambda\mu(A')\lambda\mu(B')\right) =$$

$$= \frac{1}{c\lambda}\log\left((u + \lambda\mu(A'))(u + \lambda\mu(B'))\right) =$$

$$= \frac{1}{c\lambda}\left(\log(u + \lambda\mu(A')) + \log(u + \lambda\mu(B'))\right) = m(A') + m(B').$$

We used the fact that $u+\lambda\mu(A'), u+\lambda\mu(B')$ and $(u+\lambda\mu(A'))\,(u+\lambda\mu(B'))$ are in B, because $|\lambda|\|\mu(A')\| < \sqrt{2}-1$ and $|\lambda|\|\mu(B')\| < \sqrt{2}-1$. □

Consequence 3.3. *Let $\mu : \mathcal{C} \to X$ be a λ–additive measure which is bounded and assume that*

$$M = \sup\left\{\|\mu(A)\| \middle| A \in \mathcal{C}\right\} < \frac{\sqrt{2}-1}{|\lambda|}.$$

Then $\mu = h_\lambda \circ m$, where m appears in Theorem **3**.*2.*

Proof. For any $A \in \mathcal{C}$, one has $\mu(A) \in B_M \subset B_{1/|\lambda|}$, hence $m(A) = \theta_\lambda(\mu(A)) \in \frac{1}{c\lambda}\Delta$ (as in the proof of Theorem **3**.2). This implies that $(h_\lambda \circ m)(A) =$
$= h_\lambda(\theta_\lambda(\mu(A))) = (H_\lambda \circ \Theta_\lambda)\,(\mu(A)) = \mu(A)$. □

Remark. *The preceding result can be interpreted as follows: in case a λ–additive measure μ is "flat enough", i.e.*

$$M = \sup\left\{\|\mu(A)\| \middle| A \in \mathcal{C}\right\} < \frac{\sqrt{2}-1}{|\lambda|},$$

then μ is "representable", i.e. there exists an additive measure m which "represents" μ, this meaning that $\mu = h_\lambda \circ m$.

3.3. **σ–Additivity and $\sigma - \lambda$–Additivity in Correspondence.** We begin with infinite products in Banach algebras. Our (ad–hoc) definitions will not involve any restrictions (recall that in the scalar case, the convergent infinite products must have non zero values...).

Definition 3.3. *Let $(x_n)_n \subset X$ be a sequence. We say that the infinite product $\prod\limits_{n=1}^{\infty} x_n$ is* convergent *if the sequence $(P_n)_n$, defined via $P_n = \prod\limits_{i=1}^{n} x_i$ is convergent (in X), with $\lim\limits_n P_n = P$. In this case, we write $P \overset{def}{=} \prod\limits_{n=1}^{\infty} x_n$ and we call P $\left(\text{or } \prod\limits_{n=1}^{\infty} x_n\right)$* the value of the infinite product.

Lemma 3.1. *Assume that the sequence $(x_n)_n$ has the following properties:*

a) $x_n \in B$ for any n.

b) $\prod\limits_{n=1}^{\infty} x_n$ is convergent and $\prod\limits_{n=1}^{\infty} x_n \in B$.

Then:

1) There exists n_0 such that $P_n = \prod\limits_{i=1}^{n} x_i \in B$ for any $n \geq n_0$ (because B is a neighbourhood of P).

2) $\log\left(\prod\limits_{n=1}^{\infty} x_n\right) = \sum\limits_{n=1}^{\infty} \log(x_n)$.

Proof. For any $n \geq n_0$, one has $\log(P_n) = \log\left(\prod\limits_{i=1}^{n} x_i\right) = \sum\limits_{i=1}^{n} \log(x_i)$. Then $\lim\limits_n \prod\limits_{i=1}^{n} x_i = \prod\limits_{i=1}^{\infty} x_i \in B$, hence (log is analytic): $\lim\limits_n \log\left(\prod\limits_{i=1}^{n} x_i\right) = \log\left(\prod\limits_{i=1}^{\infty} x_i\right)$, i.e. $\lim\limits_n \sum\limits_{i=1}^{n} \log(x_i) \overset{def}{=} \sum\limits_{n=1}^{\infty} \log(x_i) = \log\left(\prod\limits_{i=1}^{\infty} x_i\right)$.
□

In the sequel, we consider again a non empty set T, a ring $\mathcal{C} \subset \mathcal{P}(T)$ and a non null complex number λ.

