ON THE PROJECTIVE COVARIANT REPRESENTATIONS OF C^*-DYNAMICAL SYSTEMS ASSOCIATED WITH COMPLETELY MULTI-POSITIVE PROJECTIVE u-COVARIANT MAPS

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In această lucrare demonstrăm că o aplicaţie liniară complet multi-pozitivă proiectivă u-covariantă ρ de la A în $\mathcal{L}_B(E)$ relativ la C^*-sistemul dinamic (G, A, α) induce o reprezentare proiectivă covariantă $(\Phi_\rho, v^\rho, E_\rho)$ a lui (G, A, α) pe un C^*-modul Hilbert peste B. Apoi vom arăta că o aplicaţie liniară complet multi-pozitivă proiectivă u-covariantă nedegenerată de la o C^*-algebră A pe un C^*-modul Hilbert E peste o C^*-algebră B poate fi extinsă la o aplicaţie liniară complet multi-pozitivă pe twist-cross-produsul $A \times_\alpha^\omega G$. Ca un corolar demonstrăm că reprezentarea lui $A \times_\alpha^\omega G$ indusă de aplicaţia liniară complet multi-pozitivă proiectivă u-covariantă ρ este unitar echivalentă cu reprezentarea lui $A \times_\alpha^\omega G$ indusă de $(\Phi_\rho, v^\rho, E_\rho)$.

In this paper we prove that a completely multi-positive projective u-covariant linear map ρ from A to $\mathcal{L}_B(E)$ relative to the C^*-dynamical system (G, A, α) induces a projective covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) on a Hilbert C^*-module over B. Then we show that a completely multi-positive projective u-covariant non-degenerate linear map from a C^*-algebra A on a Hilbert C^*-module E over a C^*-algebra B can be extended to a completely multi-positive linear map on the twisted crossed product $A \times_\alpha^\omega G$. As a corollary we prove that the representation of $A \times_\alpha^\omega G$ induced by the completely multi-positive projective u-covariant linear map ρ is unitarily equivalent with the representation of $A \times_\alpha^\omega G$ induced by $(\Phi_\rho, v^\rho, E_\rho)$.

Keywords: Hilbert C^*-modules, C^*-algebras, C^*-dynamical systems, projective covariant representations, projective u-covariant completely multi-positive linear maps, twisted crossed products.

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1. Introduction

To every positive linear functional on a C^*-algebra A can be associated a cyclic representation on a Hilbert space H by GNS (Gel'fand-Naimark-Segal)

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construction. In [12], Stinespring extended this theorem for a completely positive linear map from A into $\mathcal{B}(H)$, the C^*-algebra of linear bounded operators on a Hilbert space H, in order to obtain a representation of A on another Hilbert space K. On the other hand, Paschke [11] (respectively, Kasparov [8]) showed that a completely positive linear map from A to another C^*-algebra of all adjointable operators on the Hilbert C^*-module H_B) induces a $*$-representation of A on a Hilbert B-module. Kaplan introduced in [7] the notion of multi-positive (or n-positive) linear functional on a C^*-algebra A, an $n \times n$ matrix of linear map from $M_n(A)$ (the algebra of all $n \times n$ matrices over A) to $M_n(\mathbb{C})$ and proved that a multi-positive linear functional on a C^*-algebra induces a $*$-representation of this C^*-algebra on a Hilbert space in terms of the GNS construction. Representations on Hilbert spaces are naturally generalized to representations on Hilbert C^*-modules. Heo, combined in [2] these two constructions to obtain a representation of A on a Hilbert C^*-module for completely multi-positive linear maps from A to another C^*-algebra. Using this, he obtained a representation on a Hilbert C^*-module associated with completely bounded linear maps.

By KSGNS (Kasparov-Stinespring-Gel'fand-Naimark-Segal) construction [9], to a strictly completely positive map ρ from a C^*-algebra A on a Hilbert C^*-module F over a C^*-algebra B can be associated a triple $(F_\rho, \pi_\rho, v_\rho)$ consisting of a Hilbert B-module F_ρ, a $*$-homomorphism $\pi_\rho \colon A \to \mathcal{L}_B(F_\rho)$ and an adjointable operator $v_\rho \colon F \to F_\rho$ which is unique up to a unitary equivalence. If $F = B = \mathbb{C}$, then the KSGNS construction reduces to the classical GNS construction. If $B = \mathbb{C}$ (so F is a Hilbert space), then we get the Stinespring construction. In the context of Hilbert C^*-modules the construction was given by Kasparov. In [4], Joiţa extended KSGNS construction for strict continuous completely multi-positive linear maps from a locally C^*-algebra A to $\mathcal{L}_B(E)$, the C^*-algebra of all adjointable B-module morphisms from E into E, and showed in Theorem 4.3, [4] a covariant version of this construction. In Theorem 2.1 we prove a projective generalization of this construction. Given a C^*-dynamical system (G, A, α), a completely multi-positive projective u-covariant non-degenerate linear map from A to $\mathcal{L}_B(E)$ induces a projective covariant non-degenerate representation of (G, A, α) on a Hilbert B-module which is unique up to a unitary equivalence. In Theorem 2.2 we prove that a completely multi-positive projective u-covariant linear map ρ from A to $\mathcal{L}_B(E)$ relative to the dynamical system (G, A, α) induces a projective covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) on a Hilbert C^*-module over B.

In Proposition 3.1 we show that a completely multi-positive projective u-covariant non-degenerate linear map from a C^*-algebra A on a Hilbert C^*-module E over a C^*-algebra B can be extended to a completely multi-positive linear map on the twisted crossed product $A \times_\alpha^\omega G$, as a generalization of Proposition 4.5, [4]. By Theorem 2.2 and Proposition 3.1 we prove in Corollary 3.2

that the representation of $A\times_\alpha^\omega G$ induced by the completely multi-positive projective u-covariant linear map ρ is unitarily equivalent with the representation of $A\times_\alpha^\omega G$ induced by $(\Phi_\rho, v^\rho, E_\rho)$.

