A VERSION OF THE KRONECKER LEMMA

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In lucrare se prezinta o varianta a lemei lui Kronecker relativa la siruri si serii de numere reale. Rezultatele obtinute se aplica la studiul sirurilor de variabile aleatoare.

In this work it is presented a version of Kronecker lemma concerning real number series and sequences. The results obtained are applied to the study of random variable sequences.

Key words: Stolz-Cesaro lemma**,** Kronecker lemma, random variable sequences.

1. Introduction

The Kronecker lemma concerning real number series and sequences is widely used in the field of Probabilities in the study of random variable sequences. The proofs of some theorems concerning the law of large numbers and the law of the iterated logarithm for sums of independent random variables rely on the Kronecker lemma. The theorem of R.J.Tomkins [12] that establishes a relation between the law of the iterated logarithm and the law of large numbers is proven on the basis of this lemma. In the paper of Guang-Hui Cai and Hang Wu [2] relative to the law of the iterated logarithm for sums of negatively associated random variables, results are obtained by employing the Kronecker lemma.

Kronecker lemma [9] has the following statement: if the real number series $\sum_{n\geq 1} x_n$ is convergent and $(a_n)_{n\in N^*}$, is a strictly increasing sequence heaving the limit $\lim_{n\to\infty} a_n = \infty$, then it exists

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} a_k x_k = 0 \tag{1}$$

The paper wishes to present a version of this lemma having as hypothesis the strictly decreasing sequence $(a_n)_{n\in N^*}$.

2. A version of the Kronecker lemma

The main purpose of this paper is to establish the following theorem

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Theorem 1. Let $\sum_{n=1}^{\infty} x_n$ be a convergent real number series and $(a_n)_{N^*}$, $a_n \in R$, $\forall n \in N^*$ a strictly decreasing sequence.
If

a) the sequence $(a_n)_{n\in N^*}$ is convergent and $\lim_{n\to\infty} a_n = 0$, (2)

b) the sequence $\left(\sum_{k=1}^{n} a_k x_k\right)_{n\in N^*}$ is convergent and

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k x_k = 0 \tag{3}$$

then the sequence $\left(\frac{1}{a_n}\sum_{k=1}^{n} a_k x_k\right)_{n\in N}$ is convergent and

$$\lim_{n\to\infty} \frac{1}{a_n}\sum_{k=1}^{n} a_k x_k = 0. \tag{1'}$$

For the proof of the theorem 1 it is necessary the following lemma which is a version of the Stolz-Cesaro lemma.

Lemma. Given the real number sequences $(x_n)_{n\in N^*}$ and $(y_n)_{n\in N^*}$.
If
a) the sequence $(y_n)_{n\in N^*}$ is strictly decreasing,
b) the sequences $(x_n)_{n\in N^*}$ and $(y_n)_{n\in N^*}$ are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$,
c) there exists $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = l \in \overline{R}$, (4)
then it exists

$$\lim_{n\to\infty} \frac{x_n}{y_n} = l. \tag{5}$$

The proof of this lemma is well known.

Proof of Theorem 1

Given $S = \sum_{n=1}^{\infty} x_n$, $S_1 = 0$, $S_{n+1} = \sum_{k=1}^{n} x_n$, $a_0 = 0$, $b_k = a_k - a_{k-1}$, $k \in N^*$.
We have

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \left(a_k - a_{k-1}\right) = a_n,\ \forall n \in N^*. \tag{6}$$

