STABILITY IN p-TH MOMENT FOR UNCERTAIN NONLINEAR SWITCHED SYSTEMS WITH INFINITE-TIME DOMAIN

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The uncertain nonlinear switched system, characterized by its susceptibility to subjective uncertainties, can be described through uncertain differential equations. While investigations have covered mean stability, measure stability, and almost sure stability in the context of uncertain nonlinear switched systems, these forms of stability may not be suitable for every situation. This paper seeks to present the concept of stability in the p-th moment for nonlinear switched systems, offering it as an additional form of stability. The paper also introduces a stability theorem for uncertain nonlinear switched systems that exhibit stability in the p-th moment. Moreover, it explores the connections between stability in measure and stability in the p-th moment within the framework of uncertain nonlinear switched systems. An example is offered to validate the applicability of our outcomes.

Keywords: uncertainty theory; uncertain nonlinear switched systems; p-th moment stability

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1. Introduction

Differential equations are an important means of characterizing nonlinear dynamical systems [1, 2, 3, 4]. Nonlinear switched systems represent a fascinating intersection of stability [5], stabilization [6] and applications [7, 8, 9, 10], reflecting the intricate dynamics of real-world processes. These systems encompass an array of subsystems, each delineated by distinct nonlinear dynamics, with the added complexity of switching between these subsystems based on certain rules or triggers. In practice, nonlinear switched systems are omnipresent in technology and nature, from electric power system [7], Robotic system [8], the automatic transmission in vehicles [9] to the intricate patterns of gene expression in biological cells [10]. The ongoing research in this domain is geared towards refining control strategies to be resilient to uncertainties and perturbations, often leveraging computational techniques like machine learning for adaptive control. Stochastic nonlinear switched systems are a natural extension of classical switched system theory, integrating the unpredictability of stochastic processes with the complexity of nonlinear dynamics. These systems are characterized by their ability to switch between a collection of nonlinear subsystems in a way that is not entirely predictable, often due to random disturbances, noise, or inherent uncertainties in the system or the environment. Unlike stochastic switched systems [11], the uncertain nonlinear switched system is a nonlinear switched system disturbed by subjective uncertainties, which can be illustrated by uncertain differential equations associated with belief degrees. This type of uncertainty associated with belief degrees is a distinct type of indeterminate phenomenon that can be described using uncertainty theory [12, 13] as the

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opposite of probability theory. Today, uncertainty theory is widely applied in various fields,such as uncertain variational inequalities [14, 15], stability analysis of uncertain systems [16, 17], Liu inequalities and numerical method of uncertain systems [18, 19]and so on.

Stability analysis [20] for the uncertain differential systems is fundamental and important. For example, Su et al. [21] presented the concept of stability for the multidimensional uncertain differential equation in the sense of uncertain measure, and subsequent stable in p-th moment [22], stability in mean [23], almost sure stability [24] were investigated one after another. In 2022, Su et al. [25] investigated three types of stabilities for an uncertain nonlinear switched systems. However, these cannot be applied to all cases, so this paper aims to supplement the existing research by presenting a concept of stability in p-th moment for uncertain nonlinear switched systems. Besides theoretical research, uncertain differential equations have numerous applications in dynamical systems, one of which is in nonlinear switched systems, as explored in this paper and previous works [26, 27, 28].

The paper is organized as follows: Section 2 will provide a review of some basic concepts, lemmas, and theorems. Section 3 presents the concept of p-th moment stability and proves the stability theorem. In Section 4, we provide an example to demonstrate the effectiveness of the results. Finally, a brief summary will be given in Section 5.

2. Preliminaries

Theorem 2.1. [18] *Let U_k and U_k^α be the solution and α-path of the uncertain differential equation(UDE)*

$$dU_k = f(k, U_k)dk + g(k, U_k)dC_k$$

Then

$$\mathcal{M}\{U_k \le U_k^\alpha, \forall k\} = \alpha, \mathcal{M}\{U_k > U_k^\alpha, \forall k\} = 1 - \alpha.$$

Theorem 2.2. [20] *Let C_k be a Liu process on uncertainty space. Then there exists an uncertain variable K such that $K(\gamma)$ is a Lipschitz constant of the sample path $C_k(\gamma)$ for each γ, uncertain measure $\mathcal{M}$, and uncertain distribution function $\Phi(x)$,*

$$\lim_{x\to+\infty} \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \le x\} = 1$$

and

$$\mathcal{M}\{\gamma \in \Gamma | K(\gamma) \le x\} \ge 2\Phi(x) - 1.$$

Theorem 2.3. [18] *Suppose that C_k is a canonical Liu process, and U_k is an integrable uncertain process on $[a, b]$. Then the inequality*

$$\left| \int_a^b U_k(\gamma)dC_k \right| \le K_\gamma \int_a^b |U_k(\gamma)|dk$$

holds, where K_γ is the Lipschitz constant of the sample path $U_k(\gamma)$.

Definition 2.1. [19]*Let α be a real number with $0 < \alpha < 1$. An UDE*

$$dU_k = f(U_k, k)dk + g(U_k, k)dC_k$$

is said to have an α-path U_k^α if it solves the corresponding ordinary differential equation (ODE)

$$dU_k^\alpha = f(U_k^\alpha, k)dk + \mid g(U_k^\alpha, k) \mid \varPhi^{-1}(\alpha)dk$$

where $\varPhi^{-1}(\alpha)$is the inverse standard normal uncertainty distribution, i.e.,

$$\varPhi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha}.$$

Theorem 2.4. [20] *Let C_k be a canonical Liu process. Then there exists an uncertain variable K such that for each γ, K_γ is a Lipschitz constant of the sample path $C_k(\gamma)$, and*

$$\mathcal{M}\{K \le x\} \ge 2\left(1+\exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1} - 1.$$

