ON THE HEAT TRANSFER OF HOLOGRAPHIC TYPE IN NANOSTRUCTURES

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Assimilating any nanostructure with a fractal, in the most general Mandelbrot's sense, non – differentiable behaviors in their dynamics on the heat transfer phenomena are analyzed. As such, nanostructure dynamics on the heat transfer in the form of Schrödinger – type various regimes imply "holographic implementation" of the thermal fields through groupal invariance of SL(2R) – type. Then, by means of previous groupal invariance as synchronization group between any nanostructure entities, both the phases and the amplitudes of the entities are affected from a homographic perspective. In a special case of synchronization of nanostructure entities, given by Riccati type gauge, period doubling, damping oscillations, self – modulation and chaotic regimes emerge as natural behaviors in the nanostructure dynamics of the heat transfer processes. The present model can also be applied to a large class of nanostructures (i.e. polymeric biocomposites, "liquid wood", temperature – depending drug release systems etc.).

Keywords: fractal theory of motion, groupal invariances of SL(2R) – type, heat transfer at various scale resolutions

1. Introduction

The problem of heat transfer in nanostructures has been analyzed in the better part of the past century [1 – 7].

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Usually, models used to describe nanostructure dynamics and particularly the ones related to heat transfer, are based on the uncertain hypothesis that the variables describing it are differentiable [1 – 4, 7 – 9]. The success of these models must be understood gradually on domains in which differentiability is still valid. However, the differential procedures are not suitable when describing processes related to nanostructure dynamics, which imply nonlinearity and chaos (it is reminded that this is the de – facto case [6, 7 – 9]).

To describe nanostructure dynamics in the fractal paradigm, but remaining faithful to the differentiable mathematical procedures, it is necessary to explicitly introduce scale resolutions, both in the expression of the physical variables and in the fundamental equations which govern nanostructure dynamics. This means that, instead of "working" with a single physical variable described by a strict non – differentiable function, it is possible to "work" only with approximations of these mathematical functions obtained by averaging them on different scale resolutions. As a consequence, any variable purposed to describe nanostructure dynamics will perform as the limit of a family of mathematical functions, this being non – differentiable for null scale resolutions and differentiable otherwise [10, 11].

In the present paper, considering the fractal paradigm as being functional, a non – differentiable model describing the heat transfer in nanostructures is proposed.

2. Mathematical Model

2.1 Nanostructure as a fractal medium and "holographic implementations" of its dynamics

Assimilating any nanostructure with a complex system [12 – 14], it is behaving like a fractal medium induced by the collisions process between its entities. Such assumption can be theoretically sustained by a typical example: between two successive collisions the trajectory of the nanostructure entities is a straight line that becomes non – differentiable in the impact point. Considering now that all the collision impact points are forming an uncountable set of points, it results that the trajectories of nanostructure entities become continuous but non – differentiable curves i.e. a fractal [11].

In such a context, Fractal Theories of Motion becomes functional for describing various dynamics in nanostructures and particularly the heat transfer. The fundamental assumption of these models is the one that the dynamics of any entity of nanostructures will be described by continuous but non – differentiable motion curves (fractal motion curves). These fractal motion curves exhibit the property of self – similarity in their every point, which can be translated into a property of holography (every part reflects the whole) [11]. Basically, the

discussion will be about "holographic implementations" of dynamics of any nanostructure entity for example through Schrödinger – type fractal "regimes" (i.e. describing nanstructure dynamics and particularly the heat transfer through Schrödinger – type equations at various scale resolutions – Schrödinger equation of fractal type).

