DISCRETE MULTIPLE RECURRENCE

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The aim of our paper is to formulate and solve problems concerning multitime multiple recurrence equations. Among the general things, we discuss in detail the cases of autonomous and non-autonomous recurrences, highlighting in particular the theorems of existence and uniqueness of solutions. Though the multitime multiple recurrences have occurred in analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc, the theoretical part about them needs further investigation.

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1. General statements

A multivariate recurrence relation is an equation that recursively defines a multivariate sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. Some simply defined recurrence relations can have very complex (chaotic) behaviors. We can use such recurrences including the Differential Transform Method to solve completely integrable first order PDEs system with initial conditions via discretization.

In this paper we shall refer to *discrete multitime multiple recurrence* (autonomous and non-autonomous), giving results regarding generic properties and existence and uniqueness of solutions (see also [7]-[8]). Also, we seek to provide a fairly thorough and unified exposition of recurrence relations in both univariate and multivariate settings. Some open problems raised in filters theory [1], [3], [5]-[6], [9]-[12], general recurrence theory [2], [4], [17], and multitime dynamical systems [13]-[16], receive here detailed answers.

Let $m \geq 1$ be an integer number. We denote $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^m$. Also, for each $\alpha \in \{1, 2, \ldots, m\}$, we denote $1_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$,

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i.e., 1_α has 1 on the position α and 0 otherwise. We use product order relation on $\mathbb{Z}^m$.

Let M be an arbitrary nonempty set and $t_1 \in \mathbb{Z}^m$ be a fixed element. For each $\alpha \in \{1, 2, \dots, m\}$, let $F_\alpha \colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \times M \to M$ be a function.

We fix $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$. A first order multitime recurrence of the type

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \in \mathbb{Z}^m, \ t \geq t_0, \ \forall \alpha \in \{1, 2, \dots, m\}, \tag{1}$$

is called a *discrete multitime multiple recurrence.*

This model of multiple recurrence can be justified by the fact that a completely integrable first order PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)), \, t \in \mathbb{R}^m$$

can be discretized as

$$x^i(t + 1_\alpha) = F^i_\alpha(t, x(t)), \, t \in \mathbb{Z}^m.$$

The initial (Cauchy) condition, for the PDE system, is translated into initial condition for the multiple recurrence.

Proposition 1.1. *If for any $(t_0, x_0) \in \{t \in \mathbb{Z}^m \mid t \geq t_1\} \times M$, there exists at least one m-sequence $x \colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$ which satisfies the recurrence* (1) *and the initial condition $x(t_0) = x_0$, then*

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \qquad \forall t \geq t_1, \ \forall x \in M, \tag{2}$$
$$\forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

Proof. Let $t \geq t_0$. The equality $x(t + 1_\beta + 1_\alpha) = x(t + 1_\alpha + 1_\beta)$ is equivalent to

$$F_\alpha(t + 1_\beta, x(t + 1_\beta)) = F_\beta(t + 1_\alpha, x(t + 1_\alpha))$$
$$\iff F_\alpha\big(t + 1_\beta, F_\beta(t, x(t))\big) = F_\beta\big(t + 1_\alpha, F_\alpha(t, x(t))\big).$$

For $t = t_0$, one obtains: $F_\alpha\big(t_0 + 1_\beta, F_\beta(t_0, x_0)\big) = F_\beta\big(t_0 + 1_\alpha, F_\alpha(t_0, x_0)\big)$. Since t_0 and x_0 are arbitrary, it follows the relations (2). □

2. Autonomous discrete multitime multiple recurrence

Let M be a nonempty set. For any function $G \colon M \to M$, we denote: $G^{(n)} = \underbrace{G \circ G \circ \ldots \circ G}_{n}$, if $n \geq 1$; and $G^{(0)} = \mathrm{Id}_M$.

Theorem 2.1. *For each $\alpha \in \{1, 2, \dots, m\}$, let $G_\alpha \colon M \to M$ be a function.*
a) Let $t_0 \in \mathbb{Z}^m$. If for any $x_0 \in M$, there exists at least one m-sequence

$$x \colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M,$$

which satisfies the recurrence equation

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall t \geq t_0, \ \forall \alpha \in \{1, 2, \dots, m\}, \tag{3}$$

and the initial condition $x(t_0) = x_0$, then

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \dots, m\}. \tag{4}$$

b) If, for any $\alpha, \beta \in \{1, 2, \dots, m\}$, the relations (4) *are satisfied, then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists a unique m-sequence $x \colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$*

which satisfies the recurrence (3) *and the initial condition* $x(t_0) = x_0$*; this sequence is defined by the composition*

$$x(t) = G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0), \quad \forall t \geq t_0. \tag{5}$$

Proof. a) The equality $x(t_0 + 1_\beta + 1_\alpha) = x(t_0 + 1_\alpha + 1_\beta)$ is equivalent to
$G_\alpha(x(t_0 + 1_\beta)) = G_\beta(x(t_0 + 1_\alpha)) \iff G_\alpha(G_\beta(x(t_0))) = G_\beta(G_\alpha(x(t_0)))$ $\iff G_\alpha \circ G_\beta(x_0) = G_\beta \circ G_\alpha(x_0)$.

