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LIMITS IN THE CATEGORY OF STRUCTURAL TOPOLOGICAL SPACES

M. Z. Kazemi Baneh*[1] and S. N. Hosseini[2]

We introduce the notion of topological structure, and relative to that the notions of structural topology as well as structural continuity are given. We show that for a given topological structure, structural topological spaces together with structural continuous morphisms form a concrete category. We then give the construction of the induced, coinduced, discrete and indiscrete structural topologies, proving certain results that hold for topological spaces also hold for structural topological spaces. We demonstrate that if the base category is (finitely) complete, then the category of structural topological spaces is concretely (finitely) complete. Finally we provide some illustrative examples.

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1. Introduction and Preliminaries

Structural topology as well as structural continuity are introduced in [2] based on a locally given topological structure S_X for an object X in a category, where it is shown that these notions generalize the notions of topology and continuity, see [6], as well as some of the notions of fuzzy topology and fuzzy continuity existing in the literature. In this paper in Section 2, the topological structure S is defined globally consisting of natural transformations and structural topology is defined slightly different. Also structural continuity is introduced. We then show that for a given topological structure S, structural topological spaces together with structural continuous morphisms form a concrete category that we denote by $\mathsf{S}Top$. In Section 3, under certain conditions we construct the induced, coinduced, discrete and indiscrete S-topologies in $\mathsf{S}Top$. We also prove that certain propositions similar to those in the category Top of topological spaces hold in the category $\mathsf{S}Top$. In Section 4, we show that concrete equalizers, concrete terminal objects and concrete (finite) products exist in $\mathsf{S}Top$ as soon as equalizers, terminal objects and (finite) products,

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respectively exist in the base category, proving that $\mathsf{S}Top$ has concrete (finite) limits if the base category has (finite) limits. Finally in Section 5, we furnish several illustrative examples.

For categorical notions we refer the reader to [1, 3, 4]. The following lemma will be used in the subsequent sections.

Denoting join of subobjects $a : A \rightarrowtail X$ and $b : B \rightarrowtail X$ in a category by $a \vee b : A \vee B \rightarrowtail X$ and the collection of all the monomorphisms by Mono, one can easily verify that:

Lemma 1.1. *Let $\mathcal{C}$ be an $(\mathsf{E}, \mathsf{Mono})$-structured category for some collection E of morphisms. If the binary coproduct of the subobjects $a : A \rightarrowtail X$ and $b : B \rightarrowtail X$ of X exists, then the mono part of the factorization of $a \oplus b : A \sqcup B \longrightarrow X$ is the join $a \vee b : A \vee B \rightarrowtail X$.*

2. The Category of Structural Topological Spaces

In this section we give a slightly different definition of a topological structure relative to a base than the one given in [2]. We then introduce the notions of structural topological space and structural continuity and show that the structural topological spaces with structural continuous morphisms form a concrete category.

Definition 2.1. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories.*

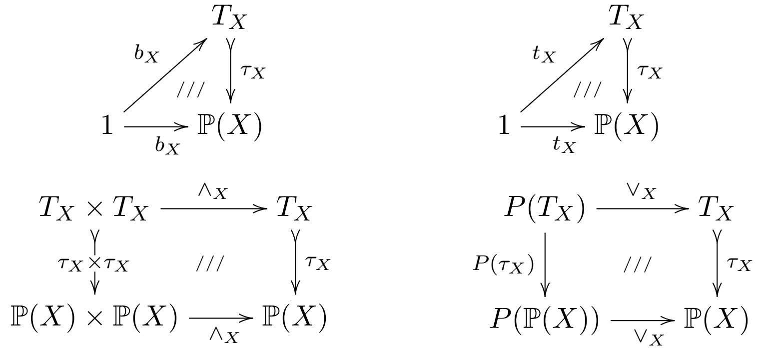
- **a)** *A pair of functors $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C} \xrightarrow{P} \mathcal{C}$, with $\mathcal{C}$ finitely complete, is called a base on $(\mathcal{E}, \mathcal{C})$.*
- **b)** *A topological structure relative to a base $(\mathbb{P}, P)$ or a $(\mathbb{P}, P)$-topological structure, is a quadruple $\mathsf{S} = (b, t, \wedge, \vee)$, where*

$$1 \xrightarrow{b} \mathbb{P},\ 1 \xrightarrow{t} \mathbb{P},\ S \circ \mathbb{P} \xrightarrow{\wedge} \mathbb{P} \ \text{ and } \ P \circ \mathbb{P} \xrightarrow{\vee} \mathbb{P}$$

  are natural transformations. Here $S : \mathcal{C} \longrightarrow \mathcal{C}$ is the square functor.
- **c)** *A structural topology on an object X of $\mathcal{E}$ relative to a structure S, or just an S-topology on X, is a $\mathcal{C}$-monomorphism $T_X \overset{\tau_X}{\rightarrowtail} \mathbb{P}(X)$ such that morphisms*

$$1 \xrightarrow{b_X} T_X,\ 1 \xrightarrow{t_X} T_X,\ T_X \times T_X \xrightarrow{\wedge_X} T_X \ \text{ and } \ P(T_X) \xrightarrow{\vee_X} T_X$$

  exist rendering commutative the following diagrams.



In this case, (X, τ_X) *is called a structural topological space or just an* S-*topological space.*

In the rest of the paper we let S be a topological structure relative to a base $(\mathbb{P}, P)$ on $(\mathcal{E}, \mathcal{C})$.

Definition 2.2. *Given* S-*topological spaces* (X, τ_X) *and* (Y, τ_Y)*, an* $\mathcal{E}$-*morphism* $f : X \longrightarrow Y$ *is said to be structurally continuous or* S-*continuous, if there exists a* $\mathcal{C}$-*morphism* $T_f : T_Y \longrightarrow T_X$ *such that following diagram commutes.*

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\rightarrowtail} & \mathbb{P}(X) \\ {\scriptstyle T_f}\uparrow & /// & \uparrow{\scriptstyle \mathbb{P}(f)} \\ T_Y & \underset{\tau_Y}{\rightarrowtail} & \mathbb{P}(Y). \end{array}$$

In this case we write $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$.