2.2 Scale covariant derivative and geodesics equations

Let it be considered that the scale covariance principle (the physics laws applied to the heat transfer in nanostructures are invariant with respect to scale resolution transformations [10]) and postulate that the transition from the standard (differentiable) heat transfer in nanostructures to the fractal (non – differentiable) heat transfer in nanostructures can be implemented by replacing the standard time derivative $\frac{d}{dt}$ by the non – differentiable operator $\frac{\hat{d}}{dt}$ [15 – 19]:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l + \frac{1}{4}(dt)^{\left(\frac{2}{D_F}\right)-1} D^{lp} \partial_l \partial_p \tag{1}$$

where

$$\begin{gathered} \hat{V}^l = V_D^l - V_F^l \\ D^{lp} = d^{lp} - i\bar{d}^{lp} \\ d^{lp} = \lambda_+^l \lambda_+^p - \lambda_-^l \lambda_-^p \\ \bar{d}^{lp} = \lambda_+^l \lambda_+^p + \lambda_-^l \lambda_-^p \\ \partial_t = \frac{\partial}{\partial t}, \partial_l = \frac{\partial}{\partial X^l}, \ \partial_l \partial_p = \frac{\partial}{\partial X^l}\frac{\partial}{\partial X^p}, i = \sqrt{-1}, \quad l, p = 1,2,3 \end{gathered} \tag{2}$$

In the above relations $\hat{V}^l$ is the complex velocity, V_D^l is the differentiable velocity independent on the scale resolution dt and V_F^l is the non – differentiable velocity dependent on the scale resolution. X^l is the fractal spatial coordinate and *t* is the non – fractal time having the role of an affine parameter of the motion curves. D^{lp} is the constant tensor associated with the differentiable – non – differentiable transition of the heat transfer processes, λ_+^l is the constant vector associated with the forward differentiable – non – differentiable of heat transfer processes and λ_-^l is the constant vector associated with the backwards differentiable – non – differentiable of heat transfer processes. D_F is the fractal dimension of the movement curve. For the fractal dimension it is possible to choose any definition: Kolmogorov type fractal dimension, Hausdorff – Besikovici type fractal dimension etc. [11, 20, 21]. But once chosen this becomes operational, it needs to be constant and arbitrary: $D_F < 2$ for the corelative physical processes, $D_F > 2$ for the non-corelative physical processes etc. [10, 11].

Now, the non – differentiable operator plays the role of the scale covariant derivative, namely it is used to write the fundamental equations of the heat transfer in the nanostructures, in the same form as in the classic (differentiable) case. Under these conditions, accepting the functionality of the scale covariant principle, i.e. applying scale covariant derivative (1) to the complex velocity field (2), in the absence of any external constraint, the geodesics equation of the nanostructure entities takes the following form [15 – 19]:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i + \frac{1}{4}(dt)^{\left(\frac{2}{D_F}\right)-1} D^{lk} \partial_l \partial_k \hat{V}^i = 0 \tag{3}$$

This means that the fractal local acceleration $\partial_t \hat{V}^i$, the fractal convection $\hat{V}^l \partial_l \hat{V}^i$ and the fractal dissipation $D^{lk} \partial_l \partial_k \hat{V}^i$ of any nanostructure entity, make their balance in any point of the motion fractal curve. Moreover, the presence of the complex coefficient of viscosity – type $4^{-1}(dt)^{\left(\frac{2}{D_F}\right)-1} D^{lk}$ in the nanostructure dynamics specifies that it is a rheological medium. So, the nanostructure's structures have memory, as a datum, by their own structure.

If the fractalization in the dynamics of nanostructures is achieved by Markov – type stochastic processes, which involve Lévy type movements [10, 11, 20, 21] of the nanostructure entities, then:

$$\lambda_+^i \lambda_+^l = \lambda_-^i \lambda_-^l = 2\lambda \delta^{il} \tag{4}$$

where λ is a coefficient associated to the differentiable – non – differentiable transition and δ^{il} is Kronecker's pseudo – tensor.

Under these conditions, the geodesics equation (motion equation) takes the simple form:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i - i\lambda(dt)^{\left(\frac{2}{D_F}\right)-1} \partial^l \partial_l \hat{V}^i = 0 \tag{5}$$

For irrotational motions of the nanostructure entities, the complex velocity field $\hat{V}^i$ takes the form:

$$\hat{V}^i = -2i\lambda(dt)^{\left(\frac{2}{D_F}\right)-1} \partial^i \ln \Psi \tag{6}$$

Then substituting (6) in (5), the geodesics equation (5) (for details see method from [15 – 19]) becomes Schrödinger – type equation at various scale resolutions (Schrödinger equation of fractal type):

$$\lambda^2 (dt)^{\left(\frac{4}{D_F}\right)-2} \partial^l \partial_l \Psi + i\lambda(dt)^{\left(\frac{2}{D_F}\right)-1} \partial_t \Psi = 0 \tag{7}$$

The variable $\Phi = -2i\lambda(dt)^{(2/D_F)-1}\, ln\, \Psi$ defines, through (6), the complex scalar potential of the complex velocity field, while Ψ corresponds to the state function of fractal type. Both variables, Φ and Ψ, have no direct physical meaning, but possible "combinations" of them can acquire it if they satisfy certain conservation laws.