Since x_0 is arbitrary, it follows the relations (4).

b) Firstly we remark that any sequence of the form (5) satisfies the relations (3) and the initial condition $x(t_0) = x_0$:

$$x(t + 1_\alpha) = G_1^{(t^1-t_0^1)} \circ \ldots \circ G_\alpha^{(t^\alpha+1-t_0^\alpha)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0); \tag{6}$$

using (4) and the relation (6), it follows

$$x(t + 1_\alpha) = G_\alpha \circ G_1^{(t^1-t_0^1)} \circ \ldots \circ G_\alpha^{(t^\alpha-t_0^\alpha)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0) = G_\alpha(x(t)).$$

The initial condition $x(t_0) = x_0$ is checked immediately.

The necessity is proved by induction on m, the number of components of the point $t = (t^1, \ldots, t^m)$.

For $m = 1$, we have $t = t^1$ and $t_0 = t_0^1$. If $t > t_0$, then

$$x(t) = x(t^1) = G_1(x(t^1 - 1)) = G_1^{(2)}(x(t^1 - 2)) =$$
$$= \ldots = G_1^{(k)}(x(t^1 - k)) = \ldots = G_1^{(t^1-t_0^1)}(x(t_0^1)) = G_1^{(t^1-t_0^1)}(x_0).$$

If $t = t_0$, then the relation $x(t) = G_1^{(t^1-t_0^1)}(x_0)$ is obvious.

Let $m \geq 2$. Suppose that the relation is true for $m-1$ and we shall prove it for m. We denote $\tilde{t} = (t^2, \ldots, t^m)$; $\tilde{t}_0 = (t_0^2, \ldots, t_0^m)$.

Let $\tilde{x}(\tilde{t}) = x(t_0^1, \tilde{t}) = x(t_0^1, t^2, \ldots, t^m)$. If $t^1 > t_0^1$, then

$$x(t) = x(t^1, \tilde{t}) = G_1(x(t^1 - 1, \tilde{t})) = G_1^{(2)}(x(t^1 - 2, \tilde{t})) =$$
$$= \ldots = G_1^{(k)}(x(t^1 - k, \tilde{t})) = \ldots = G_1^{(t^1-t_0^1)}(x(t_0^1, \tilde{t})) = G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t})).$$

We have proved that if $t^1 > t_0^1$, then $x(t) = G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t}))$; the relation is easily verified also for $t^1 = t_0^1$.

For $\alpha \in \{2, \ldots, m\}$, we denote $\tilde{1}_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{m-1}$; hence $1_\alpha = (0, \tilde{1}_\alpha)$. For $\alpha \geq 2$ and $t^1 = t_0^1$, the relations (3) become

$$x((t_0^1, \tilde{t}) + (0, \tilde{1}_\alpha)) = G_\alpha(x(t_0^1, \tilde{t})), \text{ i.e.,}$$
$$\tilde{x}(\tilde{t} + \tilde{1}_\alpha) = G_\alpha(\tilde{x}(\tilde{t})), \quad \forall \tilde{t} \geq \tilde{t}_0, \ \forall \alpha \in \{2, \ldots, m\}.$$

Obviously $\tilde{x}(\tilde{t}_0) = x(t_0^1, \tilde{t}_0) = x(t_0) = x_0$. Since $\tilde{t}$ has $m - 1$ components, from the induction hypothesis it follows

$$\tilde{x}(\tilde{t}) = G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0), \quad \forall \tilde{t} \geq \tilde{t}_0.$$

Consequently, for any $t \geq t_0$, we have

$$x(t) = G_1^{(t^1-t_0^1)}(\tilde{x}(\tilde{t})) = G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0). \qquad \square$$

Lemma 2.1. *Let* $G\colon M \to M$ *be an arbitrary function and* $t_0 \in \mathbb{Z}^m$, $\beta \in \{1, 2, \ldots, m\}$*, fixed. If, for any* $x_0 \in M$*, there exists at least one* m*-sequence* $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0 - 1_\beta\} \to M$*, which satisfies the relation*

$$x(t + 1_\beta) = G(x(t)), \quad \forall t \geq t_0 - 1_\beta, \tag{7}$$

and the condition $x(t_0) = x_0$, then G is surjective (onto).