Definition 3.4. *A generalized measure $\mu : \mathcal{C} \to X$ is called $\sigma - \lambda$*–additive (satisfies the $\sigma - \lambda$–rule) *if, for any disjoint sequence $(E_i)_i \subset \mathcal{C}$ such that $\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{C}$, one has*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(u + \lambda\mu(E_i)) - u\right).$$

The infinite product in the definition is assumed to be convergent.

Remarks. *1. (This is a definition) $\sigma - 0$–additivity means σ–additivity.*
2. If μ is $\sigma - \lambda$–additive, it follows that μ is λ–additive (the converse is false).

Theorem 3.4. *With the notations from Theorem* **3***.1 and within the same framework, let us assume that m is σ–additive.*

Then, for any $\lambda \neq 0$, it follows that $\mu = h_\lambda \circ m$ is $\sigma - \lambda$–additive.

Proof. Let $(E_i)_i \subset \mathcal{C}$ be a disjoint sequence with $\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{C}$. One must prove that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \stackrel{def}{=} (h_\lambda \circ m)\left(\bigcup_{i=1}^{\infty} E_i\right) =$$

$$= \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(u + \lambda\mu(E_i)) - u\right) = \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(u + \lambda\,(h_\lambda \circ m)\,(E_i)) - u\right).$$

Because $m\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \lim\limits_{n} \sum\limits_{i=1}^{n} m(E_i)$, we have

$$(h_\lambda \circ m)\left(\bigcup_{i=1}^{\infty} E_i\right) = h_\lambda\left(\lim_{n}\sum_{i=1}^{n} m(E_i)\right) = \lim_{n} h_\lambda\left(\sum_{i=1}^{n} m(E_i)\right) =$$

$$= \lim_{n}\frac{1}{\lambda}\left(\exp\left(c\lambda\sum_{i=1}^{n} m(E_i)\right) - u\right) = \lim_{n}\frac{1}{\lambda}\left(\prod_{i=1}^{n}\exp\left(c\lambda m(E_i)\right) - u\right) =$$

$$= \lim_{n}\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(u + \exp\left(c\lambda m(E_i)\right) - u\right) - u\right) =$$

$$= \lim_{n}\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(u + \lambda\,(h_\lambda \circ m)\,(E_i)\right) - u\right) = \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}\left(u + \lambda\,(h_\lambda \circ m)\,(E_i)\right) - u\right).$$

□

Theorem 3.5. *With the notation from Theorem* **3***.2 and within the same framework, assume that μ is $\sigma - \lambda$–additive.*

Then $m = \theta_\lambda \circ \mu$ is σ–additive and $\mu = h_\lambda \circ m$.

Proof. Let $(E_i)_i \subset \mathcal{C}$ be a disjoint sequence with $\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{C}$. Then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = (\theta_\lambda \circ \mu)\left(\bigcup_{i=1}^{\infty} E_i\right) = \theta_\lambda\left(\mu\left(\bigcup_{i=1}^{\infty} E_i\right)\right) =$$

$$= \frac{1}{c\lambda}\left(\log\left(u + \lambda\mu\left(\bigcup_{i=1}^{\infty} E_i\right)\right)\right) =$$

$$= \frac{1}{c\lambda}\left(\log\left(u + \prod_{i=1}^{\infty}(u + \lambda\mu(E_i)) - u\right)\right) = \frac{1}{c\lambda}\log\left(\prod_{i=1}^{\infty} u + \lambda\mu(E_i)\right).$$

The last equality is meaningful, because $\prod_{i=1}^{\infty}(u+\lambda\mu(E_i)) \in B$, in view of the fact that $\prod_{i=1}^{\infty}(u+\lambda\mu(E_i)) = u+\lambda\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \in B$ (see Theorem **3**.2).

