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REMARKS ON SOME GENERALIZATIONS OF θ-CONTRACTION

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The concept of θ-contraction was modified and generalized in several ways during the last decade. Some assumptions concerning the class Θ are shown to be superfluous in order to obtain a unique fixed point for a θ-type contraction, θ-Suzuki type and, consequently, θ-contraction. Improvement of several previously published results are derived with a modified contractive condition and we have presented an example of possible application. The same approach was used for the F-Suzuki contraction and numerous generalizations are made.

Keywords: θ-contraction, θ-type Suzuki contraction, Suzuki-contraction, F-contraction, fixed point.

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1. Introduction

M. Jleli and B. Samet in [12] presented, as stated, a new generalization of Banach contractive condition through defining a new class of contractions known as θ-contractions in the setting of the Branciari metric space [6]. Concept of Branciari's generalized metric space includes a replacement of a usual triangle inequality by $d(x,y) \leq d(x,z)+d(z,w)+d(w,y)$ for any pairwise distinct points known as rectangular or quadrilateral inequality. It was shown in [12] that a θ-contraction on a complete Branciari metric space has a unique fixed point. Continuing in the same manner, M. Jleli, E. Karapinar and B. Samet in [11] proved several θ-contraction results by adding another requirement to the class of θ-functions, continuity, omitting some of the previous ones and involving a more general contractive condition. Later on, several different generalizations of the θ-contraction concept were presented and existence and uniqueness of a fixed point of this class of contraction was proved in a different setting (Branciari's metric space, metric space, b-metric space, partial metric space, cone metric space, etc.). It is important to mention that several papers were committed to the applications in the area of image processing, differential and integral equations and so on. The research on this topic continues which is shown by the recent publication on this topic. (see [1, 3, 4, 9, 10], [13]-[16])

Several questions regarding the definition of θ-functions have arisen. First of them was do we, as in the definition presented in [12], really need for θ to fulfill such a strict condition as (θ_3)? Do we need to add a continuity request or is it somehow implied? Can we loosen up the requests for θ or at least redefine them? What is the relation between θ-contraction and famous Banach contraction? Are some of this conclusions different depending on setting-Branciari's metric space, metric space or some other? Some of this questions have already been answered, at least partially (par example [11],[14]) and some of them will be the main point of interest in this paper.

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2. Preliminaries

The concept of θ-contraction was introduced by M. Jleli and B. Samet.

Definition 2.1. *Let Θ be a set of functions $\theta : (0,\infty) \mapsto (1,\infty)$ such that*

(θ_1) *θ is nondecreasing, i.e., $x \le y \implies \theta(x) \le \theta(y)$;*

(θ_2) *for each sequence $(x_n) \subseteq (0,\infty)$*

$$\lim_{n\to\infty} \theta(x_n) = 1 \;\Leftrightarrow\; \lim_{n\to\infty} x_n = 0;$$

(θ_3) *there exists $0 < k < 1$ and $l \in (0,\infty]$ such that*

$$\lim_{x\to 0} \frac{\theta(x)-1}{x^k} = l.$$

Results of [12] concerning existence and uniqueness of a fixed point for the newly introduced contractive mapping are derived in a setting of a Branciari metric space. (see [6])

Definition 2.2. *Let X be a non-empty set and $d : X \times X \mapsto [0,\infty)$ a mapping such that for all $x, y, z, w \in X$ pairwise distinct points*

(d_1) *$d(x,y) = 0 \iff x = y$;*

(d_2) *$d(x,y) = d(y,x)$;*

(d_3^*) *$d(x,y) \le d(x,z) + d(z,w) + d(w,y)$.*

Evidently, (complete) metric space is a (complete) Branciari metric space, but converse does not hold in general.

Definition 2.3. *Let (X,d) be a Branciari metric space. A mapping $T : X \mapsto X$ is a θ-contraction if there exists a function $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$(\forall\, x, y \in X)\, Tx \ne Ty \implies \theta(d(Tx,Ty)) \le [\theta(d(x,y))]^k. \tag{1}$$

Theorem 2.1. [12] *Let (X,d) be a complete Branciari metric space and a mapping $T : X \mapsto X$ a θ-contraction. A mapping T has a unique fixed point on X.*

The class Θ and corresponding θ-contraction was modified in [11] by adding the continuity assumption.

Definition 2.4. *Let Θ' be a set of functions $\theta : (0,\infty) \mapsto (1,\infty)$ such that*

(θ_1) *θ is nondecreasing, i.e., $x \le y \implies \theta(x) \le \theta(y)$;*

(θ_2) *for each sequence $(x_n) \subseteq (0,\infty)$*

$$\lim_{n\to\infty} \theta(x_n) = 1 \;\Leftrightarrow\; \lim_{n\to\infty} x_n = 0;$$

(θ_3) *there exists $0 < k < 1$ and $l \in (0,\infty]$ such that $\lim_{x\to 0} \frac{\theta(x)-1}{x^k} = l$;*

(θ_4) *θ is continuous.*

This modification of θ-contraction also includes a more general contractive condition.