Proof. Let $y \in M$. There exists an m-sequence $x(\cdot)$ which satisfies (7) and the condition $x(t_0) = y$. For $t = t_0 - 1_\beta$, one obtains $x(t_0) = G(x(t_0 - 1_\beta))$, hence $G(x(t_0 - 1_\beta)) = y$. Because y is arbitrary, it follows that the function G is surjective. □

Proposition 2.1. *We consider the functions $G_\alpha \colon M \to M$, $\alpha \in \{1, 2 \ldots, m\}$.*
a) Let $t_0 \in \mathbb{Z}^m$ and $\alpha_0 \in \{1, 2, \ldots, m\}$, fixed. If for any $x_0 \in M$, there exists at least one m-sequence $x \colon \{t \in \mathbb{Z}^m \mid t \geq t_0 - 1_{\alpha_0}\} \to M$, which satisfies

$$x(t + 1_\alpha) = G_\alpha(x(t)), \tag{8}$$

$$\forall t \geq t_0 - 1_{\alpha_0},\ \forall \alpha \in \{1, 2, \ldots, m\},$$

and the condition $x(t_0) = x_0$, then G_{α_0} is surjective and

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}. \tag{9}$$

b) Suppose that, for any $\alpha \in \{1, 2, \ldots, m\}$, the functions G_α are surjective and that the relations (9) are satisfied.

Let $(t_0, x_0) \in \mathbb{Z}^m \times M$ and $s \in \mathbb{Z}^m$, $s \leq t_0$. If for $a \in M$, we have $G_1^{(t_0^1 - s^1)} \circ G_2^{(t_0^2 - s^2)} \circ \ldots \circ G_m^{(t_0^m - s^m)}(a) = x_0$, then the m-sequence

$$x \colon \{t \in \mathbb{Z}^m \mid t \geq s\} \to M,$$

$$x(t) = G_1^{(t^1 - s^1)} \circ G_2^{(t^2 - s^2)} \circ \ldots \circ G_m^{(t^m - s^m)}(a), \quad \forall t \geq s, \tag{10}$$

satisfies the recurrence (8), $\forall t \geq s$, $\forall \alpha \in \{1, 2, \ldots, m\}$, and $x(t_0) = x_0$.
c) Suppose that the functions G_α are surjective and that the relations (9) are satisfied.

Then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists at least one m-sequence $x \colon \mathbb{Z}^m \to M$ which satisfies the recurrence (8), $\forall t \in \mathbb{Z}^m$, $\forall \alpha \in \{1, 2, \ldots, m\}$, and the condition $x(t_0) = x_0$.

Proof. *a*) The surjectivity of G_{α_0} follows from Lemma 2.1. The relations (9) are obtained from Theorem 2.1, *a*), considering the restriction of $x(\cdot)$ to the set $\{t \in \mathbb{Z}^m \mid t \geq t_0\}$.

b) We observe that the function $G_1^{(t_0^1 - s^1)} \circ G_2^{(t_0^2 - s^2)} \circ \ldots \circ G_m^{(t_0^m - s^m)}$ is surjective, since $t_0^\alpha - s^\alpha \geq 0$, $\forall \alpha$, and G_α are surjective. Consequently, there exists $a \in M$ such that $G_1^{(t_0^1 - s^1)} \circ G_2^{(t_0^2 - s^2)} \circ \ldots \circ G_m^{(t_0^m - s^m)}(a) = x_0$.

From Theorem 2.1, *b*), it follows that the function defined by the formula (10) is the unique m-sequence which satisfies the recurrence (8), $\forall t \geq s$, $\forall \alpha$, and the condition $x(s) = a$. For $t = t_0$, we have

$$x(t_0) = G_1^{(t_0^1 - s^1)} \circ G_2^{(t_0^2 - s^2)} \circ \ldots \circ G_m^{(t_0^m - s^m)}(a) = x_0.$$

c) Let $G = G_1 \circ G_2 \circ \ldots \circ G_m$. Since the functions G_α are surjective, it follows that the function G is surjective. Hence, there exists a function $H \colon M \to M$ such that $G \circ H = \mathrm{Id}_M$ (right inverse).

For $n \in \mathbb{N}$, we denote $P_n = \{t \in \mathbb{Z}^m \mid t \geq t_0 - n \cdot \mathbf{1}\}$; let $a_n = H^{(n)}(x_0)$. We observe that $G(a_{n+1}) = a_n$ and $G^{(n)}(a_n) = x_0$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we consider the function $y_n \colon P_n \to M$, defined by
$y_n(t) = G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}(a_n), \quad \forall t \geq t_0 - n \cdot \mathbf{1}.$

Because $G^{(n)}(a_n) = x_0$, i.e., $G_1^{(n)} \circ G_2^{(n)} \circ \ldots \circ G_m^{(n)}(a_n) = x_0$, according to step *b*), it follows that the m-sequence y_n satisfies the recurrence (8), $\forall t \in P_n$, $\forall \alpha$ and the condition $y_n(t_0) = x_0$.