Lemma 2.1. *Given* S-*topological spaces* (X, τ_X), (Y, τ_Y) *and* (Z, τ_Z),

(1) *the identity morphism* $(X, \tau_X) \xrightarrow{1_X} (X, \tau_X)$ *is* S-*continuous.*

(2) *the composition of the* S-*continuous morphisms* $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$ *and* $(Y, \tau_Y) \xrightarrow{g} (Z, \tau_Z)$ *is* S-*continuous.*

Proof. Similar to the proof given in [2]. □

Proposition 2.1. *The* S-*topologiacl spaces together with* S-*continuous maps form a category.*

Proof. Follows from 2.1. □

The category of S-topological spaces and S-continuous maps is denoted by $\mathsf{S}Top$. It can be easily verified that,

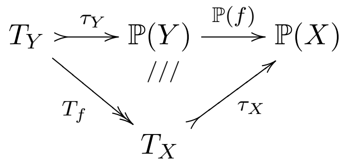
Theorem 2.1. *The mapping* $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ *taking* $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ *to* $f : X \longrightarrow Y$ *is a faithful functor.*

By the above lemma the category $\mathsf{S}Top$ is concrete over $\mathcal{E}$.

3. Induced, Coinduced, Discrete and Indiscrete S-topologies

In this section we show that under certain conditions the induced, coinduced, discrete and indiscrete S-topologies exist in $\mathsf{S}Top$ by actually constructing such entities. We also prove that certain facts similar to those in the category Top of topological spaces hold in $\mathsf{S}Top$. To this end writing an E-morphism $e : X \longrightarrow Y$ in an $(\mathsf{E}, \mathsf{Mono})$-structured category as $e : X \twoheadrightarrow Y$, we have.

Theorem 3.1. *Suppose $\mathcal{C}$ is an $(\mathsf{E}, \mathsf{Mono})$-structured category, the square functor preserves E-morphisms and the functor P preserves both E-morphisms and monos. If $f : X \longrightarrow Y$ is an $\mathcal{E}$-morphism and τ_Y is an S-topology on Y, then τ_X defined by the mono part of $\mathbb{P}(f)\tau_Y$ as shown below,*



is the smallest S-topology on X making f S-continuous.

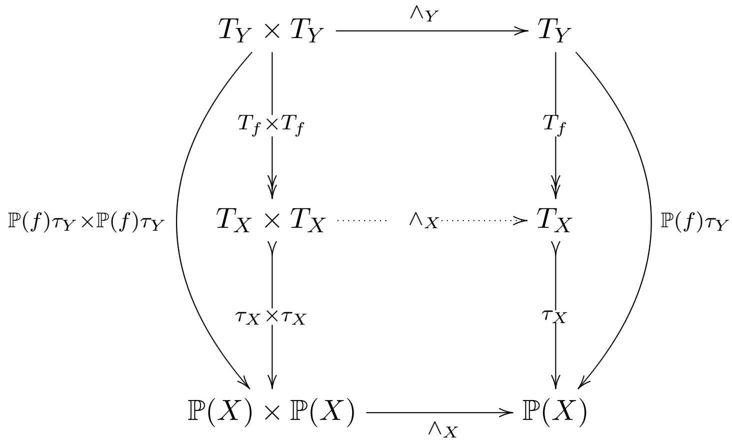
Proof. Set $b_X = T_f b_Y : 1 \longrightarrow T_X$. We have:

$$\tau_X b_X = \tau_X T_f b_Y = \mathbb{P}(f)\tau_Y b_Y = \mathbb{P}(f) b_Y = b_X$$

Similarly $t_X = T_f t_Y : 1 \longrightarrow T_Y$ satisfies the required equality. Now we have,

$$\tau_X T_f \wedge_Y = \mathbb{P}(f)\tau_Y \wedge_Y = \mathbb{P}(f) \wedge_Y (\tau_Y \times \tau_Y) = \\ \wedge_X (\mathbb{P}(f) \times \mathbb{P}(f))(\tau_Y \times \tau_Y) = \wedge_X(\tau_X \times \tau_X)(T_f \times T_f)$$

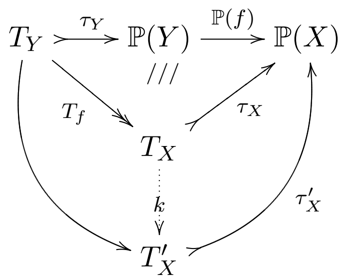
so that the following square commutes.



By the diagonal property of the factorization structure there is a (unique) morphism $\wedge_X : T_x \times T_X \longrightarrow T_X$ making the top and bottom squares commute. The existence of $\vee_X : P(T_X) \longrightarrow T_X$ follows similarly. This proves τ_X is an S-topology.

The commutativity of the above triangle defining τ_X yields the S-continuity of f.

To show that τ_X is the smallest such S-topology, suppose τ'_X also renders f S-continuous. So we have the following commutative triangles.

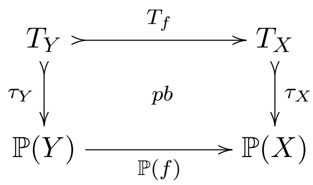


The diagonal property of the factorization structure gives a map $k : T_X \longrightarrow T'_X$ making the left ant right triangles in the above diagram commutative. The commutativity of the right one yields $\tau_X \leq \tau'_X$.

□

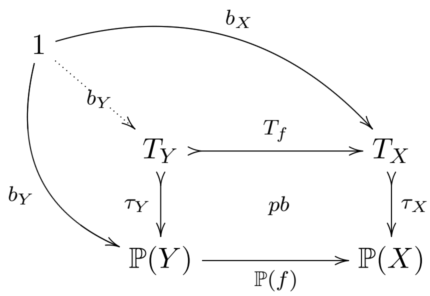
The S-topology on X in the above theorem is called the S-topology induced by f and τ_Y and is denoted by τ^i_X.