Theorem 2.2. [11] *Let (X,d) be a complete Branciari metric space and $T : X \mapsto X$ a mapping. Suppose that there exist a $\theta \in \Theta'$ and $k \in (0,1)$ such that for all $x, y \in X$*

$$\theta(d(Tx,Ty)) \le [\theta(M(x,y))]^k, \tag{2}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$, then T has a unique fixed point.

Improvement of the results of [11] was done by J. Ahmad et al. [3] by excluding (θ_3) from the definition of Θ' and by retaining the contractive condition (2). Denote with Θ^* the class of all functions $\theta : (0,\infty) \mapsto (1,\infty)$ satisfying (θ_1), (θ_2) and (θ_4). The main result of [3] is the existence and uniqueness of a fixed point for a θ-contraction fulfilling (2) for $k \in (0,1)$ and $\theta \in \Theta^*$, but complete Branciari metric space is replaced by a complete metric space.

Theorem 2.3. [3] *If (X, d) is a complete metric space and $T : X \mapsto X$ a θ^*-contraction for some $\theta \in \Theta^*$ and $k \in (0, 1)$, then T has a unique fixed point.*

X. Liu et al, [14] presented the condition $(\theta_2^*) \inf \theta(x) = 1$ as an equivalent to (θ_2) and extended the contractive condition (2). The main results of [14] differ from previously mentioned papers since they involve a concept of Suzuki contraction. We will retain the same notation as in [14] and $\widetilde{\Theta}$ will gather all functions $\theta : (0, \infty) \mapsto (1, \infty)$ such that (θ_1), (θ_2^*) and (θ_4) hold. Some of the further extensions of this concept where shown to be equivalent to some well-known fixed point results like in [8].

Definition 2.5. *If (X, d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a function $\theta \in \widetilde{\Theta}$ and $k \in (0, 1)$ such that for any $x, y \in X$ the following implication holds*

$$\left(\frac{1}{2}d(x, Tx) < d(x, y) \wedge Tx \neq Ty\right) \implies \theta(d(Tx, Ty)) \leq (\theta(M(x, y)))^k, \tag{3}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\right\}$, then the mapping T is a θ-type Suzuki contraction.

θ-type contraction will be the each mapping satisfying (3) with $\frac{1}{2}d(x, Tx) < d(x, y)$ excluded. In this way, another generalization of the results of [11] is obtained in the complete metric space.

Theorem 2.4. *If (X, d) is a complete metric space and $T : X \mapsto X$ a θ-type Suzuki contraction for some $\theta \in \widetilde{\Theta}$ and $k \in (0, 1)$, then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.*

Consequently, θ-type contraction also possesses a unique fixed point in a complete metric space.
The main idea of this article is removing superfluous assumptions regarding the function θ and acquire the unique fixed point for θ-type Suzuki contraction, θ-type contraction and some modifications. Some results are acquired for F-Suzuki contraction. Application of the theoretical results has been found in the area of integral equations.

3. Main results

Theorem 3.1. *If (X, d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the following implication holds*

$$\left(\frac{1}{2}d(x, Tx) < d(x, y) \wedge Tx \neq Ty\right) \implies \theta(d(Tx, Ty)) \leq (\theta(M(x, y)))^k, \tag{4}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\right\}$, and saltuses on the left at each discontinuity t of the function θ are less than $\theta(t) - (\theta(t))^k$, i.e., $\sup_{s<t} \theta(s) > (\theta(t))^k$, then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

Proof. If $x_0 \in X$ is arbitrary, define the sequence of the successive approximations $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T, so we will assume that $x_n \neq x_{n-1}$ for any $n \in \mathbb{N}$. Because of $\frac{1}{2}d(x_{n-1}, x_n) < d(x_{n-1}, x_n)$, the condition (4) holds for any $n \in \mathbb{N}$ where $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1}), 0\}$. If $M(x_{n-1}, x_n) = \frac{1}{2}d(x_{n-1}, x_{n+1})$ for some $n \in \mathbb{N}$, then

$$M(x_{n-1}, x_n) = \frac{1}{2}d(x_{n-1}, x_{n+1}) \leq \max\{(d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Thus it remains to discuss on $M(x_{n-1},x_n) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$. The case $M(x_{n-1},x_n) = d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$ leads to the contradiction since (4) implies $\theta(d(x_n,x_{n+1})) \leq (\theta(d(x_n,x_{n+1})))^k$. Hence, $M(x_{n-1},x_n) = d(x_{n-1},x_n)$ for any $n \in \mathbb{N}$, implies

$$\theta(d(x_n,x_{n+1})) \leq (\theta(d(x_{n-1},x_n)))^k \Longrightarrow \theta(d(x_n,x_{n+1})) \leq (\theta(d(x_0,x_1)))^{k^n},$$

for any $n \in \mathbb{N}$. Letting $n \to \infty$, we get

$$1 \leq \lim_{n\to\infty} \theta(d(x_n,x_{n+1})) \leq \lim_{n\to\infty} \theta\left(d(x_0,x_1)\right)^{k^n} = 1.$$