Theorem 2.5. [20]*Let C_k be a canonical process. Then there exists a nonnegative uncertain variable K such that K_γ is a Lipschitz constant of the sample path $C_k(\gamma)$ for each γ, and*

$$\lim_{x\to+\infty} \mathcal{M}\{K \le x\} = 1.$$

A nonlinear uncertain switched system with infinite-time domain and countable switches written as the following UDEs will be considered:

$$\begin{cases} d\mathbf{U}_k = \mathbf{f}_{i(k)}\big(k,\mathbf{U}_k\big)dk + \mathbf{g}_{i(k)}\big(k,\mathbf{U}_k\big)dC_k, \quad k\in[0,+\infty) \\ \quad i(k) \in I = \big\{1,2,\ldots,M\big\}, \\ \mathbf{U}\big|_{k=0} = \mathbf{U}_0, \end{cases} \tag{1}$$

where $\mathbf{U}_k \in R^n$ represents the state vector of the system, vector functions $\mathbf{f}_{i(k)}(k,\mathbf{u}) : [0,+\infty)\times R^n \to R^n$ and $\mathbf{g}_{i(k)}(k,\mathbf{u}) : [0,+\infty)\times R^n \to R^n$ are both bounded for any $i(k)\in I$, and C_k is a canonical process defined on an uncertainty space, representing the noise of the system. Throughout this paper, for a vector $\mathbf{u} = (u_1,u_2,\cdots,u_n)^T$, L_1-norm is employed to measure it as the following:

$$\|\mathbf{u}\| = \sum_{i=1}^{n} \big|u_i\big|. \tag{2}$$

The switching law of uncertain switched system (1) defined on the interval $[0,+\infty)$ is

$$\Lambda = \Big(\big(k_0,i(0)\big),\big(k_1,i(1)\big),\ldots,\big(k_N,i(N)\big),\ldots\Big),$$

where k_τ $(\tau = 0,1,\ldots,N,\ldots)$ stand for the switching instants whose number is countable, and $0 = k_0 \le k_1 \le \cdots \le k_N \le \cdots < +\infty$. The tuple $\big(k_\tau, i(\tau)\big)$ means that at the instant k_τ the system switches to sub-system $i(\tau)$ from sub-system $i(\tau-1)$, that is, sub-system $i(\tau)$ alone keeps active in time interval $[k_\tau,k_{\tau+1})$ for each $\tau\in\big\{0,1,\ldots,N,\ldots\big\}$. In order to analyze the stability of system (1) more concisely, assumption 2.1 about the vector functions in the system is presented in the following.

Assumption 2.1 It is assumed that the vector functions $\mathbf{f}_{i(\tau)}(k,\mathbf{u})$ and $\mathbf{g}_{i(\tau)}(k,\mathbf{u})$ satisfy the strong Lipschitz conditions

$$\big\|\mathbf{f}_{i(\tau)}(k,\mathbf{u}) - \mathbf{f}_{i(\tau)}(k,\mathbf{v})\big\| + \big\|\mathbf{g}_{i(\tau)}(k,\mathbf{u}) - \mathbf{g}_{i(\tau)}(k,\mathbf{v})\big\| \le \mathfrak{L}_{i(\tau)}(k)\|\mathbf{u}-\mathbf{v}\|,$$
$$\forall \mathbf{u},\mathbf{v}\in R^n, k\ge 0$$

for each $i(\tau) = 1,2,\ldots,M$, where $\mathfrak{L}_{i(\tau)}(k)$ are positive functions with

$$\int_0^{+\infty} \mathfrak{L}_{i(\tau)}(k)dk < +\infty.$$

The symbol $\mathfrak{L}(k)$ is employed to denote the supremum of positive functions $\mathfrak{L}_{i(\tau)}(k)$ $\big(i(\tau) = 0,1,\ldots,M\big)$, so the following equality is established for $k\in[0,+\infty)$:

$$\mathfrak{L}(k) = \sup_k \Big\{\mathfrak{L}_{i(\tau)}(k)\Big|\ \tau = 0,1,\ldots,N,\ldots\Big\} = \sup_j \Big\{\mathfrak{L}_j(k)\Big|\ j = 1,2,\ldots,M\Big\}. \tag{3}$$

Lemma 2.1. *If a vector function $\mathbf{f}(k) = \big(f_1(k),f_2(k),\ldots,f_n(k)\big)$ is differentiable when $k > 0$, then $\|\mathbf{f}(k)\|$ is differentiable almost everywhere in the interval $(0,+\infty)$.*

3. Stability Theorem

Inspired by the work in Gao et al. [22], we now consider stability in p-th moment of uncertain nonlinear switched systems (1).

Definition 3.1. *A multidimensional UDEs for uncertain nonlinear switched systems is said to be stable in p-th moment if for any solutions* $\mathbf{U}_k$ *and* $\mathbf{V}_k$ *with initial values* $\mathbf{U}_0$ *and* $\mathbf{V}_0$, *respectively, we have*

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E[\sup_{k\geq 0}\|\mathbf{U}_k-\mathbf{V}_k\|^p]=0. \tag{4}$$

Theorem 3.1. *System* (1) *is stable in p-th moment if Assumption 2.1 is true, and the integral of the supremum of* $\left\{\mathfrak{L}_{i(\tau)}(k)\middle|\ i(\tau)=0,1,\ldots,M\right\}$ *on* $[0,+\infty)$ *is smaller than* $\frac{\pi}{\sqrt{3}p}(p>1)$, *and*

$$\int_0^{+\infty}\mathfrak{L}(k)dk<\frac{\pi}{\sqrt{3}p}.$$

Proof In the framework of system (1), we observe that the difference $\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)$ proves to be differentiable for each γ within the set Γ. This holds true for all values of k in the range $(0,+\infty)$, excluding the specific instances $\{k_1,k_2,\ldots,k_N,\ldots\}$. Given that the Lebesgue measure of this set of switching instants is zero, it follows that $\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|$ is differentiable almost everywhere within the interval $(0,+\infty)$, in accordance with Lemma 2.1.