Theorem 3.2. *If $f : X \longrightarrow Y$ is an $\mathcal{E}$-morphism and τ_X is an S-topology on X, then τ_Y defined by the pullback of τ_X along $\mathbb{P}(f)$ as shown below,*



is the largest S-topology on Y making f S-continuous.

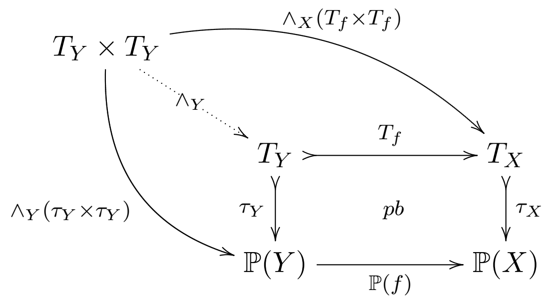
Proof. Since the outer square in the diagram



commutes, there is a unique b_Y making the upper and lower triangles commutative. The commutativity of the lower one shows that $b_Y : 1 \longrightarrow \mathbb{P}(Y)$ factors through $\tau_Y : T_Y \rightarrowtail \mathbb{P}(Y)$ as required. Similarly there is a morphism $t_Y : 1 \longrightarrow T_Y$ such that $\tau_Y t_Y = t_Y$. Since

$$\mathbb{P}(f) \wedge_Y (\tau_Y \times \tau_Y) = \wedge_X(\mathbb{P}(f) \times \mathbb{P}(f))(\tau_Y \times \tau_Y) = \wedge_X(\mathbb{P}(f)\tau_Y \times \mathbb{P}(f)\tau_Y) =$$
$$\wedge_X(\tau_X T_f \times \tau_X T_f) = \wedge_X(\tau_X \times \tau_X)(T_f \times T_f) = \tau_X \wedge_X (T_f \times T_f),$$

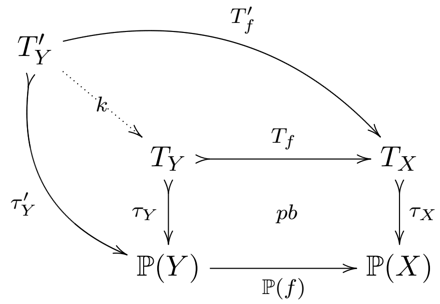
the following outer square commutes.



So there is a unique morphism $\wedge_Y : T_Y \times T_Y \longrightarrow T_Y$ making the two triangles commutative. The commutativity of the lower triangle gives the required commutative square. The existence of $\vee_Y : P(T_Y) \longrightarrow T_Y$ making the required square commutative can be proved similarly. This proves that τ_Y is an S-topology on Y.

The pullback square defining τ_Y shows that f is S-continuous.

To show that τ_Y is the largest S-topology on Y making f S-continuous, suppose τ'_Y is another such S-topology. It follows that there is a map $T'_f : T'_Y \longrightarrow T_X$ rendering commutative the outer square in the following diagram.

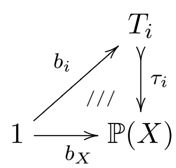


Therefore there is a morphism k making the two triangles commutative. The commutativity of the lower one gives $\tau'_Y \leq \tau_Y$ as desired. □

The S-topology on Y in the above theorem is called the S-topology coinduced by f and τ_X and is denoted by τ^c_Y.

Proposition 3.1. *Let $\mathcal{C}$ have intersections. For $X \in \mathcal{E}$, the intersection of any collection of structural topologies on X is a structural topology.*

Proof. Let $\{\tau_i : T_i \rightarrowtail \mathbb{P}(X) : i \in I\}$ be a collection of structural topologies on X. Set $\tau = \bigcap_{i \in I} \tau_i : T \rightarrowtail \mathbb{P}(X)$. Since τ_i is an S-topology, we have $b_i : 1 \longrightarrow T_i$ making the following triangle commutative.



So $b_X \leq \tau_i$ for all i, yielding $b_X \leq \tau$. Hence there is a morphism $b : 1 \longrightarrow T$ such that $b_X = \tau b$. Similarly there is a morphism $t : 1 \longrightarrow T$ such that $t_X = \tau t$. Also for each i we have $\wedge_i : T_i \times T_i \longrightarrow T_i$ rendering commutative the following square.

$$\begin{array}{ccc} T_i \times T_i & \xrightarrow{\wedge_i} & T_i \\ {\scriptstyle \tau_i \times \tau_i} \downarrow & /// & \downarrow {\scriptstyle \tau_i} \\ \mathbb{P}(X) \times \mathbb{P}(X) & \xrightarrow[\wedge_X]{} & \mathbb{P}(X) \end{array}$$

On the other hand Since $\tau \leq \tau_i$, we have a morphism k_i such that $\tau = \tau_i k_i$. Therefore $\wedge_X(\tau \times \tau) = \wedge_X(\tau_i \times \tau_i)(k_i \times k_i) = \tau_i \wedge_i (k_i \times k_i)$, implying $\wedge_X(\tau \times \tau) \leq \tau_i$ for all i. Thus $\wedge_X(\tau \times \tau) \leq \tau$. Hence there is a morphism $\wedge : T \times T \longrightarrow T$ such that $\wedge_X(\tau \times \tau) = \tau\wedge$ as desired. Similarly we have a morphism $\vee : P(T) \longrightarrow T$ such that $\vee_X P(\tau) = \tau\vee$, completing the proof. □

Remark 3.1. *Notice that in 3.1, the weaker assumption that the collection $Sub(\mathbb{P}(X))$ to have intersections does not suffice, because in $\wedge_X(\tau \times \tau) \leq \tau_i$, the morphism $\wedge_X(\tau \times \tau)$ is not a monomorphism.*

Corollary 3.1. *Let $\mathcal{C}$ have intersections. Given a subobject $m : M \rightarrowtail \mathbb{P}(X)$, there is a smallest structural topology on X containing m.*

Proof. An intersection of structural topologies that contain m is by 3.1 a structural topology; it obviously contains m. □

The S-topology in the above corollary is called the S-topology generated by m and is denoted by $\langle m \rangle$.