Also, $\theta(d(x_n,x_{n+1})) \leq (\theta(d(x_{n-1},x_n)))^k < \theta(d(x_{n-1},x_n))$, implies $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for any $n \in \mathbb{N}$. As the sequence $(d(x_{n-1},x_n))$ is a monotone decreasing sequence, its limit exists and let $a := \lim\limits_{n\to\infty} d(x_{n-1},x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1},x_n)$. We will prove that $a = 0$. If $a > 0$, then $\theta(a) \leq \lim\limits_{n\to\infty} \theta(d(x_n,x_{n+1})) = 1$. It cannot be the case, meaning that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$.

Assume contrary of what we intend to prove, that (x_n) is not a Cauchy sequence and choose $\varepsilon > 0$ out of the countable set of discontinuities of the function θ for which there exist strictly increasing sequences $(n_i),(m_i) \subseteq \mathbb{N}$ such that $n_i < m_i$ for any $i \in \mathbb{N}$ and

$$d(x_{n_i},x_{m_i}) \geq \varepsilon \text{ and } d(x_{n_i},x_{m_i-1}) < \varepsilon,$$

where $n_i = \min\{j \geq i \mid d(x_j,x_m) \geq \varepsilon \wedge m > j\}$ and $m_i = \min\{j > n_i \mid d(x_{n_i},x_j) \geq \varepsilon\}$. Hence,

$$\varepsilon \leq d(x_{n_i},x_{m_i}) \leq d(x_{n_i},x_{m_i-1}) + d(x_{m_i-1},x_{m_i}) \leq \varepsilon + d(x_{m_i-1},x_{m_i}),$$

implying $\lim_{i\to\infty} d(x_{n_i},x_{m_i}) = \varepsilon$. Moreover,

$$\begin{aligned} d(x_{n_i},x_{m_i}) &\leq d(x_{n_i},x_{n_i-1}) + d(x_{n_i-1},x_{m_i-1}) + d(x_{m_i-1},x_{m_i}) \\ d(x_{n_i-1},x_{m_i-1}) &\leq d(x_{n_i-1},x_{n_i}) + d(x_{n_i},x_{m_i}) + d(x_{m_i},x_{m_i-1}), \end{aligned}$$

leading to $\lim_{i\to\infty} d(x_{n_i-1},x_{m_i-1}) = \varepsilon$.

In a similar way we obtain $\lim_{i\to\infty} d(x_{n_i-1},x_{m_i}) = \lim_{i\to\infty} d(x_{n_i},x_{m_i-1}) = \varepsilon$. Considering (4), observe that $\frac{1}{2}d(x_{n_i-1},x_{n_i}) < d(x_{n_i-1},x_{m_i-1})$ starting from some $n_0 \in \mathbb{N}$ and

$$\theta(\varepsilon) \leq \theta\left(d(x_{n_i},x_{m_i})\right) \leq \left(\theta\left(M(x_{n_i-1},x_{m_i-1})\right)\right)^k$$

where

$$\begin{aligned} M(x_{n_i-1},x_{m_i-1}) = \max\{&d(x_{n_i-1},x_{m_i-1}), d(x_{n_i-1},x_{n_i}), d(x_{m_i-1},x_{m_i}), \\ &\frac{1}{2}d(x_{n_i-1},x_{m_i}), d(x_{m_i-1},x_{n_i})\}. \end{aligned}$$

If $M(x_{n_i-1},x_{m_i-1}) = d(x_{n_i-1},x_{m_i-1})$ for some $i \in \mathbb{N}$, then

$$\theta(\varepsilon) \leq \lim_{i\to\infty} \theta\left(d(x_{n_i},x_{m_i})\right) \leq \lim_{i\to\infty} \left(\theta\left(d(x_{n_i-1},x_{m_i-1})\right)\right)^k = \left(\theta(\varepsilon)\right)^k, \tag{5}$$

leads to the contradiction.

Limits of both subsequences $\lim_{i\to\infty} \theta(d(x_{m_i-1},x_{m_i}))$ and $\lim_{i\to\infty} \theta(d(x_{n_i-1},x_{n_i}))$ are equal to 1, so if there are infinitely many $i \in \mathbb{N}$ such that $M(x_{n_i-1},x_{m_i-1})$ belongs to the set $\{d(x_{n_i-1},x_{n_i}), d(x_{m_i-1},x_{m_i})\}$, it follows that $\theta(\varepsilon) = 1$ which is incorrect. If $M(x_{n_i-1},x_{m_i-1}) = \frac{1}{2}d(x_{n_i-1},x_{m_i})$, then

$$\theta(\varepsilon) \leq \lim_{i\to\infty} \theta\left(d(x_{n_i},x_{m_i})\right) \leq \lim_{i\to\infty} \left(\theta\left(\frac{1}{2}d(x_{n_i-1},x_{m_i})\right)\right)^k \leq \left(\theta(\varepsilon)\right)^k.$$