2. The projective covariant representation of a C^*-dynamical system on a Hilbert C^*-module induced by a completely multi-positive projective u-covariant linear map

We remind some definitions and notations that will be used throughout the paper.

A **completely multi-positive** linear map from a C^*-algebra A into a C^*-algebra B is an $n\times n$ matrix, $[\rho_{ij}]_{i,j=1}^n$, of linear maps from A into B such that the map $\rho\colon M_n(A)\to M_n(B)$ defined by $\rho([a_{ij}]_{i,j=1}^n)=[\rho_{ij}(a_{ij})]_{i,j=1}^n$ is completely positive.

Let (G,A,α) be a C^*-dynamical system and let u be a projective unitary representation of G on a Hilbert B-module E with the multiplier ω. We say that a completely positive linear map $\rho\colon A\to\mathcal{L}_B(E)$ is **projective u-covariant** relative to the C^*- dynamical system (G,A,α) if

$$\rho(\alpha_g(a))=u_g\rho(a)u_g^* \text{ for all } a\in A \text{ and } g\in G.$$

Let u be a projective unitary representation $u\colon G\to\mathcal{U}(B)$ with the multiplier ω and let $\rho=[\rho_{ij}]_{i,j=1}^n$ be a multi-positive linear map from A into B. The map ρ may be considered as a map from $M_n(A)$ into $M_n(B)$. Let $\widetilde{u}_g$ be the diagonal matrix with all the diagonal entries u_g. If the map $\rho\colon M_n(A)\to M_n(B)$ is projective $\widetilde{u}$-covariant with respect to the dynamical system $(M_n(A),G,\widetilde{\alpha})$, where the action $\widetilde{\alpha}\colon G\to\mathrm{Aut}(M_n(A))$ is induced by the action $\alpha\colon G\to\mathrm{Aut}(A)$ and defined by

$$\widetilde{\alpha}_g([a_{ij}]_{i,j=1}^n)=[\alpha_g(a_{ij})]_{i,j=1}^n, [a_{ij}]_{i,j=1}^n\in M_n(A),$$

we say that ρ is a **projective u-covariant multi-positive** linear map from A into B. We note that a multi-positive linear map $\rho=[\rho_{ij}]_{i,j=1}^n$ is projective u-covariant if and only if $\rho_{ij}(\alpha_g(a_{ij})))=u_g\rho_{ij}(a_{ij})u_g^*, i,j=1,\dots,n$ for each $[a_{ij}]_{i,j=1}^n\in M_n(A)$ and $g\in G$.

A **projective covariant representation** of a C^*- dynamical system (G,A,α) on a Hilbert B-module E is a triple (Φ,v,E), where Φ is a $*$-representation of A on E, v is a projetive unitary representation of G on E with the multiplier ω and $\Phi(\alpha_g(a))=v_g\Phi(a)v_g^*$, for all $g\in G$ and $a\in A$.

Theorem 2.1. *Let (G,A,α) be a C^*-dynamical system, let u be a projective unitary representation of G on a Hilbert module E over a C^*-algebra B with the multiplier ω and let $\rho=[\rho_{ij}]_{i,j=1}^n$ be a completely multi-positive projective u-covariant non-degenerate linear map from A to $\mathcal{L}_B(E)$.*

1. Then there is a projective covariant non-degenerate representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G,A,α), where v^ρ is a projective unitary representation with the multiplier ω and n elements $V_{\rho,i}$, $i=1,2,\dots,n$ in $\mathcal{L}_B(E,E_\rho)$ such that

(a) $\rho_{ij}(a) = V_{\rho,i}^*\Phi_\rho(a)V_{\rho,i}$, *for all* $a \in A$ *and* $i, j = 1, 2, \dots, n$;

(b) $\{\Phi_\rho(a)V_{\rho,i}\xi; a \in A, \xi \in E, i = 1, 2, \dots, n\}$ *spans a dense submodule of* E_ρ;

(c) $v_g^\rho V_{\rho,i} = V_{\rho,i}u_g$ *for all* $g \in G$ *and* $i = 1, 2, \dots, n$.

2. If F *is a Hilbert* B*-module,* (Φ, v, F) *is a projective covariant nondegenerate representation of* (G, A, α)*, where* v *is a projective unitary representation with the multiplier* ω *and* W_i, $i = 1, 2, \dots, n$ *are* n *elements in* $\mathcal{L}_B(E, F)$ *such that*

(a) $\rho_{ij}(a) = W_i^*\Phi(a)W_j$, *for all* $a \in A$ *and* $i, j = 1, 2, \dots, n$;

(b) $\{\Phi(a)W_i\xi; a \in A, \xi \in F, i = 1, 2, \dots, n\}$ *spans a dense submodule of* F;

(c) $v_gW_i = W_iu_g$ *for all* $g \in G$ *and* $i, j = 1, 2, \dots, n$,
then there is a unitary operator U *in* $\mathcal{L}_B(E_\rho, F)$ *such that*

(i) $\Phi(a)U = U\Phi_\rho(a)$, *for all* $a \in A$;

(ii) $v_gU = Uv_g^\rho$, *for all* $g \in G$;

(iii) $W_i = UV_{\rho,i}$, *for all* $i = 1, 2, \dots, n$.

Proof. 1. Let $(\Phi_\rho, V_\rho, E_\rho)$ be the KSGNS construction associated with ρ given by Theorem 3.4, [4] which satisfies 1 (a) and 1 (b).