Theorem 3.3. *Let $\mathcal{C}$ have intersections and X be an object of $\mathcal{E}$.*

a) *The identity morphism $1_{\mathbb{P}(X)} : \mathbb{P}(X) \rightarrowtail \mathbb{P}(X)$ is an S-topology on X.*

b) *The intersection of all the S-topologies on X is an S-topology on X.*

Proof. **(a)** follows easily and **(b)** follows from 3.1. □

The S-topologies in parts (a) and (b) of the above lemma are called respectively, the discrete and the indiscrete S-topologies on X and are denoted by τ_X^{dis} and τ_X^{ind}. Note that since every S-topology on X contains b_X and t_X, it contains $\langle b_X \vee t_X \rangle$. It follows that $\tau_X^{ind} = \langle b_X \vee t_X \rangle$.

Theorem 3.4. *Let $\mathcal{C}$ have intersections.*

a) *The mapping $D : \mathcal{E} \to \mathsf{S}Top$ by $D(X) = (X, \tau_X^{dis})$ and $D(f) = f$ is a full and faithful functor called the discrete functor.*

b) *The mapping $I : \mathcal{E} \to \mathsf{S}Top$ by $I(X) = (X, \tau_X^{ind})$ and $I(f) = f$ is a full and faithful functor called the indiscrete functor.*

Lemma 3.1. *The identity morphism $1_X : (X, \tau_X) \longrightarrow (X, \tau'_X)$ is S-continuous if and only if $\tau'_X \leq \tau_X$.*

Proof. Straightforward. □

Corollary 3.2. *The functor* $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ *has complete fibres.*

Proof. Follows easily from 3.1 □

Theorem 3.5. *Let* $\mathcal{C}$ *have intersections,* $f : X \longrightarrow Y$ *be an* $\mathcal{E}$*-morphism,* τ_X *and* τ_Y *be* S*-topologies on* X *and* Y*, respectively. Then*

a) $f : (X, \tau_X^{dis}) \longrightarrow (Y, \tau_Y)$ *is* S*-continuous.*

b) $f : (X, \tau_X) \longrightarrow (Y, \tau_Y^{ind})$ *is* S*-continuous.*

Proof. **(a)** The commutative square

$$\begin{array}{ccc} T_Y & \xrightarrow{\tau_Y} & \mathbb{P}(Y) \\ {\scriptstyle \mathbb{P}(f)\tau_Y}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(f)} \\ \mathbb{P}(X) & \xrightarrow[1_{\mathbb{P}(X)}]{} & \mathbb{P}(X) \end{array}$$

proves the S-continuity of f.

(b) Let τ_Y^c be the coinduced S-topology on Y by f and τ_X, so that the morphism $f : (X, \tau_X) \longrightarrow (Y, \tau_Y^c)$ is S-continuous. Since τ_Y^{ind} is the intersection of all S-topologies on Y, $\tau_Y^{ind} \leq \tau_Y^c$. Therefore by 3.1, $1_Y : (Y, \tau_Y^c) \longrightarrow (Y, \tau_Y^{ind})$ is S-continuous. Since the composition of S-continuous morphisms is S-continuous, the result follows. □

Corollary 3.3. *Let* $\mathcal{C}$ *have intersections.*

a) *The discrete functor* $D : \mathcal{E} \to \mathsf{S}Top$ *is a left adjoint of* $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ *.*

b) *The indiscrete functor* $I : \mathcal{E} \to \mathsf{S}Top$ *is a right adjoint of* $U : \mathsf{S}Top \longrightarrow \mathcal{E}$ *.*

c) *Functors* D *and* I *are full embedding and* $U \circ D = U \circ I = I_{\mathcal{E}}$*.*

Proof. Follows easily from Theorem 3.5. □

Corollary 3.4. *The concrete category* $\mathsf{S}Top$ *has an initial object and a terminal object, if* $\mathcal{E}$ *does.*

Proof. Follows from 3.3. □

For standard topology as well as types one, three and four fuzzy topologies the join $b_X \vee t_X$ is a structural topology , while for types two and five it is not, see [2]. To see when $b_X \vee t_X$ is a structural topology, we have,

Theorem 3.6. *Let* $\mathcal{C}$ *be an* $(\mathsf{E}, \mathsf{Mono})$*-structured category for some collection* E *of morphisms, the square functor* S *preserves* E*-morphisms and* P *preserves* E*-morphisms and monos. If for a structural topological space* (X, τ_X)*, the binary coproduct of* b_X *and* t_X *exists and there are morphisms* $\wedge$ *and* $\vee$ *making the following squares commutative,*

$$\begin{array}{ccc} (1\sqcup 1)\times(1\sqcup 1) & \xrightarrow{\wedge} & 1\sqcup 1 \\ \Big\downarrow{\scriptstyle (b_X\oplus t_X)\times(b_X\oplus t_X)} & & \Big\downarrow{\scriptstyle b_X\oplus t_X} \\ \mathbb{P}(X)\times\mathbb{P}(X) & \xrightarrow[\wedge_X]{} & \mathbb{P}(X) \end{array} \qquad \begin{array}{ccc} P(1\sqcup 1) & \xrightarrow{\vee} & 1\sqcup 1 \\ \Big\downarrow{\scriptstyle P(b_X\oplus t_X)} & & \Big\downarrow{\scriptstyle b_X\oplus t_X} \\ P(\mathbb{P}(X)) & \xrightarrow[\vee_X]{} & \mathbb{P}(X) \end{array}$$

then $b_X \vee t_X : 1 \vee 1 \longrightarrow \mathbb{P}(X)$ *is a structural topology. In this case* $b_X \vee t_X = \langle b_X \vee t_X\rangle$ *is the indiscrete* S*-topology on* X*.*