Let it be made explicit such a situation for Ψ. For this purpose, it is first noticed that the complex conjugate of Ψ, that is $\overline{\Psi}$, satisfies through (7) the equation:

$$\lambda^2(dt)^{\left(\frac{4}{D_F}\right)-2}\partial^l\partial_l\overline{\Psi} - i\lambda(dt)^{\left(\frac{2}{D_F}\right)-1}\partial_t\overline{\Psi} = 0 \tag{8}$$

Multiplying (7) by $\overline{\Psi}$ and (8) by Ψ, subtracting the results and introducing the notations:

$$\rho = \Psi\overline{\Psi}, \quad \boldsymbol{J} = i\lambda(dt)^{\left(\frac{4}{D_F}\right)-1}(\Psi\,\nabla\bar{\Psi} - \bar{\Psi}\nabla\Psi) \tag{9}$$

it is possible to obtain the conservation law of states density of fractal type:

$$\partial_t\rho + \nabla J = 0 \tag{10}$$

In (10) ρ corresponds to the states density of fractal type and $\boldsymbol{J}$ corresponds to the states density current of fractal type.

3. Heat transfer in nanostructures through groupal invariance of SL(2R) – type by means of Riccati gauge

The idea of motion equation has an enlarged significance, starting with the Fractal Theory of Motion under the form of Scale Relativity [10, 15, 19]. Let it be noted that Schrödinger's equation of fractal type – motion equation for the state function Ψ of fractal type – besides the fact that it is invariant with respect to the Galilei vectorial transformation group, it is also invariant, in a separate way, to time transformations and one – dimensional coordinates (let it be *x*) represent a group in themselves [22, 23]. They constitute, in the most general case of motion in a single direction, a realization of the group SL(2R) [24, 25], but with two variables and three parameters, through the action [22, 23]:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{x}{\gamma t + \delta} \tag{11}$$

Every vector in the tangent space SL(2R) is a linear combination of the three fundamental vectors, the infinitesimal action generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, \qquad X_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} \tag{12}$$

These satisfy the basic structure equations:

$$[X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \qquad [X_3, X_1] = -2X_2 \tag{13}$$

which are taken as standard commutation relations for this type of algebraic structure. The group has an invariant function, which can be obtained as the solution of a partial differential equation:

$$(cX_1 + 2bX_2 + aX_3)f(t,x) = 0 \tag{14}$$

which, in view of (12), means:

$$(at^2 + 2bt + c)\frac{\partial f(t,x)}{\partial t} + (at + b)x\frac{\partial f(t,x)}{\partial x} = 0 \tag{15}$$

The general solution of this equation is a function of the arbitrary value of the ratio:

$$\frac{x^2}{at^2 + 2bt + c} \tag{16}$$

which represent the different path of transitivity of the action described by operators from (12). In the particular case in which such a function is linear and, moreover, is a constant, it is possible to state that it represents a motion equation, be it a free entity, either a geodesic motion on the surface of a cone, or a non – standard interpretation if the statistical description is pertinent regarding the argument.

It is understood that the motion equation is linked, according to these interpretations, to the invariant functions of the SL(2R) algebra which, as an idea, may be introduced here by a generalization of the grouping procedure.