Define $\mathfrak{A}_\gamma$ as the set comprising those values of k within $(0,+\infty)$ for which $\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|$ is differentiable. That is, $\mathfrak{A}_\gamma=\left\{k\in(0,+\infty)\middle|\ \|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\| \text{ is differentiable. }\right\}$. On the other hand, $\mathfrak{B}_\gamma$ represents the residual set in $(0,+\infty)$ outside of $\mathfrak{A}_\gamma$, it's straightforward to deduce that the Lebesgue measure $m(\mathfrak{B}_\gamma)=0$. Further, let's express $\mathbf{U}_k(\gamma)=\big(u_1(\gamma),u_2(\gamma),\ldots,u_n(\gamma)\big)^T$ and $\mathbf{V}_k(\gamma)=\big(v_1(\gamma),v_2(\gamma),\ldots,v_n(\gamma)\big)^T$. For any k within $\mathfrak{A}_\gamma$, there is always an interval $[k_\tau,k_{\tau+1})$ that encompasses the moment k. This implies that the sub-system $i(\tau)$ is operational during that interval. Based on the criteria set forth in Assumption 2.1, we can deduce that

$$\begin{aligned}
&d\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|\\
&\quad=d\sum_{i=1}^n\big|u_i(\gamma)-v_i(\gamma)\big|=d\sum_{i=1}^n\pm\big(u_i(\gamma)-v_i(\gamma)\big)\\
&\quad=\sum_{i=1}^n\pm\big(du_i(\gamma)-dv_i(\gamma)\big)\leq\sum_{i=1}^n\big|du_i(\gamma)-dv_i(\gamma)\big|\\
&\quad=\big\|d\mathbf{U}_k(\gamma)-d\mathbf{V}_k(\gamma)\big\|\\
&\quad\leq\big\|\big[\mathbf{f}_{i(\tau)}\big(k,\mathbf{U}_k(\gamma)\big)-\mathbf{f}_{i(\tau)}\big(k,\mathbf{V}_k(\gamma)\big)\big]dk+\big[\mathbf{g}_{i(\tau)}\big(k,\mathbf{U}_k(\gamma)\big)-\mathbf{g}_{i(\tau)}\big(k,\mathbf{V}_k(\gamma)\big)\big]dC_k(\gamma)\big\|\\
&\quad\leq\big\|\mathbf{f}_{i(\tau)}\big(k,\mathbf{U}_k(\gamma)\big)-\mathbf{f}_{i(\tau)}\big(k,\mathbf{V}_k(\gamma)\big)\big\|dk+\big\|\mathbf{g}_{i(\tau)}\big(k,\mathbf{U}_k(\gamma)\big)-\mathbf{g}_{i(\tau)}\big(k,\mathbf{V}_k(\gamma)\big)\big\|dC_k(\gamma)\\
&\quad\leq\mathfrak{L}_{i(\tau)}(k)\cdot\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|dk+\mathfrak{L}_{i(\tau)}(k)\cdot\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|dC_k(\gamma)\\
&\quad\leq\mathfrak{L}_{i(\tau)}(k)\cdot\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|dk+K_\gamma\mathfrak{L}_{i(\tau)}(k)\cdot\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|dk\\
&\quad=\big(1+K_\gamma\big)\mathfrak{L}_{i(\tau)}(k)\cdot\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|dk,
\end{aligned}$$

Here, K_γ represents the Lipschitz constant associated with the sample path $C_k(\gamma)$, as outlined in Theorem 2.2. Thus,

$$\|\mathbf{U}_k(\gamma)-\mathbf{V}_k(\gamma)\|\leq\|\mathbf{U}_{k_\tau}(\gamma)-\mathbf{V}_{k_\tau}(\gamma)\|\cdot\exp\left(\big(1+K_\gamma\big)\int_{k_\tau}^k L_{i(\tau)}(s)ds\right).$$

For each $k \in \mathfrak{B}_\gamma$, we can pick k'_1 from the interval $\left(\left(1-\frac{1}{2}\right)k, k\right)$ such that $k'_1 \in \mathfrak{A}_\gamma$. Subsequently, we can easily find k'_2 from the interval $\left(\left(1-\frac{1}{3}\right)k, k\right) - \{k'_1\}$ such that $k'_2 \in \mathfrak{A}_\gamma$. Following the same logic, for any $n \in \mathbb{N}_+$, we can select k'_n from the interval $\left(\left(1-\frac{1}{n+1}\right)k, k\right) - \{k'_1, k'_2, \ldots, k'_{n-1}\}$ such that $k'_n \in \mathfrak{A}_\gamma$. It's clear that $k'_n \to k$ as $n \to +\infty$. For any $n \in \mathbb{N}_+$ and $k'_n \in \mathfrak{A}_\gamma$, based on the previous deduction, we obtain the following inequality:

$$\begin{aligned}\|\mathbf{U}_{k'_n}(\gamma) - \mathbf{V}_{k'_n}(\gamma)\| &\le \|\mathbf{U}_{k_\tau}(\gamma) - \mathbf{V}_{k_\tau}(\gamma)\| \cdot \exp\left(\left(1+K_\gamma\right)\int_{k_\tau}^{k'_n} L_{i(\tau)}(r)dr\right)\\ &\le \|\mathbf{U}_{k_\tau}(\gamma) - \mathbf{V}_{k_\tau}(\gamma)\| \cdot \exp\left(\left(1+K_\gamma\right)\int_{k_\tau}^{k} L_{i(\tau)}(r)dr\right).\end{aligned}$$