We transform $\frac{1}{a_n}\sum_{k=1}^{n} a_k x_k$ taking into account that $x_k = S_{k+1} - S_k$:

$$\frac{1}{a_n}\sum_{k=1}^{n} a_k x_k = \frac{1}{a_n}\sum_{k=1}^{n} a_k (S_{k+1} - S_k) = \frac{1}{a_n}\sum_{k=1}^{n} \left[a_k S_{k+1} - (a_{k-1} + b_k) S_k\right]$$

$$= \frac{1}{a_n}\sum_{k=1}^{n} \left(a_k S_{k+1} - a_{k-1} S_k\right) - \frac{1}{a_n}\sum_{k=1}^{n} b_k S_k$$

$$= \frac{1}{a_n}\left(a_1 S_2 - a_0 S_1 + a_2 S_3 - a_1 S_2 + \ldots + a_{n-1} S_n - a_{n-2} S_{n-1} + a_n S_{n+1} - a_{n-1} S_n\right) - \frac{1}{a_n}\sum_{k=1}^{n} b_k S_k.$$

Considering that $a_0 = 0$, after simplifications we get:

$$\frac{1}{a_n}\sum_{k=1}^{n} a_k x_k = S_{n+1} - \frac{1}{a_n}\sum_{k=1}^{n} b_k S_k. \tag{7}$$

We shall prove that

$$\lim_{n\to\infty} \frac{1}{a_n}\sum_{k=1}^{n} b_k S_k = S. \tag{8}$$

Because $\lim_{n\to\infty} S_{n+1} = S$, from relation (7) it results relation (1`).

To verify relation (8) we shall use the lemma.
We must prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} b_k S_k = 0 \tag{9}$$

For the calculation of the limit we use the Abel transform [5]:

If $A_n = \sum_{k=1}^{n} u_k v_k$, $V_n = \sum_{k=1}^{n} v_k$ then

$$A_n = u_n V_n - \sum_{k=1}^{n-1} (u_{k+1} - u_k) V_k \qquad (10$$

Introducing $u_k = S_k$, $v_k = b_k$ in (10) and taking into account (6) we have:

$$\sum_{k=1}^{n} b_k S_k = S_n \sum_{k=1}^{n} b_k - \sum_{k=1}^{n-1} (S_{k+1} - S_k) \sum_{i=1}^{k} b_i = S_n a_n - \sum_{k=1}^{n-1} a_k x_k \tag{11}$$

The series $\sum_{n=1}^{\infty} x_n$ being convergent, the sequence of partial sums $(S_n)_{n\geq 1}$ is bounded, therefore $\lim_{n\to\infty} S_n a_n = 0$.

By hypothesis $\lim_{n\to\infty}\sum_{k=1}^{n-1} a_k x_k = 0$, thus it results that relation (9) is fulfilled.

To verify relation (8) we apply the lemma. The conditions a) and b) are given by hypothesis and by relation (9).

We verify condition c) from the lemma:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1} b_k S_k - \sum_{k=1}^{n} b_k S_k}{a_{n+1}-a_n} = \lim_{n\to\infty}\frac{b_{n+1}S_{n+1}}{b_{n+1}} = \lim_{n\to\infty} S_{n+1} = S.$$

Relation (8) has been verified and thus the theorem has been proven.

Example. Given the convergent series $\sum_{n\geq 1}\left(\frac{2n}{3^n}-\frac{n}{2^n}\right)$ and the sequence $(a_n)_{n\in N^*}$, $a_n = \frac{1}{n}$, $n\in N^*$.

The sequence is strictly decreasing and $\lim_{n\to\infty}\frac{1}{n} = 0$. It exists $\lim_{n\to\infty}\sum_{k=1}^{n} a_k x_k = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{3^k}-\frac{1}{2^k}\right) = 0$.

The conditions for Theorem 1 are fulfilled. Relation (1`) is also satisfied:

$$\lim_{n\to\infty} n\sum_{k=1}^{n}\frac{1}{k}\left(\frac{2k}{3^k}-\frac{k}{2^k}\right) = 0.$$

3. A Kronecker type limit.

Another version of the Kronecker lemma is given by the following theorem:

Theorem 2. Given the convergent real number series $\sum_{n\geq 1} x_n$ and the real number sequences $(a_n)_{n\in N^*}$ and $(b_n)_{n\in N^*}$.