Then, in the first of equations (11), this requirement would mean that the nanostructure entities are considered simultaneously. Each entity can be located in the "swarm" (i.e. in the nanostructure) by four homogenous coordinates$(\alpha, \beta, \gamma, \delta)$, or three non – homogenous coordinates, if the equation (11) represents the context of time and a one – dimensional coordinate for the space domain, covered by this nanostructure. The simultaneity condition of the free entities of the "swarm" (i.e. in the nanostructure) can be differently characterized, from a Riccati equation in pure differentials (this will be named the Riccati type gauge) [24, 25]:

$$d\frac{\alpha t + \beta}{\gamma t + \delta} = 0, dt = \omega^1 t^2 + \omega^2 t + \omega^3 \tag{17}$$

Thus, for the description of the nanostructure dynamics as a succession of states of an ensemble of simultaneous entities, as it were, it suffices to have three differentiable 1 – forms, representing a coframe of SL(2R) algebra [26]:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma}$$

$$\omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma} \tag{18}$$

$$\omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma}$$

That this coframe refers to such algebra can be checked by direct calculation of the Maurer – Cartan equations [24, 25], which are characteristic:

$$d \wedge \omega^1 - \omega^1 \wedge \omega^2 = 0$$

$$d\omega^2 + 2(\omega^3 \wedge \omega^1) = 0 \tag{19}$$

$$d\omega^3 - \omega^2 \wedge \omega^3 = 0$$

Élie Cartan has shown that under these conditions, one can prove that the right hand side of equation (17) is an exact differential [27], therefore it should always have an integral. The Cartan – Killing metric of this coframe is given by the quadratic form [28]:

$$ds^2 = \frac{1}{4}(\omega_2^2 - 4\omega_1\omega_2) = \frac{(\alpha d\delta + \delta d\alpha - \beta d\gamma - \gamma d\beta)^2}{4(\alpha\delta - \beta\gamma)^2} - \frac{d\alpha d\delta - d\beta d\gamma}{\alpha\delta - \beta\gamma} \tag{20}$$

so that a state of a nanostructure in a given dynamic can be organized as a metric plan space, i.e. a Riemannian three – dimensional space [28]. The geodesics of this Riemannian space are given by some conservation laws of equations:

$$\omega^1 = a^1 d\tau, \qquad \omega^2 = 2a^2 d\tau, \qquad \omega^3 = a^3 d\tau \tag{21}$$

where a^1, a^2, a^3 are constant and τ is the affine parameter of the geodesics, so that, along these geodesics of differential equation (17) is an ordinary differential of Riccati type:

$$\frac{dt}{d\tau} = a^1 t^2 + 2a^2 t + a^3 \tag{22}$$

Mathematically, this requires an ensemble generated by a harmonic mapping between the positions in space and the nanostructure entities, with the

square of the coordinate x measuring the variance of the distribution which describes the spreading of nanostructure entities in space.

Because in thermodynamic systems the time, t, is inversely proportional with the temperature, T, [29, 30] under the shape of $t \equiv \frac{\mu}{T}$ with $\mu = const.$, equation (22) with the substitutions:

$$-\frac{\mu}{a^3} = A, \qquad -\frac{a^2\mu}{a^3} = B, \qquad -\frac{a_1\mu^2}{a^3} = AC \tag{23}$$

takes the shape:

$$A\frac{dT}{d\tau} - T^2 + 2BT + AC = 0 \tag{24}$$

Because the roots of the polynom:

$$P(T) = T^2 - 2BT - AC \tag{25}$$

can be written in the shape:

$$\begin{aligned} T_1 &= B + iA\Omega \\ T_2 &= B - iA\Omega \\ \Omega^2 &= \frac{C}{A} - \left(\frac{B}{A}\right)^2 \end{aligned} \tag{26}$$

the change of variable:

$$z = \frac{T - T_1}{T - T_2} \tag{27}$$

transforms the equation (24) in:

$$\dot{z} = 2i\Omega z \tag{28}$$

of solution:

$$z(\tau) = z(0)e^{2i\Omega\tau} \tag{29}$$

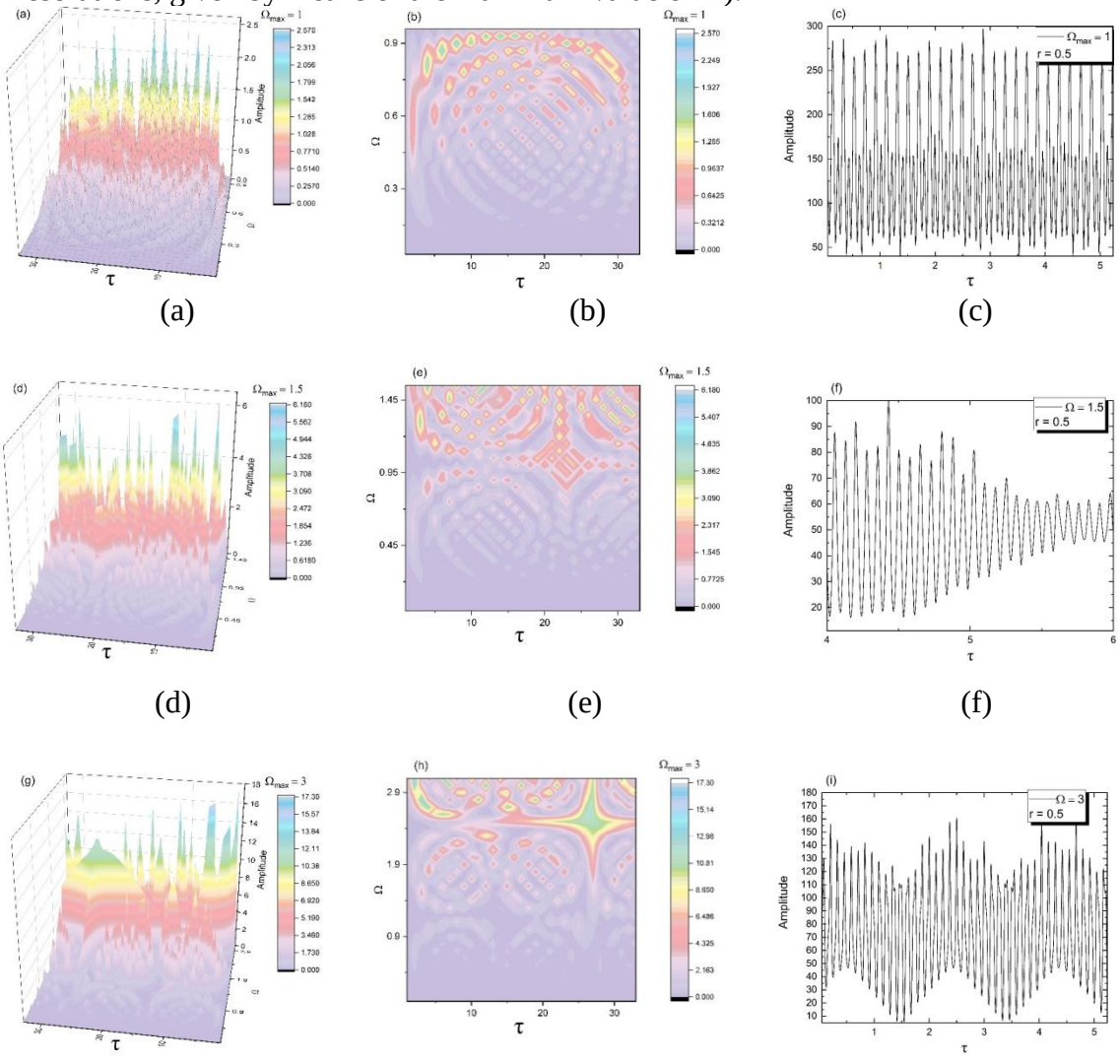
As such, if the initial condition $z(0)$ is conveniently expressed, then it is possible to construct the general solution of equation (24), by writing the transformation (27) in the shape:

$$T = \frac{T_1 + re^{2i\Omega(\tau-\tau_0)}T_2}{1 + re^{2i\Omega(\tau-\tau_0)}} \tag{30}$$