Proof. Replacing $b_X \oplus t_X$ by its $(\mathsf{E}, \mathsf{Mono})$-factorization $(b_X \vee t_X)e$, see 1.1, we get the following commutative outer squares, and so by diagonal property we get $\wedge_{1\vee 1}$ and $\vee_{1\vee 1}$ making the top and bottom squares commutative.

$$\begin{array}{ccc} (1\sqcup 1)\times(1\sqcup 1) & \xrightarrow{\wedge} & 1\sqcup 1 \\ \Big\downarrow{\scriptstyle e\times e} & & \Big\downarrow{\scriptstyle e} \\ (1\vee 1)\times(1\vee 1) & \xrightarrow{\wedge_{(1\vee 1)}} & (1\vee 1) \\ \Big\downarrow{\scriptstyle (b_X\vee t_X)\times(b_X\vee t_X)} & & \Big\downarrow{\scriptstyle b_X\vee t_X} \\ \mathbb{P}(X)\times\mathbb{P}(X) & \xrightarrow[\wedge_X]{} & \mathbb{P}(X) \end{array} \qquad \begin{array}{ccc} P(1\sqcup 1) & \xrightarrow{\vee} & 1\sqcup 1 \\ \Big\downarrow{\scriptstyle P(e)} & & \Big\downarrow{\scriptstyle e} \\ P(1\vee 1) & \xrightarrow{\vee_{(1\vee 1)}} & 1\vee 1 \\ \Big\downarrow{\scriptstyle P(b_X\vee t_X)} & & \Big\downarrow{\scriptstyle b_X\vee t_X} \\ P(\mathbb{P}(X)) & \xrightarrow[\vee_X]{} & \mathbb{P}(X) \end{array}$$

On the other hand we have $b_X = (b_X \vee t_X)e\nu_1$ and $t_X = (b_X \vee t_X)e\nu_2$. Hence $b_X \vee t_X$ is a structural topology. The last assertion follows from 3.1 and 3.3.

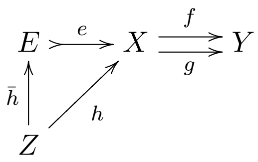
□

4. **Limits in the Category** $\mathsf{S}Top$

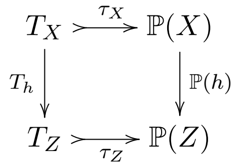
In this section we show that $\mathsf{S}Top$ has concrete (finite) limits if the category $\mathcal{E}$ has (finite) limits. We assume that the category $\mathcal{C}$ has intersections and since $\mathcal{C}$ is also assumed to be finitely complete, by dual of Theorem 14.17 of [1], it follows that $\mathcal{C}$ is $(\mathsf{ExtEpi}, \mathsf{Mono})$-structured, where ExtEpi is the collection of extremal epimorphisms. We also assume that the square functor preserves ExtEpi-morphisms and the functor P preserves both ExtEpi-morphisms and monos.

Proposition 4.1. *If* $\mathcal{E}$ *has equalizers, then* $\mathsf{S}Top$ *has concrete equalizers.*

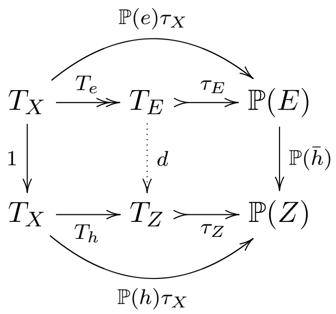
Proof. Let $(X, \tau_X) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \tau_Y)$ be parallel morphisms in $\mathsf{S}Top$ and $e : E \rightarrowtail X$ be an equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{E}$. We show that $e : (E, \tau^i_E) \longrightarrow (X, \tau_X)$ is an equalizer of $(X, \tau_X) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \tau_Y)$ in $\mathsf{S}Top$. So suppose $h : (Z, \tau_Z) \longrightarrow (X, \tau_X)$ is given such that $fh = gh$. Since $e : E \longrightarrow X$ is an equalizer of the pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{E}$, there is a unique morphism $\bar{h} : Z \longrightarrow E$ making the triangle in the following diagram commutative.



Need to show $\bar{h} : (Z, \tau_Z) \longrightarrow (E, \tau_E)$ is S-continuous. By S-continuity of h, there is a morphism T_h making the following square commutative.



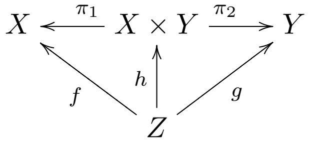
In the following diagram, the commutativity of the lower triangle is implied by the commutativity of the above square; the commutativity of the upper triangle is implied by the definition of the induced topology, see 3.1; and the commutativity of the outer square follows from the equality $\mathbb{P}(h) = \mathbb{P}(e \circ \bar{h}) = \mathbb{P}(\bar{h}) \circ \mathbb{P}(e)$.



So the diagonal morphism d exists rendering commutative the left and right squares in the above diagram. The commutativity of the right one shows that $\bar{h}$ is S-continuous. Uniqueness follows from faithfulness of $U : \mathsf{S}Top \longrightarrow \mathcal{E}$. □

Proposition 4.2. *Suppose $\mathcal{C}$ has binary coproducts. If $\mathcal{E}$ has binary products, then $\mathsf{S}Top$ has concrete binary products.*

Proof. Let (X, τ_X) and (Y, τ_Y) be two structural topological spaces and the 2-source $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ be a product in $\mathcal{E}$. Given $(X, \tau_X) \xleftarrow{f} (Z, \tau_Z) \xrightarrow{g} (Y, \tau_Y)$ in $\mathsf{S}Top$, there is a unique map $h = \langle f, g \rangle$ in $\mathcal{E}$ rendering commutative the following diagram.