If

a) The sequence $(a_n)_{n\in N^*}$ is strictly decreasing,

b) The sequences $(a_n)_{n\in N^*}$, $(b_n)_{n\in N^*}$ are convergent and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,

c) It exists $\lim\limits_{n\to\infty}\dfrac{b_{n+1}-b_n}{a_{n+1}-a_n}=l\in R$,

then the sequence $\left(\dfrac{1}{a_n}\sum\limits_{k=1}^{n}b_k x_k\right)_{n\in N^{\bullet}}$ is convergent and

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}b_k x_k=0. \tag{12}$$

Proof. The sequences $(a_n)_{n\in N^{\bullet}}$ and $(b_n)_{n\in N^{\bullet}}$ satisfy the conditions of the lemma therefore it exists $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=l$.

Let be $S_{n+1}=\sum\limits_{k=1}^{n}x_k$, $S_1=0$, $c_k=b_k-b_{k-1}$, $k\in N^{\bullet}$, $(b_0=0)$ and $S=\lim\limits_{n\to\infty}S_n$.

We transform (12) having in mind the above notations:

$$\frac{1}{a_n}\sum_{k=1}^{n}b_k x_k=\frac{1}{a_n}\sum_{k=1}^{n}b_k(S_{k+1}-S_k)=\frac{1}{a_n}\sum_{k=1}^{n}(b_kS_{k+1}-b_kS_k)=\frac{1}{a_n}\sum_{k=1}^{n}\left[b_kS_{k+1}-(c_k+b_{k-1})S_k\right]$$

$$=\frac{1}{a_n}\sum_{k=1}^{n}\left[(b_kS_{k+1}-b_{k-1}S_k)-c_kS_k\right]=\frac{1}{a_n}\left(b_nS_{n+1}-\sum_{k=1}^{n}c_kS_k\right)=\frac{b_n}{a_n}S_{n+1}-\frac{d_n}{a_n}$$

where $d_n=\sum\limits_{k=1}^{n}c_kS_k$.

But $\lim\limits_{n\to\infty}\dfrac{d_{n+1}-d_n}{a_{n+1}-a_n}=\lim\limits_{n\to\infty}\dfrac{c_{n+1}S_{n+1}}{a_{n+1}-a_n}=\lim\limits_{n\to\infty}\dfrac{b_{n+1}-b_n}{a_{n+1}-a_n}S_{n+1}=l\cdot S$.

By applying the lemma we get:

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}b_k x_k=\lim_{n\to\infty}\left(\frac{b_n}{a_n}S_{n+1}-\frac{d_n}{a_n}\right)=l\cdot S-l\cdot S=0.$$

4. Application Limits of sums of random variables

In [9] is given and proven

Theorem 3. Let be $(X_n)_{n\in N^*}$ an independent random variable sequence having the expectation $E(X_n)=0,\forall\, n\in N^{\bullet}$. and $(g_n(x))_{n\in N^{\bullet}}$ a function sequence, even and non-decreasing for $x>0$ and that satisfy one of the following conditions:

a) the function $\dfrac{x}{g_n(x)}$ does not decrease on the interval $(0,\infty)$.

b) the functions $\frac{x}{g_n(x)}$ and $\frac{g_n(x)}{x^2}$ are not increasing on the interval $(0,\infty)$.

If $(a_n)_{n\in N^*}$ is a convergent strictly positive number sequence and if the series

$$\sum_{n\geq 1}\frac{E\left(g_n\left(X_n\right)\right)}{g_n\left(a_n\right)} \tag{13}$$

is convergent then the series

$$\sum_{n\geq 1}\frac{X_n}{a_n} \tag{14}$$

is convergent *a.s.* (almost sure).