Eventually, if $M(x_{n_i-1},x_{m_i-1}) = d(x_{m_i-1},x_{n_i})$, then

$$\theta(\varepsilon) \leq \lim_{i\to\infty} \theta\left(d(x_{n_i},x_{m_i})\right) \leq \lim_{i\to\infty} \left(\theta\left(d(x_{m_i-1},x_{n_i})\right)\right)^k \leq \left(\theta(\varepsilon)\right)^k.$$

Since we got the contradiction in all discussed cases, (x_n) must be a Cauchy sequence and having in mind that (X,d) is a complete metric space, there exists some $x^* \in X$ satisfying $\lim\limits_{n\to\infty} x_n = x^*$.
Additionally, $d(Tx^*,x^*) \le d(Tx^*,x_{n+1}) + d(x_{n+1},x^*) \le M(x_n,x^*) + d(x_{n+1},x^*)$, where

$$M(x^*,x_n) = \max\{d(x^*,x_n), d(x^*,Tx^*), d(x_n,x_{n+1}), \frac{1}{2}d(x^*,x_{n+1}), d(x_n,Tx^*)\}.$$

In order to justify the last inequality and the use of (4), estimate $\frac{1}{2}d(x_n,x_{n+1})$ and $d(x_n,x^*)$ by comparing their convergence rates. Assume that there are infinitely many $n_j \in \mathbb{N}$ such that $\frac{1}{2}d(x_{n_j},x_{n_j+1}) \ge d(x_{n_j},x^*)$, then

$$d(x_{n_j},x^*) \le \frac{1}{2}d(x_{n_j},x_{n_j+1}) \le \frac{1}{2}\left(d(x_{n_j},x^*) + d(x^*,x_{n_j+1})\right)$$

implying $d(x_{n_j},x^*) < d(x^*,x_{n_j+1})$ for any $j \in \mathbb{N}$ which is in a direct conflict with $\lim_{j\to\infty} x_{n_j} = x^*$. Hence, $\frac{1}{2}d(x_n,x_{n+1}) < d(x_n,x^*)$ for any $n \in \mathbb{N}$ starting from some $n_0 \in \mathbb{N}$ and (4) is applicable.
Therefore, we will analyze the several options depending on the value of $M(x^*,x_n)$.
(i) If $M(x^*,x_n) = d(x^*,x_n)$ for infinitely many $n \in \mathbb{N}$, then

$$d(Tx^*,x^*) \le d(x^*,x_n) + d(x_{n+1},x^*).$$

Letting $n \to \infty$ leads to the conclusion that x^* is a fixed point of T.
(ii) In the case that $M(x^*,x_n) = d(x_n,x_{n+1})$ for infinitely many $n \in \mathbb{N}$, we have

$$d(Tx^*,x^*) \le d(x_n,x_{n+1}) + d(x_{n+1},x^*),$$

and $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = \lim\limits_{n\to\infty} d(x_{n+1},x^*)$ implies $Tx^* = x^*$.
(iii) If $M(x^*,x_n) = \frac{1}{2}d(x^*,x_{n+1})$ for infinitely many $n \in \mathbb{N}$, then

$$d(Tx^*,x^*) \le \frac{1}{2}d(x^*,x_{n+1}) + d(x_{n+1},x^*),$$

implies $Tx^* = x^*$.
(iv) Assume that $M(x^*,x_n) = d(x^*,Tx^*)$ for infinitely many $n \in \mathbb{N}$. As $\lim\limits_{n\to\infty} d(Tx^*,x_n) = d(Tx^*,x^*)$, from the estimation of

$$\theta(d(Tx^*,x_n)) \le \left(\theta(d(x^*,Tx^*))\right)^k, \tag{6}$$

we obtain the contradiction due to the presumption made in the statement of the theorem regarding the saltuses on the left. Since, $\lim\limits_{n\to\infty} x_n = x^*$ and $d(x_{n_j},Tx^*) < d(x^*,Tx^*)$ for some subsequence $(x_{n_j}) \subseteq (x_n)$, we have that $\lim_{j\to\infty} d(x_{n_j},Tx^*) = d(x^*,Tx^*)$ and furthermore

$$\theta(d(x^*,Tx^*)) - \lim_{j\to\infty} \theta(d(x_{n_j},Tx^*)) \le \theta(d(x^*,Tx^*)) - \left(\theta(d(x^*,Tx^*))\right)^k$$

which is inconsistent with (6).
(v) Remaining, $M(x^*,x_n) = d(x_n,Tx^*)$ starting from some $n_0 \in \mathbb{N}$. Further estimations up to n_0 gives us

$$\theta(d(Tx^*,x_n)) \le \left(\theta(d(x_{n_0},Tx^*))\right)^{k^{(n-n_0)}},$$

and by letting $n \to \infty$, $\theta(d(Tx^*,x^*) \le 1$, so $Tx^* = x^*$.
From all of the previous considerations, x^* is a fixed point of the mapping T. Uniqueness easily follows. If $Ty = y$ and $y \ne x^*$, then

$$\theta(d(x^*,y)) = \theta(d(Tx^*,Ty) \le \left(M(x^*,y)\right)^k,$$

where $M(x^*,y) = \max\{d(x^*,y), 0\}$. Thus, x^* is a unique fixed point of the mapping T. □