Following the proof of Theorem 4.3, [4], we define for each $g \in G$, a linear map v_g^ρ from $(A \otimes_{alg} E)^n$ to $(A \otimes_{alg} E)^n$ by $v_g^\rho(\bigoplus_{i=1}^n (a_i \otimes \xi_i)) = \bigoplus_{i=1}^n (\alpha_g(a_i) \otimes u_g\xi_i)$. Using the fact that $[\rho_{ij}]_{i,j=1}^n$ is projective u-covariant, it is not difficult to check that v_g^ρ extends to a bounded linear map v_g^ρ from E_ρ to E_ρ and since

$$\left\langle v_g^\rho(\bigoplus_{i=1}^n (a_i \otimes \xi_i) + \mathcal{N}), \bigoplus_{i=1}^n (b_i \otimes \eta_i) + \mathcal{N} \right\rangle =$$

$$\left\langle \bigoplus_{i=1}^n (a_i \otimes \xi_i) + \mathcal{N}, v_{g^{-1}}^\rho(\bigoplus_{i=1}^n (b_i \otimes \eta_i) + \mathcal{N}) \right\rangle$$

for all $\bigoplus_{i=1}^n (a_i \otimes \xi_i), \bigoplus_{i=1}^n (b_i \otimes \eta_i) \in (A \otimes_{alg} E)^n$, $v_g^\rho \in \mathcal{L}_B(E_\rho)$ and moreover, $(v_g^\rho)^* = v_{g^{-1}}^\rho$. Also it is easy to check that the map $g \longmapsto v_g^\rho$ is a unitary representation of G on E_ρ.

We prove that v_g^ρ is a projective representation with the multiplier ω. Let $g_1, g_2 \in G$ and $\bigoplus_{i=1}^n (a_i \otimes \xi_i) \in (A \otimes_{alg} E)^n$. We have

$$\omega(g_1, g_2)v_{g_1}^\rho v_{g_2}^\rho(\bigoplus_{i=1}^n (a_i \otimes \xi_i)) = \omega(g_1, g_2)v_{g_1}^\rho(\bigoplus_{i=1}^n (\alpha_{g_2}(a_i) \otimes u_{g_2}\xi_i)) =$$

$$\omega(g_1, g_2)(\bigoplus_{i=1}^n (\alpha_{g_1}(\alpha_{g_2}(a_i)) \otimes u_{g_1}u_{g_2}\xi_i)) = \bigoplus_{i=1}^n (\alpha_{g_1g_2}(a_i) \otimes \omega(g_1, g_2)u_{g_1}u_{g_2}\xi_i) =$$

$$\bigoplus_{i=1}^{n}(\alpha_{g_1g_2}(a_i)\otimes u_{g_1g_2}\xi_i)=v^{\rho}_{g_1g_2}(\bigoplus_{i=1}^{n}(a_i\otimes\xi_i)).$$

To show that $(\Phi_\rho, v_\rho, E_\rho)$ is a covariant projective representation of (G, A, α) it remains to prove that $\Phi_\rho(\alpha_g(a)) = v^\rho_g\Phi_\rho(a)v^\rho_{g^{-1}}$ for all $g\in G$ and $a\in A$.

Let $g\in G$ and $a\in A$. We have

$$(v^\rho_g\Phi_\rho(a)v^\rho_{g^{-1}})(\bigoplus_{i=1}^{n}(a_i\otimes\xi_i)+\mathcal{N})=(v^\rho_g\Phi_\rho(a))(\bigoplus_{i=1}^{n}(\alpha_{g^{-1}}(a_i)\otimes u_{g^{-1}}\xi_i+\mathcal{N})=$$

$$v^\rho_g(\bigoplus_{i=1}^{n}(a\alpha_{g^{-1}}(a_i)\otimes u_{g^{-1}}\xi_i+\mathcal{N})=\bigoplus_{i=1}^{n}(\alpha_g(a\alpha_{g^{-1}}(a_i))\otimes u_gu_{g^{-1}}\xi_i+\mathcal{N})=$$

$$\bigoplus_{i=1}^{n}\alpha_g(a)\alpha_g(\alpha_{g^{-1}}(a_i))\otimes u_gu_{g^*}\xi_i+\mathcal{N}=\bigoplus_{i=1}^{n}\alpha_g(a)a_i\otimes\xi_i+\mathcal{N}=$$

$$\Phi_\rho(\alpha_g(a))(\bigoplus_{i=1}^{n}(a_i\otimes\xi_i)+\mathcal{N})$$

for all $\bigoplus_{i=1}^{n}(a_i\otimes\xi_i)\in(A\otimes_{alg}E)^n$. Hence $\Phi_\rho(\alpha_g(a))=v^\rho_g\Phi_\rho(a)v^\rho_{g^{-1}}=v^\rho_g\Phi_\rho(a)(v^\rho_g)^*$, so $(\Phi_\rho, v^\rho, E_\rho)$ is a covariant representation.

To show that condition (c) is verified, let $\xi\in E$, $g\in G$ and $i=1,\dots,n$. Then we have

$$\|v^\rho_gV_{\rho,i}\xi-V_{\rho,i}u_g\xi\|^2=\lim_\lambda\|v^\rho_g\xi^\lambda_i-V_{\rho,i}u_g\xi\|^2=$$

$$\lim_\lambda\|\left\langle\xi,\rho_{ii}(e^2_\lambda)\xi\right\rangle+\langle\xi,\xi\rangle-\langle\rho_{ii}(\alpha_g(e_\lambda))u_g\xi,u_g\xi\rangle-\langle u_g\xi,\rho_{ii}(\alpha_g(e_\lambda))u_g\xi\rangle\|\le$$

$$\lim_\lambda\|\left\langle\xi,\rho_{ii}(e_\lambda)\xi\right\rangle+\langle\xi,\xi\rangle-\langle\rho_{ii}(e_\lambda)\xi,\xi\rangle-\langle\xi,\rho_{ii}(e_\lambda)\xi\rangle\|=\lim_\lambda\|\left\langle\xi-\rho_{ii}(e_\lambda)\xi,\xi\right\rangle\|=0.$$

Hence condition (c) is also verified.