As n approaches $+\infty$, and since U_k is sample-continuous for the event γ, we can conclude that

$$\|\mathbf{U}_{k}(\gamma) - \mathbf{V}_{k}(\gamma)\| \le \|\mathbf{U}_{k_\tau}(\gamma) - \mathbf{V}_{k_\tau}(\gamma)\| \cdot \exp\left(\left(1+K_\gamma\right)\int_{k_\tau}^{k} L_{i(\tau)}(r)dr\right).$$

In conclusion, for any $k \in (0, +\infty)$, coupled with the fact that γ is arbitrary, the inequality

$$\|\mathbf{U}_{k} - \mathbf{V}_{k}\| \le \|\mathbf{U}_{k_\tau} - \mathbf{V}_{k_\tau}\| \cdot \exp\left(\left(1+K\right)\int_{k_\tau}^{k} L_{i(\tau)}(r)dr\right) \tag{5}$$

is almost surely satisfied, where K is a nonnegative uncertain variable such that

$$\mathcal{M}\Big\{K(\gamma) \ge x\Big\} = 1 - \mathcal{M}\Big\{K(\gamma) < x\Big\} \le 2\left(1+\exp\left(\frac{\pi x}{\sqrt{3}}\right)\right)^{-1} \tag{6}$$

according to Theorem 2.4. In order to establish links with the initial states of the uncertain nonlinear switched system (1), it is necessary to utilize its inherent characteristics. Subsequently, by making use of the definition of the function $\mathfrak{L}(k)$, we can deduce the following

$$\begin{aligned}\|\mathbf{U}_{k} - \mathbf{V}_{k}\| &\le \|\mathbf{U}_{k_\tau} - \mathbf{V}_{k_\tau}\| \cdot \exp\left(\left(1+K\right)\int_{k_\tau}^{k} \mathfrak{L}_{i(\tau)}(r)dr\right)\\ &\le \|\mathbf{U}_{k_{\tau-1}} - \mathbf{V}_{k_{\tau-1}}\| \cdot \exp\left(\left(1+K\right)\int_{k_{\tau-1}}^{k_\tau} \mathfrak{L}_{i(\tau-1)}(r)dr\right)\\ &\quad \cdot \exp\left(\left(1+K\right)\int_{k_\tau}^{k} \mathfrak{L}_{i(\tau)}(r)dr\right)\\ &\le \|\mathbf{U}_{k_0} - \mathbf{V}_{k_0}\| \cdot \exp\left(\left(1+K\right)\sum_{j=0}^{\tau-1}\int_{k_j}^{k_{j+1}} \mathfrak{L}_{i(j)}(r)dr\right)\\ &\quad \cdot \exp\left(\left(1+K\right)\int_{k_\tau}^{k} \mathfrak{L}_{i(\tau)}(r)dr\right)\\ &\le \|\mathbf{U}_0 - \mathbf{V}_0\| \cdot \exp\left(\left(1+K\right)\sum_{j=0}^{\tau-1}\int_{k_j}^{k_{j+1}} \mathfrak{L}(r)dr\right)\\ &\quad \cdot \exp\left(\left(1+K\right)\int_{k_\tau}^{k} \mathfrak{L}(r)dr\right)\\ &\le \|\mathbf{U}_0 - \mathbf{V}_0\| \cdot \exp\left(\left(1+K\right)\int_{0}^{+\infty} \mathfrak{L}(r)dr\right)\end{aligned} \tag{7}$$

holds almost surely, which implies that

$$\sup_{k>0} \|\mathbf{U}_k - \mathbf{V}_k\|^p \le \|\mathbf{U}_0 - \mathbf{V}_0\|^p \cdot \exp\left(p \cdot \big(1+K\big) \int_0^{+\infty} \mathfrak{L}(r)dr\right) \tag{8}$$

almost surely. Taking expected value on both sides of (8), then the following result

$$\begin{aligned} E\left[\sup_{k>0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] &\le \|\mathbf{U}_0 - \mathbf{V}_0\|^p \cdot E\left[\exp\left(p \cdot \big(1+K\big) \int_0^{+\infty} \mathfrak{L}(r)dr\right)\right] \\ &= \|\mathbf{U}_0 - \mathbf{V}_0\|^p \cdot \exp\left(p \cdot \int_0^{+\infty} \mathfrak{L}(r)dr\right) \cdot E\left[\exp\left(p \cdot K \int_0^{+\infty} \mathfrak{L}(r)dr\right)\right] \end{aligned} \tag{9}$$

holds. Since $\mathfrak{L}(k)$ is integrable on $[0, +\infty)$, we obviously get

$$\exp\left(p \cdot \int_0^{+\infty} \mathfrak{L}(r)dr\right) < +\infty.$$

For the expected value $E\left[\exp\left(p \cdot K \int_0^{+\infty} \mathfrak{L}(r)dr\right)\right]$, we denote that $q = \int_0^{+\infty} \mathfrak{L}(r)dr < \frac{\pi}{\sqrt{3}p}$. It follows from the definition of expected value and Theorem 2.4 that

$$\begin{aligned} E\left[\exp\left(p \cdot K \int_0^{+\infty} \mathfrak{L}(r)dr\right)\right] &= E\Big[\exp(pqK)\Big] = \int_0^{+\infty} \mathcal{M}\Big\{\exp(pqK) \ge x\Big\}dx \\ &= \int_0^{+\infty} \mathcal{M}\left\{K \ge \frac{\ln x}{pq}\right\}dx \\ &\le 2\int_0^{+\infty} \left(1 + \exp\left(\frac{\pi \ln x}{\sqrt{3}pq}\right)\right)^{-1} dx \\ &= 2\int_0^{+\infty} \left(1 + x^{\frac{\pi}{\sqrt{3}pq}}\right)^{-1} dx < +\infty. \end{aligned}$$

Combining with (9), the following equality is derived:

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} E\left[\sup_{k>0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0.$$

When $k = 0$, it is easy to know that

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} E\Big[\|\mathbf{U}_0 - \mathbf{V}_0\|^p\Big] = \lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} \|\mathbf{U}_0 - \mathbf{V}_0\|^p = 0.$$

Combining the above two equalities, we get

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} E\left[\sup_{k\ge 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0.$$

In conclusion, uncertain nonlinear switched system (1) is stable in p-th moment. The theorem is verified.