So the diagram.

$$\begin{array}{ccccc} \mathbb{P}(X) & \xrightarrow{\mathbb{P}(\pi_1)} & \mathbb{P}(X\times Y) & \xleftarrow{\mathbb{P}(\pi_2)} & \mathbb{P}(Y) \\ & {\scriptstyle \mathbb{P}(f)}\searrow & \downarrow{\scriptstyle \mathbb{P}(h)} & \swarrow{\scriptstyle \mathbb{P}(g)} & \\ & & \mathbb{P}(Z) & & \end{array}$$

commutes. Since f and g are S-continuous, there are morphisms T_f and T_g making the following squares commutative.

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\rightarrowtail} & \mathbb{P}(X) \\ {\scriptstyle T_f}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(f)} \\ T_Z & \overset{\tau_Z}{\rightarrowtail} & \mathbb{P}(Z) \\ {\scriptstyle T_g}\uparrow & & \uparrow{\scriptstyle \mathbb{P}(g)} \\ T_Y & \overset{\tau_Y}{\rightarrowtail} & \mathbb{P}(Y) \end{array}$$

Putting the S-topologies τ_1 and τ_2 on $X \times Y$ induced by π_1 and π_2 respectively, by 3.1 we have the following commutative diagrams.

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\rightarrowtail} & \mathbb{P}(X) \\ {\scriptstyle T_{\pi_1}}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(\pi_1)} \\ T_1 & \underset{\tau_1}{\rightarrowtail} & \mathbb{P}(X\times Y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T_Y & \overset{\tau_Y}{\rightarrowtail} & \mathbb{P}(Y) \\ {\scriptstyle T_{\pi_2}}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(\pi_2)} \\ T_2 & \underset{\tau_2}{\rightarrowtail} & \mathbb{P}(X\times Y) \end{array}$$

The diagonal property of the factorization system yields morphisms n_1 and n_2 making triangles in the following diagram commutative.

$$\begin{array}{ccc} T_X & \overset{T_{\pi_1}}{\twoheadrightarrow} & T_1 \\ {\scriptstyle T_f}\downarrow & \swarrow{\scriptstyle n_1} & \downarrow{\scriptstyle \mathbb{P}(h)\circ\tau_1} \\ T_Z & \underset{\tau_Z}{\rightarrowtail} & \mathbb{P}(Z) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T_Y & \overset{T_{\pi_2}}{\twoheadrightarrow} & T_2 \\ {\scriptstyle T_g}\downarrow & \swarrow{\scriptstyle n_2} & \downarrow{\scriptstyle \mathbb{P}(h)\circ\tau_2} \\ T_Z & \underset{\tau_Z}{\rightarrowtail} & \mathbb{P}(Z) \end{array}$$

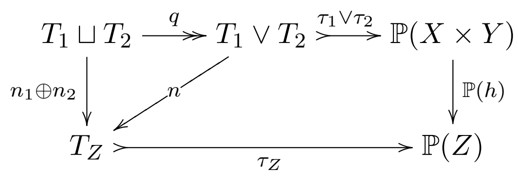
The coproduct of T_1 and T_2 yields the morphisms $\tau_{\pi_1} \oplus \tau_{\pi_2}$ and $n_1 \oplus n_1$ as follows.

$$\begin{array}{ccccc} T_1 & \xrightarrow{\nu_1} & T_1 \sqcup T_2 & \xleftarrow{\nu_2} & T_2 \\ & {\scriptstyle \tau_1}\searrow & \downarrow{\scriptstyle \tau_1\oplus\tau_2} & \swarrow{\scriptstyle \tau_2} & \\ & & \mathbb{P}(X\times Y) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} T_1 & \xrightarrow{\nu_1} & T_1 \sqcup T_2 & \xleftarrow{\nu_2} & T_2 \\ & {\scriptstyle n_1}\searrow & \downarrow{\scriptstyle n_1\oplus n_2} & \swarrow{\scriptstyle n_2} & \\ & & \mathbb{P}(Z) & & \end{array}$$

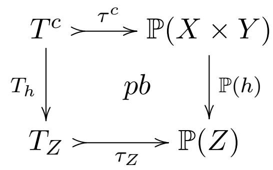
Since $\tau_Z \circ n_1 = \mathbb{P}(h) \circ \tau_1$ and $\tau_Z \circ n_2 = \mathbb{P}(h) \circ \tau_2$, we get the commutative square:

$$\begin{array}{ccc} T_1 \sqcup T_2 & \xrightarrow{\tau_1\oplus\tau_2} & \mathbb{P}(X\times Y) \\ {\scriptstyle n_1\oplus n_2}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(h)} \\ T_Z & \underset{\tau_Z}{\rightarrowtail} & \mathbb{P}(Z) \end{array}$$

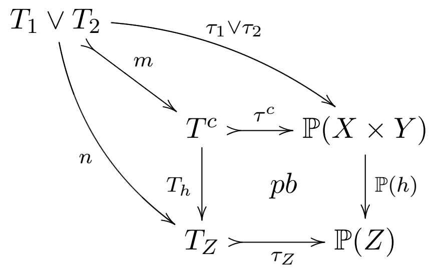
By 1.1 the mono part of the factorization of $\tau_1 \oplus \tau_2$ is the join $\tau_1 \vee \tau_2$, so that $\tau_1 \oplus \tau_2 = (\tau_1 \vee \tau_2)q$ for an epimorphism q. The above commutative square now yields the commutativity of the outer square in following diagram and the diagonal property of the factorization system gives the morphism n making the triangle and the right square commutative.



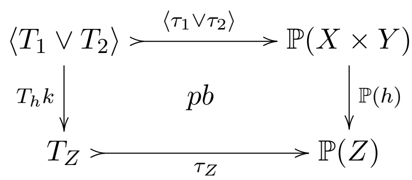
Putting the S-topology T^c on $X \times Y$ coinduced by h, by 3.2 we have the following pullback square.



Since the outer square in the following diagram commutes, there is a unique morphism m rendering commutative the two triangles.