Corollary 3.1. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous on the left function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the condition (4) holds, then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.*

Corollary 3.2. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the condition (4) holds, then T has a unique fixed point in and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.*

The Corollary 3.2 is in fact the main result of [14] (Theorem 2.1), but obtained without the explicit request that (θ_2) is fulfilled. Consequently, Theorem 3.1 and listed corollaries are the generalizations of the main result in [17].
Both the previous and the following result are a generalization of Corollary 3.6 of [11] since (θ_3) is omitted.

Corollary 3.3. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing continuous function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for all $x, y \in X$*

$$\theta(d(Tx,Ty)) \leq (\theta(M(x,y)))^k ,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

By further simplification of the contractive condition, we acquire the Theorem 2.2 in [3] as the main result of that paper.

Corollary 3.4. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing continuous function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the condition that for all $x, y \in X$*

$$\theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^k ,$$

then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

Corollary 3.5. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the following implication holds*

$$\left(\frac{1}{2}d(x,Tx) < d(x,y) \wedge Tx \neq Ty\right) \implies \theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^k ,$$

and saltuses on the left at each discontinuity t of the function θ are less than $\theta(t) - (\theta(t))^k$, i.e., $\sup_{s<t} \theta(s) > (\theta(t))^k$, then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

Corollary 3.6. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing continuous function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the following implication holds*

$$\left(\frac{1}{2}d(x,Tx) < d(x,y) \wedge Tx \neq Ty\right) \implies \theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^k , \tag{7}$$

then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

The mapping T fulfilling (7) for a nondecreasing continuous function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ will be called a simple θ-Suzuki contraction.

Theorem 3.2. *If (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that exist a nondecreasing function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that for any $x, y \in X$ the following implication holds*

$$\left(\frac{1}{2}d(x,Tx) < d(x,y) \wedge Tx \neq Ty\right) \Longrightarrow \theta(d(Tx,Ty)) \leq (\theta(N(x,y)))^k , \tag{8}$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left(d(x,Ty) + d(y,Tx)\right)\right\},$$

and saltuses on the left at each discontinuity t of the function θ are less than $\theta(t) - (\theta(t))^k$, i.e., $\sup_{s<t}\theta(s) > (\theta(t))^k$, then T has a unique fixed point in X and for any $x_0 \in X$ the sequence of successive approximations $(T^n x_0)$ converges to the fixed point of a mapping T.

Proof. If $x_0 \in X$ is arbitrary, define the iterative sequence $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. Thus, assume that $x_n \neq x_{n-1}$ for any $n \in \mathbb{N}$. Since $\frac{1}{2}\left(d(x_{n-1},x_n) + d(x_n,x_n)\right) < d(x_{n-1},x_n)$, the condition (8) holds for any $n \in \mathbb{N}$ for

$$N(x_{n-1},x_n) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{2}d(x_{n-1},x_{n+1})\}.$$

Further estimations of $d(x_{n-1},x_n)$ are similar as in the proof of Theorem 3.1 leading to $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Assume that the sequence (x_n) is not a Cauchy sequence, so for some $\varepsilon > 0$ out of the set of discontinuities of the function θ and the sequence $(n_i), (m_i)$ such that $m_i > n_i \geq i$, $i \in \mathbb{N}$, we have $d(x_{n_i}, x_{m_i}) \geq \varepsilon$ and $d(x_{n_i}, x_{m_i-1}) < \varepsilon$, where n_i and m_i are chosen to be minimal for each $i \in \mathbb{N}$ as previously described. Considerations regarding $\lim_{n\to\infty} d(x_{n_i \pm i}, x_{m_i \pm i})$ for $i \in \{0,1\}$ do not include the direct use of the contractive condition, hence the conclusion will be the same as in the proof of Theorem 3.1, meaning

$$\varepsilon = \lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \lim_{i\to\infty} d(x_{n_i-1}, x_{m_i-1}) = \lim_{i\to\infty} d(x_{n_i}, x_{m_i-1}) = \lim_{i\to\infty} d(x_{n_i-1}, x_{m_i})$$

Impacted by (8), observe that $\frac{1}{2}d(x_{n_i-1}, x_{n_i}) < d(x_{n_i-1}, x_{m_i-1})$ starting from some $n_0 \in \mathbb{N}$ and $\theta(\varepsilon) \leq \theta\left(d(x_{n_i}, x_{m_i})\right) \leq \left(\theta\left(N(x_{n_i-1}, x_{m_i-1})\right)\right)^k$, where

$$\begin{aligned} N(x_{n_i-1}, x_{m_i-1}) \quad = \quad & \max\{d(x_{n_i-1}, x_{m_i-1}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{m_i}), \\ & \frac{1}{2}\left(d(x_{n_i-1}, x_{m_i}) + d(x_{m_i-1}, x_{n_i})\right)\}. \end{aligned}$$