2. Using the fact that $\rho_{ij}(a)=V^*_{\rho,i}\Phi_\rho(a)V_{\rho,j}=W^*_i\Phi(a)W_j$ for all $a\in A$ and $i,j=1,\dots,n$, it is not difficult to check that $\|\sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi_\rho(a_s)V_{\rho,i}\xi_s\|=\|\sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi(a_s)W_i\xi_s\|$ for all $\alpha_1,\dots,\alpha_n\in\mathbf{C}$, $a_1,\dots,a_m\in A$ and $\xi_1,\dots,\xi_m\in E$. Therefore, the linear map $\Phi_\rho(a)V_\rho\xi\longmapsto\Phi(a)W\xi$ from the submodule of E_ρ generated by $\left\{\Phi_\rho(a)V_{\rho,i}\xi; a\in A,\xi\in E, i=1,\dots,n\right\}$ to the submodule of F generated by $\left\{\Phi(a)W_i\xi; a\in A,\xi\in E, i=1,\dots,n\right\}$ extends to a surjective isometric B-linear map U from E_ρ onto F. Then, by Theorem 3.5, [9], U is unitary. We define this unitary operator U in $\mathcal{L}(E_\rho, E)$ by

$$U(\sum_{s=1}^{m}\alpha\Phi_\rho(a_s)V_{\rho,i}\xi_s)=\sum_{s=1}^{m}\alpha\Phi(a_s)W_i\xi_s,\forall a_1,\dots,a_m\in A,\forall\xi_1,\dots,\xi_m\in E.$$

Let $a \in A$. From

$$\Phi(a)U(\Phi_\rho(b)V_{\rho,i}\xi) = \Phi(a)\Phi(b)W_i\xi = \Phi(ab)W_i\xi =$$

$$U(\Phi_\rho(ab)V_{\rho,i}\xi) = U\Phi_\rho(a)(\Phi_\rho(b)V_{\rho,i}\xi)$$

for all $b \in A$, $\xi \in E$ and $i = 1, 2, \ldots, n$, we conclude that $\Phi(a)U = U\Phi_\rho(a)$.

Since Φ and Φ_ρ are non-degenerate, by Proposition 4.2, [3], we have $UV_{\rho,i}\xi = \lim_\lambda U\Phi_\rho(e_\lambda)V_{\rho,i}\xi = \lim_\lambda \Phi(e_\lambda)W_i\xi = W_i\xi$ for all $\xi \in E$ and $i = 1, 2, \ldots, n$. Therefore $W_i = UV_{\rho,i}$.

Let $g \in G, a \in A, \xi \in E, i = 1, 2, \ldots, n$. We have

$$(v_gU)(\Phi_\rho(a)V_{\rho,i}\xi) = v_g(\Phi(a)UV_{\rho,i}\xi)v_g(\Phi(a)W_i\xi) = \Phi(a)v_gW_i\xi =$$

$$\Phi(a)W_iu_g\xi = U(\Phi_\rho(a)V_{\rho,i}u_g\xi) = U(\Phi_\rho(a)v_g^\rho V_{\rho,i}\xi) = (Uv_g^\rho)(\Phi_\rho(a)V_{\rho,i}\xi).$$

This implies that $v_gU = Uv_g^\rho$ and thus the assertion 2 is proved. □

Let E be a Hilbert C^*-module over a C^*-algebra B. The algebraic tensor product $E \otimes_{alg} B^{**}$, where B^{**} is the enveloping W^*-algebra of B, becomes a right B^{**}-module ([10], [11], [13]) if one defines

$$(\xi \otimes b)c = \xi \otimes bc, \text{ for } \xi \in E \text{and } b, c \in B^{**}.$$

The map $[\cdot,\cdot]\colon (E \otimes_{alg} B^{**}) \times (E \otimes_{alg} B^{**}) \to B^{**}$ defined by

$$\left[\sum_{i=1}^n \xi_i \otimes b_i, \sum_{j=1}^m \eta_j \otimes c_j\right] = \sum_{i=1}^n\sum_{j=1}^m b_i^* \langle \xi_i, \eta_j\rangle c_j$$

is a B^{**}-valued inner-product on $E \otimes_{alg} B^{**}$ and the quotient module $E \otimes_{alg} B^{**}/N_E$, where $N_E = \{\zeta \in E \otimes_{alg} B^{**}; [\zeta,\zeta] = 0\}$, becomes a pre-Hilbert B^{**}-module. The Hilbert C^*-module $\overline{E \otimes_{alg} B^{**}/N_E}$ obtained by the completion of $E \otimes_{alg} B^{**}/N_E$ with respect to the norm induced by the inner product $[\cdot,\cdot]$ is called the extension of E by the C^*-algebra B^{**}. Moreover, E can be regarded as a B-submodule of $\overline{E \otimes_{alg} B^{**}/N_E}$, since the map $\xi \longrightarrow \xi \otimes 1_{B^{**}} + N_E$ from E to $\overline{E \otimes_{alg} B^{**}/N_E}$ is an isometric inclusion. The self-dual Hilbert B^{**}-module $(\overline{E \otimes_{alg} B^{**}/N_E})^\sharp$ is denoted by $\widetilde{E}$ and we can consider E as embedded in $\widetilde{E}$ without making distinction.

For an element $T \in \mathcal{L}_{B^{**}}(\widetilde{E})$ we denote by $T|_E$ the restriction of the map T on E.

We denote by $SCP^{u,p}(A, \mathcal{L}_B(E))$ the set of all strict completely positive projective u-covariant linear maps from A to $\mathcal{L}_B(E)$ and by $CP^{u,p}$ the set of all completely positive projective u-covariant linear maps from A to $\mathcal{L}_B(E)$.