By commutativity of the above upper triangle, we get $\tau_1 \vee \tau_2 \leq \tau^c$, and so by 3.1, $\langle \tau_1 \vee \tau_2 \rangle \leq \tau^c$. Therefore there is a morphism k such that $\langle \tau_1 \vee \tau_2 \rangle = \tau^c k$. We have $T_Z T_h k = \mathbb{P}(h)\tau^c k = \mathbb{P}(h)\langle \tau_1 \vee \tau_2 \rangle$, proving the commutativity of the square,



which proves S-continuity of the morphism h. Uniqueness follows from faithfulness of $U : \mathsf{S}Top \longrightarrow \mathcal{E}$. □

So with the assumptions made on the category $\mathcal{C}$ at the beginning of the section, we have:

Theorem 4.1. *Suppose $\mathcal{C}$ has binary coproducts. If $\mathcal{E}$ has finite limits, then $\mathsf{S}Top$ has concrete finite limits.*

An investigation of the proof of 4.2 shows that:

Proposition 4.3. *Suppose $\mathcal{C}$ has coproducts. If $\mathcal{E}$ has products, then $\mathsf{S}Top$ has concrete products.*

So we have:

Theorem 4.2. *Suppose $\mathcal{C}$ has coproducts. If $\mathcal{E}$ has limits, then $\mathsf{S}Top$ has concrete limits.*

5. Examples

It is clear that by taking different bases $(\mathbb{P}, P)$ and then considering different $(\mathbb{P}, P)$-topological structures $\mathsf{S} = (b, t, \wedge, \vee)$, the category $\mathsf{S}Top$ covers many different categories. In this section we give some examples that include some known and some not so familiar categories.

Example 5.1. *For the standard topology and the types one, two and three fuzzy topologies given in* [2]*, one can easily verify that b, t, $\wedge$ and $\vee$ are natural transformations, while t is not natural for types four and five. the categories $\mathcal{E}$ and $\mathcal{C}$ are the category Set in all the former cases and so satisfy the required conditions given in this paper. Also the functors P and S being the covariant and square functors satisfy the required conditions. So for the topological structure and the types one, two and three fuzzy topological structures, the category $\mathsf{S}Top$ is complete, implying that the category of topological spaces as well as the categories of type one, type two and type three fuzzy topological spaces are all complete; some of these are of course known facts.*

Example 5.2. *As mentioned in the previous example, the map t for type four (and five) fuzzy topologies defined in* [2] *is not a natural transformation. We give a modified type four structure by redefining $\mathbb{P} : (I^U)^{op} \longrightarrow Set$ as $\mathbb{P}(X) = \mathcal{F}_X$ and for $f : X \longrightarrow Y$ (i.e., $X \leq Y$), $\mathbb{P}(f) = - \wedge X$. Now the maps b, t, $\wedge$ and $\vee$ as defined in* [2] *are all natural. It is not hard to verify that $T_X \subseteq \mathcal{F}_X$ is a type four fuzzy topology on X if and only if $\tau_X : T_X \longrightarrow \mathcal{F}_X$ is an S-topology with respect to the modified structure $\mathsf{S} = (b, t, \wedge, \vee)$. Since I^U is a complete category, it follows that the category $\mathsf{S}Top$ is complete and therefore so is the category of type four fuzzy topological spaces.*

Since the structure for type five fuzzy topologies is the same as type four, see [2]*, similar arguments hold for the category of type five fuzzy topological spaces.*

The above example is a special case of:

Example 5.3. *Let $(X, \leq)$ be a partially ordered class such that for each $x \in X$, $\downarrow x = \{a \in X : a \leq x\}$ is a set. Suppose $(X, \leq)$ is small complete and binary meet distributes over arbitrary join. Let $\mathbb{P} : (X, \leq)^{op} \longrightarrow Set$ be the functor sending the object $x \in X$ to the set $\downarrow x$ and the morphism $x \leq y$ in $(X, \leq)$ to the function*

$-\wedge x :\downarrow y \longrightarrow \downarrow x$, *that takes each* $b \in\downarrow y$ *to* $b \wedge x \in\downarrow x$. *Let* $P : Set \longrightarrow Set$ *and* $S : Set \longrightarrow Set$ *be the covariant powerset and the square functors, respectively. For* $x \in X$, *define* $b_x : 1 \longrightarrow \downarrow x$ *by* $b_x(1) = 0$, $t_x : 1 \longrightarrow \downarrow x$ *by* $t_x(1) = x$, $\wedge_x :\downarrow x \times \downarrow x \longrightarrow \downarrow x$ *by* $\wedge_x(a, b) = a \wedge b$ *and* $\vee_x : P(\downarrow x) \longrightarrow \downarrow x$ *by* $\vee_x(A) = \bigvee_{a\in A} a$. *One can easily verify that* $b_x, t_x, \wedge_x$ *and* $\vee_x$ *form natural transformations over* x. *It is also easy to see that* $T_x \subseteq\downarrow x$ *is a structural topology on* x *if and only if* $0, x \in T_x$ *and* T_x *is closed under binary meet and arbitrary join. A morphism* $f : x \longrightarrow y$ *in* $(X, \leq)^{op}$ *(i.e.* $y \leq x$*) yields an* S*-continuous* $f : (x, t_x) \longrightarrow (y, t_y)$ *with* $T_x \subseteq\downarrow x$ *and* $T_y \subseteq\downarrow y$ *if and only if* $T_y \subseteq T_x$. *So the category* $\mathsf{S}Top$ *is equivalent to the category whose objects are* (x, t_x) *with* $t_x : T_x \subseteq\downarrow x$ *and* S*-continuous maps as explained above. Since* $(X, \leq)$ *is a complete category, so is* $\mathsf{S}Top$.