If $N(x_{n_i-1}, x_{m_i-1}) \in \{d(x_{n_i-1}, x_{m_i-1}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{m_i})\}$ for infinitely many $n \in \mathbb{N}$, we have the same estimations as in the previous case since $\lim_{i\to\infty}\theta(d(x_{m_i-1}, x_{m_i}))$ and $\lim_{i\to\infty}\theta(d(x_{n_i-1}, x_{n_i}))$ are equal to 1 and the case $N(x_{n_i-1}, x_{m_i-1}) = d(x_{n_i-1}, x_{m_i-1})$ is impossible as it has been proven in (5).
It remains to discuss if $N(x_{n_i-1}, x_{m_i-1}) = \frac{1}{2}\left(d(x_{n_i-1}, x_{m_i}) + d(x_{m_i-1}, x_{n_i})\right)$ for $n \geq n_0 \in \mathbb{N}$. Observe that

$$\frac{1}{2}\left(d(x_{n_i-1}, x_{m_i}) + d(x_{m_i-1}, x_{n_i})\right) \leq \max\{d(x_{n_i-1}, x_{m_i}), d(x_{m_i-1}, x_{n_i})\}$$

and denote that maximum with $u(x_{n_i-1}, x_{m_i-1})$, then $\theta(\varepsilon) \leq (\theta(\varepsilon))^k$. Consequently, (x_n) is a Cauchy sequence and there exists some $x^* \in X$ satisfying $\lim_{n\to\infty} x_n = x^*$. Furthermore,

$d(Tx^*, x^*) \leq N(x_n, x^*) + d(x_{n+1}, x^*)$ for some

$$N(x^*, x_n) = \max\{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1}), \frac{1}{2}\left(d(x^*, x_{n+1}) + d(x_n, Tx^*)\right)\}$$

because (8) holds as $\frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x^*)$ for $n \geq n_1$ is deduced in a same way as in the proof of Theorem 3.1. As first three options for $N(x^*, x_n)$ are equivalent to those of the proof of Theorem 3.1 (i)-(iii), we will refer the readers to the previous proof and consider only the case that differs. Indeed, if $N(x^*, x_n) \in \{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1})\}$ for infinitely many n, it follows $Tx^* = x^*$.
Elseways, suppose that $N(x^*, x_n) = \frac{1}{2}\left(d(x^*, x_{n+1}) + d(x_n, Tx^*)\right)$ for any $n \geq n_2$, then

$$d(Tx^*, x^*) \leq \frac{1}{2}\left(d(x^*, x_{n+1}) + d(x_n, Tx^*)\right) + d(x_{n+1}, x^*),$$

for any $n \geq n_2$. Letting $n \to \infty$, we acquire $d(Tx^*, x^*) \leq \frac{1}{2}d(x^*, Tx^*)$, meaning $Tx^* = x^*$. All derived cases guarantee the existence of the fixed point of T and the uniqueness is obtained analogously as in the proof of Theorem 3.1. □

We can state the analogous corollaries to Corollary 3.1 and 3.2 when (8) is fulfilled.

4. F-Suzuki contraction

As a corollary we may gather several results for F-contraction and F-Suzuki contraction on a complete metric space. Recall that the idea of F-contraction came from [18].

Definition 4.1. [18] *Let $F : (0, \infty) \to \mathbb{R}$ be a function fulfilling the following conditions:*

(F_1) *F is strictly increasing, i.e., $0 < x < y \implies F(x) < F(y)$;*
(F_2) *For each sequence $(x_n) \subseteq (0, \infty)$,*

$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) *There exists $k \in (0, 1)$, such that $\lim_{x\to 0^+} x^k F(x) = 0$.*

Denote by $\mathcal{F}$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying $(F_1) - (F_3)$, then the F-contraction is defined as follows:

Definition 4.2. *Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

then a mapping T is called a F-contraction.

In [15], (F_3) was replaced by the continuity presumption and the authors have proven that F-Suzuki contraction has a unique fixed point on a complete metric space.