According to Lemma **3**.1, one has $\log\left(\prod_{i=1}^{\infty}(u+\lambda\mu(E_i))\right) = \sum_{i=1}^{\infty}\log(u+\lambda\mu(E_i))$, because $u+\lambda\mu(E_i) \in B$ for any i.

We got finally:

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{1}{c\lambda}\sum_{i=1}^{\infty}\log\left(u+\lambda\mu(E_i)\right) = \sum_{i=1}^{\infty}\frac{1}{c\lambda}\log\left(u+\lambda\mu(E_i)\right) = \sum_{i=1}^{\infty} m(E_i).$$

The equality $\mu = h_\lambda \circ m$ follows from Consequence **3**.3. □

3.4. **Applications.** We shall deal with the commutative unital Banach algebra $X = C(T)$.

A. Take $T = [a,b]$ (where $-\infty < a < b < \infty$), $\mathcal{C}$ = the Lebesgue measurable sets of $[a,b]$ and let $\ell : \mathcal{C} \to \mathbb{R}_+$ be the Lebesgue measure on $[a,b]$.

For any $A \in \mathcal{C}$ we have the continuous function $W_A : [a,b] \to \mathbb{C}$, acting via $W_A(t) = \ell(A\cap[a,t])$ for any $t \in [a,b]$, hence $W_A \in X = C([a,b])$. The generalized measure $m : \mathcal{C} \to X$ given via $m(A) = W_A$ is σ–additive (direct verification). Then, for any non null complex numbers c and λ, we apply Theorem **3**.4, obtaining the non trivial example of $\sigma-\lambda$–additive measure $\mu = h_\lambda \circ m : \mathcal{C} \to X$. Namely, for any $A \in \mathcal{C}$, one has $\mu(A) = \frac{1}{\lambda}(\exp(c\lambda m(A)) - u)$. So, for any $A \in \mathcal{C}$ and $t \in [a,b]$, one has $\mu(A) = \frac{1}{\lambda}(\exp(c\lambda m(A))(t) - 1)$. Because, for any $f \in X$ which is real valued, for any λ and c real and for any $t \in [a,b]$, one has (using the action of exp in the real domain): $\exp(f)(t) =$
$= \exp(f(t)) = e^{f(t)}$ we get, for any $A \in \mathcal{C}$ and $t \in [a,b]$:

$$\mu(A)(t) = \frac{1}{\lambda}\left(e^{c\lambda\ell(A\cap[a,t])} - 1\right)$$

Remark. *Let us work for $T = [a,b] = [0,1], \lambda = c = 1$ and for real valued functions in $X = C(T)$. One has, for any $A \in \mathcal{C}$ and any $t \in [0,1]$*

$$\mu(A)(t) = e^{\ell(A\cap[0,t])} - 1 \geq \ell(A\cap[0,t])$$

hence

$$\|\mu(A)\| \geq \sup_{t\in[0,1]} \ell(A\cap[0,t]) = \ell(A).$$

This implies that

$$M = \sup\left\{\|\mu(A)\| \big| A \in \mathcal{C}\right\} \geq \ell([0,1]) = 1 > \frac{\sqrt{2}-1}{|\lambda|} = \sqrt{2}-1.$$

*Clearly μ is representable, being "represented" by m. So, this example shows that the condition $M < \frac{\sqrt{2}-1}{|\lambda|}$ in Theorem **3**.2 is only sufficient, not being necessary for the "representability" of μ.*

B. Now, take $T = \{1,2,\ldots,n\}$ with the discrete topology, hence we work in the commutative unital Banach algebra $X = \mathbb{C}^n$.