As another special case of the above example we have:

Example 5.4. *Consider the class of all the sets under the inclusion. As a category it is the subcategory of Set with sets as objects and inclusions as morphisms. The category* $\mathsf{S}Top$ *is (equivalent to) the subcategory of topological spaces with topological spaces as objects and continuous inclusions as morphisms; and it is a complete category.*

As yet another spacial case of Example 5.3 consider,

Example 5.5. *Using the frame* $(P(X), \subseteq)$, *where* X *is a set, the category* $\mathsf{S}Top$ *is (equivalent to) the small category whose objects are topological spaces* (A, τ_A) *with* $A \subseteq X$; *and whose morphisms are continuous inclusions. This small category is complete and thus a complete lattice.*

Example 5.6. *Let* $\mathcal{E}$ *be a well-powered category so that for each object* $X \in \mathcal{E}$, *the collection* $Sub(X)$ *of isomorphism classes of monos to* X *is a set. Assume that* $\mathcal{E}$ *is (finitely) complete with a pullback stable initial object. Let* $\mathbb{P} : \mathcal{E}^{op} \longrightarrow Set$ *be the functor taking an object* X *to* $Sub(X)$ *and a morphism* f *to the pullback function* f^{-1}; *and let* $P : Set \longrightarrow Set$ *be the covariant powerset functor. Let binary meet in* $Sub(X)$ *be obtained by pullback and assume* $Sub(X)$ *has pullback stable arbitrary joins. Let* $b_X : 1 \longrightarrow Sub(X)$ *take the point to* $[!_X]$, *where* $!_X : 0 \longrightarrow X$ *is the unique morphism from the initial object* 0 *to* X; *and let* $t_X : 1 \longrightarrow Sub(X)$ *take the point to* $[1_X]$, *where* 1_X *is the identity morphism on* X. *One can easily verify that* $\mathsf{S} = (b, t, \wedge, \vee)$ *is a topological structure. So the category* $\mathsf{S}Top$ *whose objects are pairs* (X, t_X), *where* $t_X : T_X \rightarrowtail Sub(X)$ *with* T_X *containing* b_X, t_X *and is closed under binary meets and arbitrary joins is (finitely) complete.*

Since a topos satisfies the given conditions, $\mathcal{E}$ *can be taken to be a topos.*

Example 5.7. *Let* $\mathcal{E}$ *be a topos,* $\mathbb{P}$ *and* P *be the contravariant and covariant power object functors. Let* $\wedge_X : \mathbb{P}(X) \times \mathbb{P}(X) \longrightarrow \mathbb{P}(X)$ *be the internal binary meet,* *see* [4]*, page 201, Theorem 1 (Internal). Let* $\vee_X : \mathbb{P} \circ P(X) \longrightarrow \mathbb{P}(X)$ *be the internal union, see* [5]*, Corollaries 2.7 and 2.8 for definition and naturality. By Proposition 7, on page 196 of* [4]*, the square funcor preserves epis.* P *preserves monos, see Corollary 3 on page 175 of* [4]*. Assuming* P *preserves epis, since* $\mathcal{E}$ *is finitely complete, so is* $\mathsf{S}Top$*.*

Example 5.8. *With* Top *the category of topological spaces, let* $\mathbb{P} : Top^{op} \longrightarrow Set$ *take a topological space* (X, τ_X) *to* τ_X *and a continuous function* f *to the inverse image function* f^{-1}*; and let* $P : Set \longrightarrow Set$ *be the covariant powerset functor. Let* $b_{(X,\tau_X)} : 1 \longrightarrow \tau_X$ *and* $t_{(X,\tau_X)} : 1 \longrightarrow \tau_X$ *take the point to* $\emptyset$ *and* X*, respectively. With binary intersection as meet and arbitrary union as join, one can verify that* $\mathsf{S} = (b, t, \wedge, \vee)$ *is a topological structure. Thus* $\mathsf{S}Top$ *is a category with objects triples* $(X, \tau_X, t_{(X,\tau_X)})$ *with* $t_{(X,\tau_X)} : T_{(X,\tau_X)} \rightarrowtail \tau_X$*, where* $T_{(X,\tau_X)}$ *is isomorphic (in* Set*) to a topology on* X *that is contained in* τ_X*; and with morphisms, those functions* $f : (X, \tau_X, t_{(X,\tau_X)}) \longrightarrow (Y, \tau_Y, t_{(Y,\tau_Y)})$ *that are continuous with respect to both* (τ_X, τ_Y) *and* $(T_{(X,\tau_X)}, T_{(Y,\tau_Y)})$*. Since* Top *is complete, so is* $\mathsf{S}Top$*.*

Example 5.9. *With Rmod the category of R-modules, let* $\mathbb{P} : Rmod \longrightarrow Rmod$ *be the identity functor and* $P : Rmod \longrightarrow Rmod$ *be the square functor* S*. For a module* M*, let* $b_M : 1 \longrightarrow M$ *and* $t_M : 1 \longrightarrow M$ *both be the inclusions from the zero module* 1 *to* M*. With the R-module homomorphism* $+ : M \times M \longrightarrow M$ *as both meet and join, one can easily show that* $\mathsf{S} = (b, b, +, +)$ *is a topological structure. The category* $\mathsf{S}Top$ *is equivalent to the category whose objects are pairs of R-modules* (M, M') *with* $M' \subseteq M$*; and morphisms* $f : (M, M') \longrightarrow (N, N')$ *those module homomorphisms* $f : M \longrightarrow N$ *that restrict to a module homomorphism* $f' : M' \longrightarrow N'$*. Since Rmod is cocomplete, the category* $\mathcal{E} = Rmod^{op}$ *is complete and therefore so is the category* $\mathsf{S}Top$*.*

6. **Conclusion**

After forming the category $\mathsf{S}Top$ of structural topological spaces and showing that it is a concrete category, we introduce and prove the existence of induced, coinduced, discrete and indiscrete structural topologies. We then prove the existence of certain limits provided that the base category has such limits and conclude that $\mathsf{S}Top$ has limits if the base category does. Finally we provide many examples to illustrate the results obtained in the manuscript and to show the diversity of the category $\mathsf{S}Top$.

For further research, one can work on colimits as well as introducing and investigating the closed objects and closure operators.

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