Let $\rho \in SCP^{u,p}(A, \mathcal{L}_B(E))$. We denote by $C^p(\rho)$ the C^*-subalgebra of $\mathcal{L}_{B^{**}}(\widetilde{E_\rho})$ generated by

$$\left\{T \in \mathcal{L}_{B^{**}}(\widetilde{E_\rho}); T\widetilde{\Phi_\rho}(a) = \widetilde{\Phi_\rho}(a)T, T\widetilde{v_g^\rho} = \widetilde{v_g^\rho}T, \widetilde{V_\rho}^*T\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}|_E \in \mathcal{L}_B(E), \forall a \in A\right\}.$$

Theorem 2.2. *Let (G, A, α) be C^*-dynamical system, let u be a projective unitary representation of G on a Hilbert module E over a C^*-algebra B with the multiplier ω, let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely multi-positive projective u-covariant linear map with respect to the dynamical system (G, A, α) from A to $\mathcal{L}_B(E)$.*

(1) *Then there is $(\Phi_\rho, v^\rho, E_\rho)$ a projective covariant representation of (G, A, α) on a Hilbert C^*-module E_ρ over B, where v^ρ is a projective unitary representation with the multiplier ω, an isometry $V_\rho : E \to E_\rho$ and $[T_{ij}^\rho]_{i,j=1}^n \in M_n(\widetilde{\Phi_\rho}(A)' \cap \widetilde{v^\rho}(G)')$ such that*
   - (a) $\widetilde{V_\rho}^* T_{ij}^\rho \widetilde{\Phi_\rho}(a)\widetilde{V_\rho}|_E \in \mathcal{L}_B(E)$ *for all $a \in A$ and for all $i, j = 1, 2, ...n$, $\left[\widetilde{V_\rho}^* T_{ij}^\rho \widetilde{V_\rho}|_E\right]_{i,j=1}^n$ is a positive element in $M_n(\mathcal{L}_B(E))$ and $\sum_{k=1}^n T_{kk}^\rho = nI_{\mathcal{L}_{B^{**}}(\widetilde{E_\rho})}$;*
   - (b) $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ *is dense in E_ρ ;*
   - (c) $\rho_{ij}(a) = \widetilde{V_\rho}^* T_{ij}^\rho \widetilde{\Phi_\rho}(a)\widetilde{V_\rho}|_E$ *for all $a \in A$ and $i, j = \overline{1, n}$;*
   - (d) $v_g^\rho V_\rho = V_\rho u_g$ *for all $g \in G$.*

(2) *If (Ψ, w, F) is another projective covariant representation of (G, A, α) on a Hilbert C^*-module F over B, where w is a projective unitary representation with the multiplier ω, $W : E \to F$ is an isometry and $[S_{ij}]_{i,j=1}^n \in M_n(\widetilde{\Psi}(A)' \cap \widetilde{w}(G)')$ such that*
   - (a) $\widetilde{W}^* S_{ij} \widetilde{\Psi}(a)\widetilde{W}|_E \in \mathcal{L}_B(E)$ *for all $a \in A$ and for all $i, j = 1, 2, ...n$, $\left[\widetilde{W}^* S_{ij} \widetilde{W}|_E\right]_{i,j=1}^n$ is a positive element in $M_n(\mathcal{L}_B(E))$, and $\sum_{k=1}^n S_{kk} = nI_{\mathcal{L}_{B^{**}}(\widetilde{F})}$;*
   - (b) $\{\Psi(a)W\xi; a \in A, \xi \in E\}$ *is dense in F ;*
   - (c) $\rho_{ij}(a) = \widetilde{W}^* S_{ij} \widetilde{\Psi}(a)\widetilde{W}|_E$ *for all $a \in A$ and $i, j = 1, 2, ..., n$,*
   - (d) $w_g W = W u_g$ *for all $g \in G$*

   then there is a unitary operator $U : E_\rho \to F$ such that
   - (i) $\Psi(a) = U\Phi_\rho(a)U^*$ *for all $a \in A$;*
   - (ii) $W = UV_\rho$;
   - (iii) $S_{ij} = UT_{ij}^\rho U^*$ *for all $i, j = 1, 2, ...n$.*
   - (iv) $w_g = Uv_g^\rho U^*$ *for all $g \in G$.*

Proof. Setting $\rho = \frac{1}{n}\sum_{i=1}^n \rho_{ii}$, the map ρ is completely positive projective u-covariant. Then, by Theorem 2.1, there is a projective covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α), where v^ρ is a projective unitary representation with the multiplier ω and $V_\rho \in \mathcal{L}_B(E, E_\rho)$ such that

(a) $\rho(a) = V_\rho^* \Phi_\rho(a)V_\rho$, for all $a \in A$;
(b) $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ;
(c) $v_g^\rho V_\rho = V_\rho u_g$ for all $g \in G$.

For $i \neq j$ by the proof of Theorem 2.2, [6], the maps $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Re}\rho_{ij}\right)$ and $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Im}\rho_{ij}\right)$ are completely positive and $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Re}\rho_{ij}\right) \leq \rho$, $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Im}\rho_{ij}\right) \leq \rho$. On the other hand, the maps $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Re}\rho_{ij}\right)$ and $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Im}\rho_{ij}\right)$ are projective u-covariant, because ρ_{ij} are projective u- covariant. Hence $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Re}\rho_{ij}\right)$, $\frac{1}{2}\left(\rho - \frac{2}{n}\mathrm{Im}\rho_{ij}\right) \in [0,\rho]$, where $[0,\rho] = \left\{\theta \in CP^{u,p} |\ \theta \leq \rho\right\}$ with $\theta \leq \rho$ meaning that $\rho - \theta \in CP^{u,p}$. Applying Theorem 2.2, [5], there is a unique positive $T_{ij}^{\rho} \in C^p(\rho)$ such that $\rho_{ij}(a) = \widetilde{V}_\rho^* T_{ij}^{\rho} \widetilde{\Phi}_\rho(a) \widetilde{V}_\rho|_E$ for all $a \in A$.