Theorem 3.6. *Consider a non null complex number λ, a non empty set Ω, a σ–algebra $\Sigma \subset \mathcal{P}(\Omega)$ and a $\sigma-\lambda$–additive measure $\mu : \Sigma \to \mathbb{C}^n$ such that (see Theorem **3**.2)*

$$M = \sup\left\{\|\mu(A)\| \big| A \in \Sigma\right\} < \frac{\sqrt{2}-1}{|\lambda|}.$$

Assume that μ is non atomic.

Then the range

$$\mu(\Sigma) = \{\mu(A) \big| A \in \Sigma\}$$

is compact and the set

$$E(\lambda, \mu) = \{u + \lambda\mu(A) \big| A \in \Sigma\}$$

is compact and logarithmically convex.

Proof. According to Theorem **3**.5, it follows that $\mu = h_\lambda \circ m$, where $m : \Sigma \to \mathbb{C}^n$ is the σ–additive measure given via $m = \theta_\lambda \circ \mu$. First, we notice that m is non atomic. Indeed, if there exists an atom A of m, then $m(A) \neq 0$ and, for any $\Sigma \ni B \subset A$, one has either $m(B) = 0$ or $m(B) = m(A)$. Then, either $\mu(B) = h_\lambda(m(B)) = h_\lambda(0) = 0$ or $\mu(B) = h_\lambda(m(B)) = h_\lambda(m(A)) = \mu(A)$. Besides, $\mu(A) \neq 0$. Indeed, h_λ is injective on $\theta_\lambda\left(B_{1/|\lambda|}\right) \supset \{\theta_\lambda(\mu(C)) \big| C \in \Sigma\}$ and $\mu(A) = 0$, i.e. $h_\lambda(m(A)) = 0$ would imply that $h_\lambda(m(A)) = h_\lambda(0)$, i.e. $m(A) = 0$, which is false. This completes the proof of the fact that A should be an atom of μ, impossible. Applying the famous convexity theorem of A. Lyapounov to the non atomic $m : \Sigma \to \mathbb{C}^n$ (see e.g. [9], [14], [3]), it follows that the range $m(\Sigma) = \{m(A) \big| A \in \Sigma\}$ is compact and convex. Because $\mu = h_\lambda \circ m$, it follows that $\mu(\Sigma) = h_\lambda(m(\Sigma))$ is compact.

Finally, we prove that the set $E(\lambda, \mu)$ is logarithmically convex (the fact that this set is compact is true because $\mu(\Sigma)$ is compact).

First, it is seen that $E(\lambda, \mu) \subset B$, because $\|\lambda\mu(C)\| < 1$ for any $C \in \Sigma$ (i.e. the enunciation is meaningful).

Next, take $U \in \Sigma, V \in \Sigma, t \in [0, 1]$ and let us find $D \in \Sigma$ such that

$$(u + \lambda\mu(U))^{1-t}(u + \lambda\mu(V))^t = u + \lambda\mu(D). \tag{3.5}$$

Because $m(\Sigma)$ is convex, we can find $D \in \Sigma$ such that

$$(1 - t)m(U) + tm(V) = m(D). \tag{3.6}$$

and we shall prove (**3**.5) for this D. From (**3**.6), it follows that

$$\exp((1 - t)m(U) + tm(V)) = \exp((1 - t)m(U)) \cdot \exp(tm(V)) = \exp(m(D)). \tag{3.7}$$

We have successively

$$m(D) = \frac{1}{c\lambda} \log(u + \lambda\mu(D)), \text{ hence } \exp(m(D)) = (u + \lambda\mu(D))^{\frac{1}{c\lambda}}$$

$$m(U) = \frac{1}{c\lambda} \log(u + \lambda\mu(U)), \text{ hence } \exp(m(U)) = (u + \lambda\mu(U))^{\frac{1}{c\lambda}}$$

and $\exp((1 - t)m(U)) = (u + \lambda\mu(U))^{\frac{1-t}{c\lambda}}$. Similarly $\exp(tm(V)) = (u + \lambda\mu(V))^{\frac{t}{c\lambda}}$.

It follows that the true relation (**3**.7) is exactly relation (**3**.5) $\left(\text{at power } \frac{1}{c\lambda}\right)$ and this finished the proof. □

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