Definition 4.3. *Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be an F-Suzuki contraction if there exist $\tau > 0$ and continuous mapping $F \in \mathcal{F}^*$ fulfilling (F_1) and (F_2) such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),\ x, y \in X. \tag{9}$$

Theorem 4.1. [15] *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F^*-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

Lemma 4.1. *If a mapping $T : X \mapsto X$ is a F-Suzuki contraction on a complete metric space, then T is a simple θ-Suzuki contraction on X.*

Proof. Assume that $T : X \mapsto X$ is a F-contraction on a complete metric space (X, d) fulfilling (9). Define $\theta : (0, +\infty) \mapsto (1, \infty)$ as

$$\theta(x) = e^{e^{F(x)}}, \ x \in (0, \infty). \tag{10}$$

Mapping is well-defined because $e^{e^{F(x)}} > 1$ for $x > 0$ and non-decreasing since (F_1) holds. Furthermore, if $\frac{1}{2}d(x,y) < d(x,Tx)$, then $\theta(d(Tx,Ty)) = e^{e^{F(d(Tx,Ty))}} \le \left(e^{e^{F(d(x,y))}}\right)^{e^{-\tau}}$. Accordingly, T is a simple θ-Suzuki contraction. □

Taking into the account Lemma 4.1, we derive Theorem 4.1 as a direct consequence of Theorem 3.1. Additionally, we may state the result for a newly defined class of F-Suzuki type contractions

Definition 4.4. *Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be an F-Suzuki type contraction if there exist $\tau > 0$ and continuous mapping $F \in \mathcal{F}^*$ fulfilling (F_1) and (F_2) such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies \tau + F(d(Tx,Ty)) \le F(Md(x,y)), \tag{11}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}d(x,Ty), d(y,Tx)\right\}.$$

Corollary 4.1. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F^*-Suzuki type contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

Proof. If $T : X \mapsto X$ is a F-Suzuki type contraction on a complete metric space (X, d), then, analogously as in the proof of Lemma 4.1, we have that T is a θ-type Suzuki contraction for θ defined by (10). As a matter of fact, $\theta(d(Tx,Ty)) = e^{e^{F(d(Tx,Ty))}} \le e^{e^{F(M(x,y))-\tau}} = (\theta(M(x,y)))^{e^{-\tau}}$ for

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}d(x,Ty), d(y,Tx)\right\}.$$

Hence, T is a θ-type Suzuki contraction. □

Theorem 4.2. *Let (X, d) be a complete metric space and a mapping $T : X \mapsto X$. If there exist $\tau > 0$ and a continuous nondecreasing mapping $F \in \mathcal{F}^*$ such that for all $x, y \in X$ with $Tx \neq Ty$ and $\frac{1}{2}d(x,Tx) < d(x,y)$, we have*

$$\tau + F(d(Tx,Ty)) \le F\left(K(x,y)\right), \tag{12}$$

where

$$K(x,y) = \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta_1 d(x,Ty) + \delta_2 d(y,Tx)$$

for $\alpha + \beta + \gamma + 2\delta_1 + \delta_2 \le 1$ for $\alpha, \beta, \gamma, \delta_1, \delta_2 \ge 0$, then a mapping T has a unique fixed point in X.

Proof. Obviously, if (12) is fulfilled, then

$$\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta_1 d(x,Ty) + \delta_2 d(y,Tx) \le M(x,y),$$

whenever $Tx \neq Ty$ for $x, y \in X$. The reaslon for this conclusion lyes in $\delta_1 \le \frac{1}{2}$ or $\delta_2 \le \frac{1}{2}$ and the symmetry of the inequalities (4) and (12). □

Observe that the Theorem 4.2 is generalization of the results of [7] concerning F-Hardy-Rogers contractive condition, but also F-Kannan and F-Chatterjea only with the additional assumption that F is continuous. We may notice, that even some requests are omitted. we still have better results for the class $\mathcal{F}^*$, then for $\mathcal{F}$.

5. **Applications**

The applications of presented results connected to θ-contractions, F-contractions and also Hardy-Rogers (Kannan, Chaterrjea) contractions have been easily found mostly in the area of differential, difference and integral equations. We will discuss on Fredholm integral equation of the second kind but on time scales. For that purpose, we collect some basic definitions regarding time scales.

Definition 5.1. *A time scale is an nonempty closed subset of the set of real numbers.*

A time scale is usually denoted by the symbol $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \mapsto \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \mapsto \mathbb{T}$ are defined as usual along with a Hilger derivative. (see [5])

Definition 5.2. *A continuous function $f : \mathbb{T} \longmapsto \mathbb{R}$ is pre-differentiable with region of differentiation D if*

(i) $D \subset \mathbb{T}^\kappa$,
(ii) $\mathbb{T}^\kappa \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$,
(iii) f *is differentiable at each* $t \in D$.

Theorem 5.1. [5] *Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $f : \mathbb{T}^\kappa \longmapsto \mathbb{R}$ be a given regulated function. Then there exists the unique pre-differentiable function F satisfying*

$$F^\Delta(t) = f(t) \quad \text{for} \quad \text{all} \quad t \in D, \quad F(t_0) = x_0.$$

Definition 5.3. *Assume that $f : \mathbb{T} \mapsto \mathbb{R}$ is a regulated function. Any function F by Theorem 5.1 is a pre-antiderivative of f. The indefinite integral of the regulated function f is defined by*

$$\int f(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f.
The Cauchy integral of f is defined by

$$\int_\tau^s f(t)\Delta t = F(s) - F(\tau) \quad \text{for} \quad \text{all} \quad \tau, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \longmapsto \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \mapsto \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds} \quad \text{for} \quad \text{all} \quad t \in \mathbb{T}^\kappa.$$