For each $i = 1, 2, \ldots, n$, clearly $\frac{1}{n}\rho_{ii} \leq \rho$ and by the proof of Theorem 2.2, [5], there is a unique element $T_{ij}^0 \in C^p(\rho)$ such that $\frac{1}{n}\rho_{ii}(a) = \widetilde{V}_\rho^* T_{ii}^0 \widetilde{\Phi}_\rho(a) \widetilde{V}_\rho|_E$ for all $a \in A$. Let $T_{ii}^\rho = nT_{ii}^0$. Then $T_{ii}^\rho \in C^p(\rho)$ and $\rho_{ii}(a) = \widetilde{V}_\rho^* T_{ii}^\rho \widetilde{\Phi}_\rho(a) \widetilde{V}_\rho\,|_E$ for all $a \in A$. From

$$\rho\,(a) = \frac{1}{n}\sum_{k=1}^{n} \rho_{kk}\,(a) = \frac{1}{n}\sum_{k=1}^{n} \widetilde{V}_\rho^* T_{kk}^\rho \widetilde{\Phi}_\rho(a) \widetilde{V}_\rho\,|_E = \widetilde{V}_\rho^* \sum_{k=1}^{n} \frac{1}{n} T_{kk}^\rho \widetilde{\Phi}_\rho(a) \widetilde{V}_\rho\,|_E$$

for all $a \in A$ and by the proof of Theorem 2.2, [5], we deduce that $\sum\limits_{k=1}^{n}\frac{1}{n}T_{kk} = I_{\mathcal{L}_{B^{**}}(\widetilde{F})}$.

Following the proof of Theorem 2.2, [6], we show that $\left[\widetilde{V}_\rho^* T_{ij} \widetilde{V}_\rho\,|_E\right]_{i,j=1}^{n}$ is a positive element in $M_n\left(\mathcal{L}_B(E)\right)$. Let $a \in A$ and $\xi_k \in E$, $k = 1, 2, ..., n$. We have

$$\sum_{i,j=1}^{n}\left[T_{ij}\left(\Phi_\rho(a_i)V_\rho\xi_i\right), \Phi_\rho(a_j)V_\rho\xi_j\right] = \sum_{i,j=1}^{n}\left[\widetilde{V}_\rho^* T_{ij} \widetilde{\Phi}_\rho(a_j^* a_i)\widetilde{V}_\rho\xi_i, \xi_j\right] =$$

$$\sum_{i,j=1}^{n}\left\langle \widetilde{V}_\rho^* T_{ij} \widetilde{\Phi}_\rho(a_j^* a_i)\widetilde{V}_\rho\xi_i, \xi_j\right\rangle = \sum_{i,j=1}^{n}\left\langle \rho_{ij}\left(a_j^* a_i\right)\xi_i, \xi_j\right\rangle \geq 0$$

since $[\rho_{ij}]_{i,j=1}^{n}$ is completely multi-positive. From this and taking into consideration that the subspace generated by $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ is dense in E_ρ, we deduce that $\sum\limits_{i,j=1}^{n}\left[T_{ij}\left(\eta_i\right), \eta_j\right] \geq 0$ for all $\eta_1, ..., \eta_n \in E$ and then

$$\sum_{i,j=1}^{n}\left\langle \widetilde{V}_\rho^* T_{ij}\widetilde{V}_\rho\xi_i, \xi_j\right\rangle = \sum_{i,j=1}^{n}\left[\widetilde{V}_\rho^* T_{ij}\widetilde{V}_\rho\xi_i, \xi_j\right] = \sum_{i,j=1}^{n}\left[T_{ij}\widetilde{V}_\rho\xi_i, \widetilde{V}_\rho\xi_j\right] \geq 0$$

for all $\xi_1, ..., \xi_n \in E$. So we showed that $\left[\widetilde{V}_\rho^* T_{ij} \widetilde{V}_\rho\,|_E\right]_{i,j=1}^{n}$ is a positive element in $M_n\left(\mathcal{L}_B(E)\right)$.

Assertion 2. $(a), (b), (c)$ results from the proof of Theorem 2.2, [6].

We consider the linear map

$$U : \mathrm{sp}\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\} \to \mathrm{sp}\{\Psi(a)W\xi; a \in A, \xi \in E\}$$

defined by

$$U\left(\Phi_\rho(a)V_\rho\xi\right) = \Psi(a)W\xi.$$

From

$$\left\langle U\left(\Phi_\rho(a)V_\rho\xi\right), U\left(\Phi_\rho(a)V_\rho\xi\right)\right\rangle = \left\langle \Psi(a)W\xi, \Psi(a)W\xi\right\rangle = \left\langle W^*\Psi\left(a^*a\right)W\xi,\xi\right\rangle =$$

$$\left\langle \widetilde{W}^* I_{\mathcal{L}(\widetilde{F})}\widetilde{\Psi}(a^*a)\widetilde{W}\xi,\xi\right\rangle = \sum_{k=1}^{n}\frac{1}{n}\left\langle \widetilde{W}^* S_{kk}\widetilde{\Psi}(a^*a)\widetilde{W}\xi,\xi\right\rangle =$$

$$\sum_{k=1}^{n}\frac{1}{n}\left\langle \rho_{kk}(a^*a)\xi,\xi\right\rangle = \sum_{k=1}^{n}\frac{1}{n}\left\langle \widetilde{V}_\rho^* T_{kk}\widetilde{\Phi_\rho}(a^*a)\widetilde{V}_\rho\xi,\xi\right\rangle =$$

$$\left\langle \widetilde{V}_\rho^* I_{\mathcal{L}(\widetilde{E}_\rho)}\widetilde{\Phi}_\rho(a^*a)\widetilde{V}_\rho\xi,\xi\right\rangle = \left\langle \Phi_\rho(a)V_\rho\xi, \Phi_\rho(a)V_\rho\xi\right\rangle$$

for all $a \in A$ and $\xi \in E$ and taking into account that $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ is dense in E_ρ and $\{\Psi(a)W\xi; a \in A, \xi \in E\}$ is dense in F, we deduce that U extends to an unitary operator from E_ρ to F. It can be easily verified that $U\Phi_\rho\left(a\right) = \Psi\left(a\right)U$ for all $a \in A$ and $UV_\rho = W$.