Theorem 3.2. *If uncertain nonlinear switched system* (1) *is stable in p-th moment, then it is stable in measure.*

Proof *From Definition 3.1, for two solutions* $\mathbf{U}_k$ *and* $\mathbf{V}_k$ *with different initial values* $\mathbf{U}_0$ *and* $\mathbf{V}_0$*, respectively. Then it follows from the definition of stability in p-th moment that*

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} E\left[\sup_{k\ge 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0, \forall p > 0. \tag{10}$$

By Markov inequality, for any given real number $\epsilon > 0$*, we have*

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} \mathcal{M}\{\|\mathbf{U}_k - \mathbf{V}_k\| > \epsilon\} \le \lim_{\|\mathbf{U}_0 - \mathbf{V}_0\| \to 0} \frac{E[\|\mathbf{U}_k - \mathbf{V}_k\|^p]}{\epsilon^p} \le$$

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} \frac{E\left[\sup\limits_{k\geq 0}\|\mathbf{U}_k-\mathbf{V}_k\|^p\right]}{\epsilon^p} \to 0, \forall k \geq 0.$$

Thus, p-th moment stability implies the stability in measure. This concludes the theorem.

Remark 3.1. *Nevertheless, when dealing with uncertain switched systems, it's crucial to note that stability in measure doesn't necessarily guarantee stability in the p-th moment. To illustrate this point, let's examine the following uncertain switched system*

$$\begin{cases} dU_k = U_k/(k+1)^2\, dC_k, & k \in [0,T) \\ dU_k = 2U_k/(k+1)^2\, dC_k, & k \in [T,+\infty), \end{cases} \tag{11}$$

consisting of two sub-systems, where the state variable $U_k \in R$. Obviously, $f_1(k,u) = f_2(k,u) = 0,\ g_1(k,u) = u/(k+1)^2$ and $g_2(k,u) = 2u/(k+1)^2$, so we have

$$\begin{aligned} \|f_1(k,u)-f_1(k,v)\| + \|g_1(k,u)-g_1(k,v)\| &\leq \frac{1}{(k+1)^2}\cdot\|u-v\|, \\ \|f_2(k,u)-f_2(k,v)\| + \|g_2(k,u)-g_2(k,v)\| &\leq \frac{2}{(k+1)^2}\cdot\|u-v\| \end{aligned} \tag{12}$$

for any $k \geq 0,\ u,v \in R$. And it is easy to find that $\mathfrak{L}(k) = 4/(k+1)^2$ which is integrable on $[0,+\infty)$. Therefore, system (11) is stable in measure according to Theorem 3.1. Observing system (12), we know that it has a solution $U_k \equiv 0$ with the initial state $U_0 = 0$, and it has a solution

$$U_k = \begin{cases} U_0\cdot\exp\left(\int_0^k \frac{1}{(r+1)^2}dC_r\right), & 0 \leq k < T, \\ U_0\cdot\exp\left(\int_0^T \frac{1}{(r+1)^2}dC_r\right)\cdot\exp\left(\int_T^k \frac{2}{(r+1)^2}dC_r\right), & k \geq T \end{cases} \tag{13}$$

with an initial state $U_0 \neq 0$. Then, for $p > 1$, we have

$$\begin{aligned} \sup_{k\geq 0}\|U_k - V_k\|^p &= \sup_{k\geq T}\|U_k - V_k\|^p \\ &= \|U_0\|^p\cdot\exp\left(p\cdot\int_0^T\frac{1}{(r+1)^2}dC_r\right)\cdot\sup_{k\geq T}\exp\left(p\cdot\int_T^k\frac{2}{(r+1)^2}dC_r\right) \\ &\geq \|U_0\|^p\cdot\sup_{k\geq T}\exp\left(p\cdot\int_0^k\frac{2}{(r+1)^2}dC_r\right) \end{aligned} \tag{14}$$

almost surely, and we obtain

$$\begin{aligned} E\left[\sup_{k\geq 0}\|U_k-V_k\|^p\right] &\geq \|U_0\|^p\cdot E\left[\sup_{k\geq T}\exp\left(p\cdot\int_0^k\frac{1}{(r+1)^2}dC_r\right)\right] \\ &\geq \|U_0\|^p\cdot E\left[\exp\left(p\cdot\int_0^{+\infty}\frac{1}{(r+1)^2}dC_r\right)\right] \end{aligned} \tag{15}$$

Because

$$\int_0^{+\infty}\frac{1}{(r+1)^2}dC_r \sim \mathcal{N}\left(0, \int_0^{+\infty}\frac{1}{(r+1)^2}dr\right) = \mathcal{N}(0,1), \tag{16}$$

by applying related conclusions in uncertainty theory, we get that

$$E\left[\exp\left(p\cdot\int_0^{+\infty}\frac{1}{(r+1)^2}dC_r\right)\right] = +\infty. \tag{17}$$

That is to say,

$$E\left[\sup_{k\geq 0}\|U_k - V_k\|^p\right] = +\infty \tag{18}$$

is established due to that $U_0 \neq 0$. *Hence, system* (11) *is not stable in p-th moment according to Definition 3.1. In short, uncertain switched system* (11) *is stable in measure but not stable in p-th moment. Combining such result with Theorem 3.1, it is concluded that stability in p-th moment is a sufficient and unnecessary condition of stability in p-th moment for uncertain switched systems.*