Taking into account Theorem 1,we can change the statement of Theorem 3 as following:

Theorem 3^***.** Given the conditions of Theorem 3, if $(a_n)_{n\in N^*}$ is a convergent and strictly decreasing positive number sequence having $\lim\limits_{n\to\infty} a_n = 0$, then, if $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} X_k = 0$ *a.s.*, then it exists

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} X_k = 0 \quad \text{a.s.} \tag{15}$$

Proof . (If $\{\Omega, K, P\}$ is a probability space and $X_n : \Omega \to R, n \in N^*$, then $\left(X_n(\omega)\right)_{n\in N}$, for whatever $\omega\in\Omega$, (ω fixed) is a real number sequence, so the above results can be applied to $\left(X_n(\omega)\right)_{n\in N}$ on a subset of probability 1 of Ω)

Appling theorem 3 it results that the series $\sum\limits_{n=1}^{\infty}\frac{X_n}{a_n}$ is convergent (*a.s.)*

We verify the conditions of Theorem 1.

Condition a) is given by hypothesis; for the verification of condition b) we consider $x_k = \frac{X_k}{a_k}, k \in N^*$ in Theorem 1.

It results that $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k x_k = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k \frac{X_k}{a_k} = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} X_k = 0$ *a.s.* .

Thus condition b) is satisfied.The conclusion of theorem 1 shows that it exists $\lim\limits_{n\to\infty}\frac{1}{a_n}\sum\limits_{k=1}^{n} a_k x_k = \lim\limits_{n\to\infty}\frac{1}{a_n}\sum\limits_{k=1}^{n} a_k\frac{X_k}{a_k} = \lim\limits_{n\to\infty}\frac{1}{a_n}\sum\limits_{k=1}^{n} X_k = 0,$ *a.s.* which is exactly relation (15).

Corollary. Let $(X_n)_{n\in N^\bullet}$ be an independent random variable sequence having $E(X_n)=0, \forall n\in N^\bullet$ and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} X_k = 0$, *a.s.*

If $(a_n)_{n\in N^\bullet}$ is a strictly decreasing positive number sequence having $\lim\limits_{n\to\infty} a_n = 0$ and the series

$$\sum_{n\geq 1}\frac{E|X_n|^p}{a_n^p}, \quad 0<p<2, \tag{16}$$

is convergent, then

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} X_k = 0, \ a.s..$$

Proof. The conclusion of the corollary is obtained from Theorem 3^* if we consider $g_n(x)=g(x)=|x|^p$.

If $0<p<1$ then condition a) from Theorem 3 is verified. If $1<p<2$ then condition b) from Theorem 3 is verified. If $p=1$, then the relations (13) and (14) are identical and we apply directly Theorem 3^*.

Theorem 4. Let $(X_n)_{n\geq 1}$ and $(g_n(x))_{n\geq 1}$ satisfying the conditions of Theorem 3.

If $(a_n)_{n\in N^\bullet}$ and $(b_n)_{n\in N^\bullet}$ are real number sequences fulfilling the conditions

a) the sequence $(a_n)_{n\in N^\bullet}$ is strictly decreasing and $\lim\limits_{n\to\infty} a_n = 0$

b) the sequence $(b_n)_{n\in N^\bullet}$ is convergent and $\lim\limits_{n\to\infty} b_n = 0$

c) it exist $\lim\limits_{n\to\infty}\dfrac{b_{n+1}-b_n}{a_{n+1}-a_n} = l\in R.$

d) the series $\sum\limits_{n\geq 1}\dfrac{E(g_n(X_n))}{g_n(b_n)}$ is convergent,

then $\lim\limits_{n\to\infty}\dfrac{1}{a_n}\sum\limits_{k=1}^{n} X_k = 0, \ a.s$.

Proof: From Theorem 3 it results that the series $\sum\limits_{n=1}^{\infty}\dfrac{X_n}{b_n}$ is convergent a.s.

In Theorem 2 we consider $x_k = \frac{X_k}{b_k}$. The conditions of Theorem 2 are fulfilled, thus

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} b_k x_k = \lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} b_k \frac{X_k}{b_k} = \lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n} X_k = 0,\ a.s..$$

R E F E R E N C E S

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