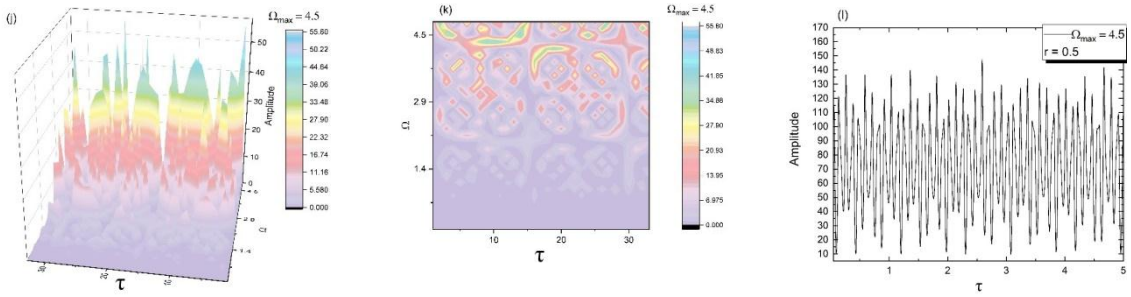
where r and τ_0 are two integration constants. Using relations (26), it is possible to write this solution in real terms:

$$z = B + A\Omega\left\{\frac{2r\sin[2\Omega(\tau-\tau_0)]}{1+r^2+2r\cos[2\Omega(\tau-\tau_0)]} + i\frac{1-r^2}{1+r^2+2r\cos[2\Omega(\tau-\tau_0)]}\right\} \tag{31}$$

Therefore, synchronization of phase – amplitude type between nanostructure entities in the heat transfer process implies groupal invariance of SL(2R) – type. Then, period doubling, damping oscillations, self – modulation and chaotic regimes emerge as natural behaviors in the nanostructure heat transfer (see Figures 1 a – l for $r = 0.5$ and Real $[(z-B)/A]$ ≡Amplitude at various scale resolutions, given by means of the maximum value of Ω).



(g) (h) (i)



(j) (k) (l)

Figs. 1 a-l. Various types of evolutions during thermal transfer in nanostructures (3D, contour plot and time series) representation: period doubling (a, b, c), damped oscillation regimes (d, e, f), signal modulation (g, h, i) and chaotic behavior (j, k, l)).

A similar work, with a rigorous mathematical model and using a fractal method, can be found in the paper [31].

4. Conclusions

In the motion fractal paradigm, a new model on the heat transfer in nanostructures is established. So:

i) Assimilating any nanostructure with a complex system, the said system is behaving as a fractal medium. In other words, the nanostructure becomes a fractal in the most general Mandelbrot's sense.
ii) The fundamental assumption of the author's model is the one that the dynamics of any entity of nanostructures are described by continuous but non – differentiable motion curves (fractal motion curves). These fractal motion curves exhibit the property of self – similarity in its every point, which can be translated into a property of holography (every part reflects the whole).
iii) In the previously – mentioned context, the authors discuss about "holographic implementations" of heat transfer phenomena in any nanostructure through Schrödinger – type fractal "regimes" (i.e. describing heat transfer through Schrödinger – type equations at various scale resolutions).
iv) Through a special invariance of SL(2R) – type of the Schrödinger – type fractal equation, various heat transfer regimes, in the form of synchronization between any entities of the nanostructure, are highlighted. More precisely, by means of SL(2R) – type group, the phase is only moved with a quantity depending on

the amplitude of the nanostructure entity at the transition among various entities of the nanostructure. More than that, the amplitude of the entity of the nanostructure is also affected from a homographic perspective. The usual "synchronization" manifested through the delay of the amplitudes and phases of the entities of the nanostructure must represent here only a fully particular case.

v) In a particular case of synchronization of nanostructure entities, given by Riccati type gauge, period doubling, damping oscillations, self – modulation and chaotic regimes emerge as natural behaviors in the nanostructure dynamics.

vi) According to the presented model, it can be observed that two temporal scales exist: it is not necessary to postulate them. It is about the time of describable phenomena in the nanostructure landmark τ, one one hand, and the time corresponding to the 20 metrics 20, t, on the other hand.

An observation is made, in that this model may be developed for use to other classes of materials, such as biomaterials, biocomposites and other high – end materials. Particularly, this model holds true for "liquid wood", a material which lies at the boundary between natural and artificial. More than that, the same model can be used in the dynamics analysis which involve drug release systems which are temperature – depending.

Authors' contributions:

All authors have equally contributed to this paper.

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