4. Numerical example

Example 4.1. *In order to illustrate the effectiveness of Theorem 3.1, a numerical example about stability will be presented. Now we consider the following uncertain nonlinear switched system with infinite-time domain:*

$$\begin{cases} d\mathbf{U}_k = \mathbf{f}_{i(\tau)}\big(k,\mathbf{U}_k\big)dk + \mathbf{g}_{i(\tau)}\big(k,\mathbf{U}_k\big)dC_k, \quad k\in[0,+\infty) \\ \quad i(\tau)\in I = \big\{1,2,3,4,5\big\}, \\ \mathbf{U}_0 = \big(u_1(0),u_2(0)\big)^T, \end{cases} \tag{19}$$

where $\mathbf{U}_k = \big(u_1(k),u_2(k)\big)^T \in R^2$ *represents the state vector of the system, and*

$$\begin{aligned}
\mathbf{f}_1\big(k,\mathbf{u}\big) &= e^{-k}\cdot\mathbf{u}, \quad \mathbf{g}_1\big(k,\mathbf{u}\big) = \frac{1}{5+k^2}\cdot\exp\big(-\mathbf{u}\big),\\
\mathbf{f}_2\big(k,\mathbf{u}\big) &= e^{-\frac{k}{2}}\cdot\mathbf{u}, \quad \mathbf{g}_2\big(k,\mathbf{u}\big) = \frac{2}{4+k^2}\cdot\exp\big(-\mathbf{u}\big),\\
\mathbf{f}_3\big(k,\mathbf{u}\big) &= e^{-\frac{k}{3}}\cdot\mathbf{u}, \quad \mathbf{g}_3\big(k,\mathbf{u}\big) = \frac{3}{3+k^2}\cdot\exp\big(-\mathbf{u}\big),\\
\mathbf{f}_4\big(k,\mathbf{u}\big) &= e^{-\frac{k}{4}}\cdot\mathbf{u}, \quad \mathbf{g}_4\big(k,\mathbf{u}\big) = \frac{4}{2+k^2}\cdot\exp\big(-\mathbf{u}\big),\\
\mathbf{f}_5\big(k,\mathbf{u}\big) &= e^{-\frac{k}{5}}\cdot\mathbf{u}, \quad \mathbf{g}_5\big(k,\mathbf{u}\big) = \frac{5}{1+k^2}\cdot\exp\big(-\mathbf{u}\big).
\end{aligned}$$

The switching law of system (19) *defined on the interval* $[0,+\infty)$ *is*

$$\Lambda = \Big(\big(k_0,4\big),\big(k_1,1\big),\big(k_2,3\big),\big(k_3,2\big),\big(k_4,5\big),\big(k_5,3\big),\big(k_6,4\big),\big(k_7,2\big),\big(k_8,1\big),\big(k_9,5\big)\Big), \tag{20}$$

where the switching moments k_τ $(\tau = 0,1,\ldots,9)$ *are given as follows:*

$$\begin{gathered}
k_0 = 0,\ k_1 = 9,\ k_2 = 19,\ k_3 = 29,\ k_4 = 39,\ k_5 = 49,\\
k_6 = 59,\ k_7 = 69,\ k_6 = 79,\ k_7 = 89,\ k_8 = 99,\ k_9 = 109.
\end{gathered}$$

For any $k \geq 0,\ \mathbf{u},\mathbf{v}\in R^2$, *we have*

$$\begin{aligned}
\|\mathbf{f}_1(k,\mathbf{u}) - \mathbf{f}_1(k,\mathbf{v})\| &\leq e^{-k}\cdot\|\mathbf{u}-\mathbf{v}\|, \quad \|\mathbf{g}_1(k,\mathbf{u}) - \mathbf{g}_1(k,\mathbf{v})\| \leq \frac{1}{5+k^2}\cdot\|\mathbf{u}-\mathbf{v}\|,\\
\|\mathbf{f}_2(k,\mathbf{u}) - \mathbf{f}_2(k,\mathbf{v})\| &\leq e^{-\frac{k}{2}}\cdot\|\mathbf{u}-\mathbf{v}\|, \quad \|\mathbf{g}_2(k,\mathbf{u}) - \mathbf{g}_2(k,\mathbf{v})\| \leq \frac{2}{4+k^2}\cdot\|\mathbf{u}-\mathbf{v}\|,\\
\|\mathbf{f}_3(k,\mathbf{u}) - \mathbf{f}_3(k,\mathbf{v})\| &\leq e^{-\frac{k}{3}}\cdot\|\mathbf{u}-\mathbf{v}\|, \quad \|\mathbf{g}_3(k,\mathbf{u}) - \mathbf{g}_3(k,\mathbf{v})\| \leq \frac{3}{3+k^2}\cdot\|\mathbf{u}-\mathbf{v}\|,\\
\|\mathbf{f}_4(k,\mathbf{u}) - \mathbf{f}_4(k,\mathbf{v})\| &\leq e^{-\frac{k}{4}}\cdot\|\mathbf{u}-\mathbf{v}\|, \quad \|\mathbf{g}_4(k,\mathbf{u}) - \mathbf{g}_4(k,\mathbf{v})\| \leq \frac{4}{2+k^2}\cdot\|\mathbf{u}-\mathbf{v}\|,\\
\|\mathbf{f}_5(k,\mathbf{u}) - \mathbf{f}_5(k,\mathbf{v})\| &\leq e^{-\frac{k}{5}}\cdot\|\mathbf{u}-\mathbf{v}\|, \quad \|\mathbf{g}_5(k,\mathbf{u}) - \mathbf{g}_5(k,\mathbf{v})\| \leq \frac{5}{1+k^2}\cdot\|\mathbf{u}-\mathbf{v}\|,
\end{aligned}$$