Obviously, $\widetilde{U}^* S_{ij}\widetilde{U} \in C^p\left(\rho\right)$ for all $i, j = 1, 2, ..., n$. From

$$\rho_{ij}\left(a\right) = \widetilde{W}^* S_{ij}\widetilde{\Psi}(a)\widetilde{W}\,|_E = \widetilde{V}_\rho^*\widetilde{U}^* S_{ij}\widetilde{\Psi}(a)\widetilde{U}\widetilde{V}_\rho\,|_E = \widetilde{V}_\rho^*(\widetilde{U}^* S_{ij}\widetilde{U})\widetilde{\Phi}_\rho\left(a\right)\widetilde{V}_\rho\,|_E$$

for all $a \in A$ and $i, j = 1, 2, ..., n$, and from the uniqueness of the operators $T_{ij} \in C^p(\rho)$ such that $\rho_{ij}\left(a\right) = \widetilde{V}_\rho^* T_{ij}\widetilde{\Phi}_\rho\left(a\right)\widetilde{V}_\rho\,|_E$ for all $a \in A$, we deduce that $T_{ij} = \widetilde{U}^* S_{ij}\widetilde{U}$ for all $i, j = 1, 2, ..., n,$.

Let $g \in G$ and $a \in A$. Then

$$w_g U(\Phi_\rho(a)V_\rho\xi) = w_g\Psi(a)W\xi = w_g U\Phi_\rho(a)U^*UV_\rho\xi = Uv_g^\rho\Phi_\rho(a)V_\rho\xi.$$

So $w_g U = Uv_g^\rho$ for all $g \in G$, because $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ is dense in E_ρ.

Let $g \in G$. We have $w_g W = w_g UV_\rho = Uv_g^\rho V_\rho = UV_\rho u_g = Wu_g$. □

3. Extension on the twisted crossed product $A \times_\alpha^\omega G$ of a completely multi-positive projective u-covariant linear map

Proposition 3.1. *Let (G, A, α) be a unital C^*-dynamical system, let E be a Hilbert module over a C^*-algebra B and let u be a projective unitary representation of G on E with the multiplier ω. If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a completely multi-positive projective u-covariant non-degenerate linear map from A to $\mathcal{L}_B(E)$, then there is a completely multi-positive linear map $[\varphi_{ij}]_{i,j=1}^n$ from $A \times_\alpha^\omega G$ to $\mathcal{L}_B(E)$ uniquely given by*

$$\varphi_{ij}(f) = \int_G \rho_{ij}(f(g))u_g d\mu, \text{ for all } f \in C_c(G, A) \text{ and } i, j = 1, 2, \ldots, n,$$

where $C_c(G, A)$ is the set of continuous functions from G to A with compact supports. Moreover, $[\varphi_{ij}]_{i,j=1}^n$ is non-degenerate.

Proof. By Theorem 2.1, there are a projective covariant non-degenerate representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) and n elements $V_{\rho,i}$, $i = 1, \dots, n$ in $\mathcal{L}_B(E, E_\rho)$ such that $\rho_{ij}(a) = V_{\rho,i}^* \Phi_\rho(a) V_{\rho,i}$ and $v_g^\rho V_{\rho,i} = V_{\rho,i} u_g$, for all $a \in A$ and $g \in G$ and for all $i, j = 1, \dots, n$.

Let $\Phi_\rho \times v^\rho$ be the representation of $A \times_\alpha^\omega G$ associated with $(\Phi_\rho, v^\rho, E_\rho)$ (Theorem 3.3, [1]). For all $i, j = 1, \dots, n$ we define $\varphi_{ij}\colon A \times_\alpha^\omega G \to \mathcal{L}_B(E)$ by $\varphi_{ij}(f) = V_{\rho,i}^*(\Phi_\rho \times v^\rho)(f) V_{\rho,i}$. It is clear that $[\varphi_{ij}]_{i,j=1}^n$ is a completely multi-positive linear map from $A \times_\alpha^\omega G$ into $\mathcal{L}_B(E)$.

Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A and let $\xi \in E$. Then, since $\Phi_\rho \times v^\rho$ and ρ are non-degenerate,

$$\lim_\lambda \varphi_{ii}(e_\lambda)\xi = \lim_\lambda V_{\rho,i}^*(\Phi_\rho \times v^\rho)(e_\lambda) V_{\rho,i} = V_{\rho,i}^* V_{\rho,i} \xi = \xi$$

for all $i = 1, \dots, n$. So $[\varphi_{ij}]_{i,j=1}^n$ is non-degenerate.

If $f \in C_c(G, A)$, then

$$\varphi_{ij}(f) = V_{\rho,i}^*(\Phi_\rho \times v^\rho)(f) V_{\rho,i} = \int_G V_{\rho,i}^* \Phi_\rho(f(g)) v_g^\rho V_{\rho,i} dg =$$

$$\int_G V_{\rho,i}^* \Phi_\rho(f(g)) V_{\rho,i} u_g dg = \int_G \rho_{ij}(f(g)) u_g dg$$

and since $C_c(G, A)$ is dense in $A \times_\alpha^\omega G$, $[\varphi_{ij}]_{i,j=1}^n$ is unique. □

Corollary 3.2. *Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely multi-positive projective u-covariant non-degenerate linear map and let $\varphi = [\varphi_{ij}]_{i,j=1}^n$ be the unique completely multi-positive linear map from $A \times_\alpha^\omega G$ into $\mathcal{L}_B(E)$ given by Proposition 3.1. Then $(\Phi_\varphi, E_\varphi, V_\varphi, [T_{ij}^\varphi]_{i,j=1}^n)$ and $(\Phi_\rho \times v^\rho, E_\rho, V_\rho, [T_{ij}^\rho]_{i,j=1}^n)$ are unitarily equivalent.*

Proof. We show that $(\Phi_\rho \times v^\rho, E_\rho, V_\rho, [T_{ij}^\rho]_{i,j=1}^n)$ verifies statement 2 in Theorem 2.2.