Consider the homogenous nonlinear Fredholm integral equation of the second kind as in [2]:

$$x(t) = \lambda \int_a^b K(s, t, \sigma(s), \sigma(t), x(s))\Delta s, \; t \in [a, b]_{\mathbb{T}}, \tag{13}$$

where $K : ([a, b]_{\mathbb{T}})^4 \times \mathbb{R} \to \mathbb{R}$. If $X = C[a, b]_{\mathbb{T}}$ is the set of all continuous real-valued functions with the domain $[a, b]_T$ equipped with the metric $d(x, y) = \max_{a \le s \le b} |x(s) - y(s)|$, for any $x, y \in X$, then (X, d) is a complete metric space. Define the mapping $T : X \mapsto X$ such that

$$Tx(t) = \lambda \int_a^b K(s, t, \sigma(s), \sigma(t), x(s))\Delta s, \tag{14}$$

for any $t \in [a, b]_{\mathbb{T}}$ and $x \in X$. Clearly, the fixed point of T is a solution of the integral equation (13) and vice versa.

Theorem 5.2. *Assume that the function $K : ([a, b]_{\mathbb{T}})^4 \times \mathbb{R} \mapsto \mathbb{R}$ is Δ-integrable and $T : X \mapsto X$ is defined by (14). If, for any $x, y \in X$ such that $Tx \neq Ty$,*

$$|K(s, t, \sigma(s), \sigma(t), x(s)) - K(s, t, \sigma(s), \sigma(t), y(s))| \le \frac{1}{\lambda(b - a)} e^{\alpha(s)} |x(s) - y(s)|, \tag{15}$$

where $\alpha(s) = -\frac{1}{|x(s)-y(s)|+1}$*, holds for any* $s \in [a,b]_{\mathbb{T}}$ *and* $x, y \in X$*, then* T *has a unique fixed point in* X*.*

Proof. Evidently, $T : X \mapsto X$ is a well-defined function and let $x, y \in X$ be arbitrary, then

$$\begin{aligned}|Tx(y) - Ty(t)| &= \lambda \left| \int_a^b K(s,t,\sigma(s),\sigma(t),x(s))ds - \int_0^t K(s,t,\sigma(s),\sigma(t),y(s))ds \right| \\ &\leq \frac{1}{b-a} \int_a^b e^{-\frac{1}{(|x(s)-y(s)|+1)}} |x(s)-y(s)|\, ds \\ &\leq e^{-\frac{1}{d(x,y)+1}} d(x,y),\end{aligned}$$

further implies $d(Tx,Ty) \leq e^{-\frac{1}{d(x,y)+1}} d(x,y)$. Moreover, if $\theta(t) = e^t$ for $t > 0$ and $k = e^{-1}$, then for any $x, y \in X$ such that $Tx \neq Ty$, we have

$$\theta\left(d(Tx,Ty)\right) \leq \left(\theta\left(d(x,y)\right)\right)^k.$$

□

Theorem 5.3. *Assume that the function* $K : ([a,b]_{\mathbb{T}})^4 \times \mathbb{R} \mapsto \mathbb{R}$ *is* Δ*-integrable and* $T : X \mapsto X$ *is defined by (14). If for any* $x, y \in X$ *such that* $Tx \neq Ty$

$$|K(s,t,\sigma(s),\sigma(t),x(s)) - K(s,t,\sigma(s),\sigma(t),y(s))| \leq \frac{1}{\lambda(b-a)} e^{\alpha(s)} M(x(s),y(s)), \qquad (16)$$

where $\alpha(s) = -\frac{1}{|x(s)-y(s)|+1}$*, holds for any* $s \in [a,b]_{\mathbb{T}}$ *and* $x, y \in X$*, where*

$$M_{x,y}(s) = \max\left\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|\right\},$$

then T *has a unique fixed point in* X*.*

Proof. Similarly to the estimations made in the proof of Theorem 5.2 for $\theta(t) = e^t$ for $t > 0$ and $k = e^{-1}$ such that for any $x, y \in X$ such that $Tx \neq Ty$, we have

$$\theta\left(d_\lambda(Tx,Ty)\right) \leq \left(\theta\left(M(x,y)\right)\right)^k.$$

where $M(x,y) = \max\left\{d_\lambda(x,y), d_\lambda(x,Tx), d_\lambda(y,Ty)\right\}$ and all assumptions of Theorem 3.1 are fulfilled, hence T has a unique fixed point in X. □

6. **Conclusions**

Presented results are generalizations of many results like [3, 7, 11, 14, 17] among others. The definitions of both Θ and $\mathcal{F}$ are relaxed of superfluous assumptions under more complex contractive conditions. Obviously, the same approach and similar proof techniques may be used in other settings like Branciari metric space ([12]), partial metric spaces, cone metric spaces, etc. The question remaining open is can the request regarding saltuses be omitted or replaced by less demanding assumption.

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