which follows that

$$\|\mathbf{f}_1(k,\mathbf{u})-\mathbf{f}_1(k,\mathbf{v})\|+\|\mathbf{g}_1(k,\mathbf{u})-\mathbf{g}_1(k,\mathbf{v})\| \le \left(e^{-k}+\frac{1}{5+k^2}\right)\cdot\|\mathbf{u}-\mathbf{v}\|,$$
$$\|\mathbf{f}_2(k,\mathbf{u})-\mathbf{f}_2(k,\mathbf{v})\|+\|\mathbf{g}_2(k,\mathbf{u})-\mathbf{g}_2(k,\mathbf{v})\| \le \left(e^{-\frac{k}{2}}+\frac{2}{4+k^2}\right)\cdot\|\mathbf{u}-\mathbf{v}\|,$$
$$\|\mathbf{f}_3(k,\mathbf{u})-\mathbf{f}_3(k,\mathbf{v})\|+\|\mathbf{g}_3(k,\mathbf{u})-\mathbf{g}_3(k,\mathbf{v})\| \le \left(e^{-\frac{k}{3}}+\frac{3}{3+k^2}\right)\cdot\|\mathbf{u}-\mathbf{v}\|,$$
$$\|\mathbf{f}_4(k,\mathbf{u})-\mathbf{f}_4(k,\mathbf{v})\|+\|\mathbf{g}_4(k,\mathbf{u})-\mathbf{g}_4(k,\mathbf{v})\| \le \left(e^{-\frac{k}{4}}+\frac{4}{2+k^2}\right)\cdot\|\mathbf{u}-\mathbf{v}\|,$$
$$\|\mathbf{f}_5(k,\mathbf{u})-\mathbf{f}_5(k,\mathbf{v})\|+\|\mathbf{g}_5(k,\mathbf{u})-\mathbf{g}_5(k,\mathbf{v})\| \le \left(e^{-\frac{k}{5}}+\frac{5}{1+k^2}\right)\cdot\|\mathbf{u}-\mathbf{v}\|.$$

Therefore $\mathbf{f}_j(k,\mathbf{u})$ *and* $\mathbf{g}_j(k,\mathbf{u})$ $\big(j=1,2,3,4\big)$ *satisfy the strong Lipschitz conditions in Assumption 2.1. And we have*

$$\mathfrak{L}_1(k)=e^{-k}+\frac{1}{5+k^2},\quad \mathfrak{L}_2(k)=e^{-\frac{k}{2}}+\frac{2}{4+k^2},\quad \mathfrak{L}_3(k)=e^{-\frac{k}{3}}+\frac{3}{3+k^2},$$
$$\mathfrak{L}_4(k)=e^{-\frac{k}{4}}+\frac{4}{2+k^2},\quad \mathfrak{L}_5(k)=e^{-\frac{k}{5}}+\frac{5}{1+k^2},$$

so according to the definition of $\mathfrak{L}(k)$ *in Eq.* (3)*, the following expression is derived:*

$$\mathfrak{L}(k)=\sup_j\left\{\mathfrak{L}_j(k)\,\Big|\; j=1,2,3,4,5\right\}=e^{-\frac{k}{5}}+\frac{5}{1+k^2},\quad k\in[0,+\infty).$$

Through calculating, we have

$$\int_0^{+\infty}\mathfrak{L}(k)dk=\int_0^{+\infty}\left(e^{-\frac{k}{5}}+\frac{5}{1+k^2}\right)dk=5+\frac{5}{2}\pi<+\infty,$$

which means that $\mathfrak{L}(k)$ *is integrable on* $[0,+\infty)$*. To sum up, uncertain nonlinear switched system* (19) *is stable in p-th moment based on Theorem 3.1. Obviously, there exist four sub-systems in uncertain nonlinear switched system* (19)*, which can be written by the following UDEs with initial state* $\big(u_1(0),u_2(0)\big)$ *according to the switching law* Λ *provided in* (20)*:*

$$\textit{sub-system 1:}\begin{cases} du_1(k)=e^{-k}\cdot u_1(k)dk+\frac{1}{5+k^2}\cdot\exp\big(-u_1(k)\big)dC_k,\\ du_2(k)=e^{-k}\cdot u_2(k)dk+\frac{1}{5+k^2}\cdot\exp\big(-u_2(k)\big)dC_k,\\ \qquad k\in[19,29)\cup[89,99),\end{cases}$$

$$\textit{sub-system 2:}\begin{cases} du_1(k)=e^{-\frac{k}{2}}\cdot u_1(k)dk+\frac{2}{4+k^2}\cdot\exp\big(-u_1(k)\big)dC_k,\\ du_2(k)=e^{-\frac{k}{2}}\cdot u_2(k)dk+\frac{2}{4+k^2}\cdot\exp\big(-u_2(k)\big)dC_k,\\ \qquad k\in[0,9)\cup[39,49)\cup[79,89)\cup[109,+\infty),\end{cases}$$

$$\textit{sub-system 3:}\begin{cases} du_1(k)=e^{-\frac{k}{3}}\cdot u_1(k)dk+\frac{3}{3+k^2}\cdot\exp\big(-u_1(k)\big)dC_k,\\ du_2(k)=e^{-\frac{k}{3}}\cdot u_2(k)dk+\frac{3}{3+k^2}\cdot\exp\big(-u_2(k)\big)dC_k,\\ \qquad k\in[29,39)\cup[59,69),\end{cases}$$

$$\textit{sub-system 4:}\begin{cases} du_1(k)=e^{-\frac{k}{5}}\cdot u_1(k)dk+\frac{4}{2+k^2}\cdot\exp\big(-u_1(k)\big)dC_k,\\ du_2(k)=e^{-\frac{k}{5}}\cdot u_2(k)dk+\frac{4}{2+k^2}\cdot\exp\big(-u_2(k)\big)dC_k,\\ \qquad k\in[9,19)\cup[69,79),\end{cases}$$

$$\textit{sub-system 5:}\begin{cases} du_1(k)=e^{-\frac{k}{5}}\cdot u_1(k)dk+\frac{5}{1+k^2}\cdot\exp\big(-u_1(k)\big)dC_k,\\ du_2(k)=e^{-\frac{k}{5}}\cdot u_2(k)dk+\frac{5}{1+k^2}\cdot\exp\big(-u_2(k)\big)dC_k,\\ \qquad k\in[49,59)\cup[99,109).\end{cases}$$