We must prove that $(\Phi_\rho \times v^\rho, v_\rho, E_\rho)$ is a projective covariant representation, $V_\rho\colon E \to E_\rho$ is an isometry and $[T_{ij}^\rho]_{i,j=1}^n \in M_n(\widetilde{\Phi_\rho \times v^\rho}(A \times_\alpha^\omega G)' \cap \widetilde{v}^\rho(G)')$ that verifies $(a) - (d)$ in Theorem 2.2.

By Theorem 2.2, $[T_{ij}^\rho]_{i,j=1}^n \in M_n(\widetilde{\Phi}_\rho(A)' \cap \widetilde{v}^\rho(G)')$, so

$$T_{ij}^\rho(\widetilde{\Phi_\rho \times v^\rho})(f) = T_{ij}^\rho \int_G \widetilde{\Phi}_\rho(f(g)) \widetilde{v}_g^\rho dg = \int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g)) \widetilde{v}_g^\rho dg =$$

$$\int_G \widetilde{\Phi}_\rho(f(g)) T_{ij}^\rho \widetilde{v}_g^\rho dg = \int_G \widetilde{\Phi}_\rho(f(g)) \widetilde{v}_g^\rho T_{ij}^\rho dg =$$

$$\left(\int_G \widetilde{\Phi}_\rho(f(g)) \widetilde{v}_g^\rho dg\right) T_{ij}^\rho = (\widetilde{\Phi_\rho \times v^\rho})(f) T_{ij}^\rho,$$

which means that $[T_{ij}^\rho]_{i,j=1}^n \in M_n(\widetilde{\Phi_\rho \times v^\rho}(A \times_\alpha^\omega G)' \cap \widetilde{v}^\rho(G)')$.

By Theorem 2.2, applied to ρ, results that V_ρ is an isometry.

We verify condition 2.(a) in Theorem 2.2:

$$\widetilde{V}_\rho^* T_{ij}^\rho(\widetilde{\Phi_\rho \times v^\rho})(f)\widetilde{V}_\rho \xi = \widetilde{V}_\rho^* T_{ij}^\rho\left(\int_G \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho dg V_\rho \xi\right) =$$

$$\left(\int_G \widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho V_\rho dg\right)\xi = \left(\int_G \widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho dg\right)\xi,$$

for all $\xi \in E$.

$$\left[\widetilde{V}_\rho^*\left(\int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho dg\right)\widetilde{V}_\rho \xi, \eta\right] = \int_G \left[\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho \xi, \eta\right] dg =$$

$$\int_G \left[(\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)\xi, \eta\right] dg = \int_G \left[(\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)|_E \xi, \eta\right] dg =$$

$$\left[\int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)|_E dg \xi, \eta\right],$$

for all $\xi \in E, \eta \in \widetilde{E}$.

For an arbitrary $\eta \in \widetilde{E}$, we have

$$\widetilde{V}_\rho^*\left(\int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho dg\right)\widetilde{V}_\rho \xi = \int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)|_E dg \xi,$$

for all $\xi \in E$. So $\widetilde{V}_\rho^*\left(\int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho dg\right)\widetilde{V}_\rho E \subseteq E$.

But $\left(\widetilde{V}_\rho^*\left(\int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho dg\right)\widetilde{V}_\rho\right)|_E$ is a B-module morphism, thus $\left(\widetilde{V}_\rho^*\left(\int_G T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho dg\right)\widetilde{V}_\rho\right)|_E = \int_G \left(\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho\right)|_E dg \in \mathcal{L}_B(E)$.

The rest of the condition 2.(a) is verified by applying Theorem 2.2 to ρ.

Conditions 2.(b) and 2.(d) are obviously satisfied by Theorem 2.2 applied to ρ.

We verify condition 2.(c):

$$(\widetilde{V}_\rho^* T_{ij}^\rho(\widetilde{\Phi_\rho \times v^\rho})(f)\widetilde{V}_\rho)|_E = \widetilde{V}_\rho^*|_E T_{ij}^\rho(\widetilde{\Phi_\rho \times v^\rho})(f)|_E \widetilde{V}_\rho|_E =$$

$$\widetilde{V}_\rho^*|_E\left[T_{ij}^\rho \int_G \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho dg\right]|_E \widetilde{V}_\rho|_E = \widetilde{V}_\rho^*|_E\left(\int_G T_{ij}^\rho \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho dg\right)|_E \widetilde{V}_\rho|_E =$$

$$\widetilde{V}_\rho^*|_E \int_G (T_{ij}^\rho \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho|_E dg \widetilde{V}_\rho)|_E = \int_G \widetilde{V}_\rho^*|_E (T_{ij}^\rho \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho)|_E \widetilde{V}_\rho|_E dg =$$

$$\int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi_\rho}(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)|_E dg.$$

For $\xi \in E$ we have

$$(\widetilde{V}_\rho^* T_{ij}^\rho(\widetilde{\Phi_\rho \times v^\rho})(f)\widetilde{V}_\rho)|_E \xi = \left[\int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{v}_g^\rho \widetilde{V}_\rho)|_E dg\right]\xi =$$

$$= \left[\int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{V}_\rho \widetilde{u}_g)|_E dg\right]\xi = \left[\int_G (\widetilde{V}_\rho^* T_{ij}^\rho \widetilde{\Phi}_\rho(f(g))\widetilde{V}_\rho)|_E u_g dg\right]\xi =$$

$$\left[\int_G \rho_{ij}(f(g))u_g dg\right]\xi = \varphi_{ij}(f)\xi.$$

Thus, by Theorem 2.2, we obtain that $(\Phi_\varphi, E_\varphi, V_\varphi, [T_{ij}^\varphi]_{i,j=1}^n)$ and $(\Phi_\rho \times v^\rho, E_\rho, V_\rho, [T_{ij}^\rho]_{i,j=1}^n)$ are unitarily equivalent. □

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