On the basis of Definition 2.1, the corresponding ODEs of the above uncertain sub-systems are listed as follows:

$$\begin{cases} du_1^\alpha(k) = e^{-k} \cdot u_1^\alpha(k)dk + \frac{1}{5+k^2} \cdot \exp\big(-u_1^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ du_2^\alpha(k) = e^{-k} \cdot u_2^\alpha(k)dk + \frac{1}{5+k^2} \cdot \exp\big(-u_2^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ \qquad k \in [19,29) \cup [89,99), \end{cases}$$

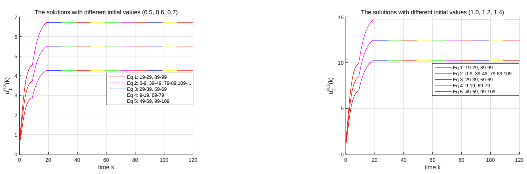
$$\begin{cases} du_1^\alpha(k) = e^{-\frac{k}{2}} \cdot u_1^\alpha(k)dk + \frac{2}{4+k^2} \cdot \exp\big(-u_1^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ du_2^\alpha(k) = e^{-\frac{k}{2}} \cdot u_2^\alpha(k)dk + \frac{2}{4+k^2} \cdot \exp\big(-u_2^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ \qquad k \in [0,9) \cup [39,49) \cup [79,89) \cup [109,+\infty), \\ \quad \big(u_1^\alpha(0), u_2^\alpha(0)\big) = \big(u_1(0), u_2(0)\big), \end{cases}$$

$$\begin{cases} du_1^\alpha(k) = e^{-\frac{k}{3}} \cdot u_1^\alpha(k)dk + \frac{3}{3+k^2} \cdot \exp\big(-u_1^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ du_2^\alpha(k) = e^{-\frac{k}{3}} \cdot u_2^\alpha(k)dk + \frac{3}{3+k^2} \cdot \exp\big(-u_2^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ \qquad k \in [29,39) \cup [59,69), \end{cases}$$

$$\begin{cases} du_1^\alpha(k) = e^{-\frac{k}{4}} \cdot u_1^\alpha(k)dk + \frac{4}{2+k^2} \cdot \exp\big(-u_1^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ du_2^\alpha(k) = e^{-\frac{k}{4}} \cdot u_2^\alpha(k)dk + \frac{4}{2+k^2} \cdot \exp\big(-u_2^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ \qquad k \in [9,19) \cup [69,79), \end{cases}$$

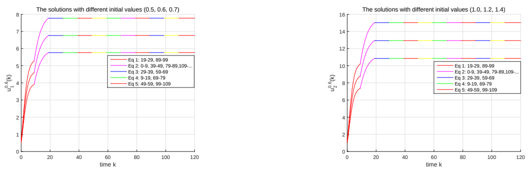
$$\begin{cases} du_1^\alpha(k) = e^{-\frac{k}{5}} \cdot u_1^\alpha(k)dk + \frac{5}{1+k^2} \cdot \exp\big(-u_1^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ du_2^\alpha(k) = e^{-\frac{k}{5}} \cdot u_2^\alpha(k)dk + \frac{5}{1+k^2} \cdot \exp\big(-u_2^\alpha(k)\big) \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dk, \\ \qquad k \in [49,59) \cup [99,109). \end{cases}$$

Figures 1 and 2 are both provided for above (ODEs) when $\alpha = 0.3$ *and* $\alpha = 0.6$. *In (a) of Figures 1, the lines represent the trajectories of* $u_1^{0.3}(k)$ *with different initial values* $u_1^{0.3}(0) = (0.5, 0.6, 0.7)$. *In (b) of Figures 1, the trajectories of* $u_2^{0.3}(k)$ *with different initial values* $u_1^{0.3}(0) = (1.0, 1.2, 1.4)$. *are illustrated by three curves from top to bottom. When* $\alpha = 0.6$, *the trajectories of* $u_1^{0.6}(k)$ *and* $u_2^{0.6}(k)$ *with before-mentioned initial values are presented in Figure 2.*



(a) (b)

FIGURE 1. The trajectories of $u_1^{0.3}(k)$ and $u_2^{0.3}(3)$ with different initial values (0.5, 1.0), (0.6, 1.2), (0.7, 1.4).



(a) (b)

FIGURE 2. The trajectories of $u_1^{0.6}(k)$ and $u_2^{0.6}(k)$ with different initial values (0.5, 1.0), (0.6, 1.2), (0.7, 1.4).

5. Conclusions

In this study, we delved into the analysis of stability in p-th moment for uncertain nonlinear switched systems. By extending the concept of stability in in p-th moment, we established sufficient conditions for determining stability in p-th moment under certain assumptions. Furthermore, we explored the interaction between stability in measure and stability in p-th moment for uncertain nonlinear switched systems. Notably, we provided a counterexample to illustrate that stability in measure does not universally guarantee stability in p-th moment. To illustrate the findings, we further presented an example that illustrates the system's stability in p-th moment.

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