We remark that $P_n \subseteq P_{n+1}$. For $t \in P_n$, we have
$y_{n+1}(t) = G_1^{(t^1-t_0^1+n+1)} \circ G_2^{(t^2-t_0^2+n+1)} \circ \ldots \circ G_m^{(t^m-t_0^m+n+1)}(a_{n+1})$
$= G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}\big(G(a_{n+1})\big)$
$= G_1^{(t^1-t_0^1+n)} \circ G_2^{(t^2-t_0^2+n)} \circ \ldots \circ G_m^{(t^m-t_0^m+n)}(a_n) = y_n(t).$
We showed that $y_{n+1}(t) = y_n(t)$, $\forall t \in P_n$. Inductively, one deduces that, for any $q \in \mathbb{N}$, we have $y_{n+q}(t) = y_n(t)$, $\forall t \in P_n$. Consequently, $y_n(t) = y_k(t)$, $\forall t \in P_{\min\{n,k\}}$.

Let us define the m-sequence $x \colon \mathbb{Z}^m \to M$: let $t \in \mathbb{Z}^m$; since $\mathbb{Z}^m = \bigcup_{n \in \mathbb{N}} P_n$, there exists $n \in \mathbb{N}$, such that $t \in P_n$. The value of the function x at t will be $x(t) = y_n(t)$.

The function $x(\cdot)$ is well defined since if $t \in P_n$ and $t \in P_k$, we have showed that $y_n(t) = y_k(t)$.

If $t \in P_n$, then $t + 1_\alpha \in P_n$. We have $x(t+1_\alpha) = y_n(t+1_\alpha) = G_\alpha(y_n(t)) = G_\alpha(x(t))$ and $x(t_0) = y_n(t_0) = x_0$. □

Proposition 2.2. *Suppose that, for the functions $G_\alpha \colon M \to M$, the relations* (9) *are satisfied.*

Let $t_0 \in \mathbb{Z}^m$ and $\alpha_0 \in \{1, 2, \ldots, m\}$, fixed. If, for any $x_0 \in M$, there exists at most one m-sequence $x \colon \{t \in \mathbb{Z}^m \mid t \geq t_0 - 1_{\alpha_0}\} \to M$, which satisfies

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall t \geq t_0 - 1_{\alpha_0},\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{11}$$

and the condition $x(t_0) = x_0$, then G_{α_0} is injective (one-to-one).

Proof. Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$. We select $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$. The functions $x, y \colon \{t \in \mathbb{Z}^m \mid t \geq t_0 - 1_{\alpha_0}\} \to M$,

$$x(t) = G_1^{(t^1-t_0^1)} \circ \ldots \circ G_{\alpha_0}^{(t^{\alpha_0}-t_0^{\alpha_0}+1)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(p), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{12}$$

$$y(t) = G_1^{(t^1-t_0^1)} \circ \ldots \circ G_{\alpha_0}^{(t^{\alpha_0}-t_0^{\alpha_0}+1)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(q), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{13}$$

are well defined (since $t^{\alpha_0} - t_0^{\alpha_0} + 1 \geq 0$), satisfy the relations (11) and $x(t_0) = G_{\alpha_0}(p) = x_0$, $y(t_0) = G_{\alpha_0}(q) = x_0$. It follows that $x(t) = y(t)$, $\forall t \geq t_0 - 1_{\alpha_0}$. For $t = t_0 - 1_{\alpha_0}$, we obtain $x(t_0 - 1_{\alpha_0}) = y(t_0 - 1_{\alpha_0})$, relation which is equivalent to $p = q$ (according to (12), (13)). Hence, the function G_{α_0} is injective. □

If $G \colon M \to M$ is a bijective function, we denote $G^{(-k)} = \left(G^{-1}\right)^{(k)}$, for $k \in \mathbb{N}$; we have $G^{(-k)} = \left(G^{(k)}\right)^{-1}$.

Proposition 2.3. *Suppose that the functions $G_\alpha\colon M \to M$ are bijective and the relations* (9) *hold. Then, for any $(t_0, x_0) \in \mathbb{Z}^m \times M$, there exists a unique solution $x\colon \mathbb{Z}^m \to M$, of the recurrence equation*

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall t \in \mathbb{Z}^m,\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{14}$$

with the condition $x(t_0) = x_0$. The m-sequence $x(\cdot)$ is defined by the relation

$$x(t) = G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0) \qquad (\forall t \in \mathbb{Z}^m). \tag{15}$$

Proof. The existence follows from Proposition 2.1, *c*).

Let $x\colon \mathbb{Z}^m \to M$ be a solution of the recurrence (14), with $x(t_0) = x_0$. For proving the uniqueness, it is sufficient to show that $x(t)$ satisfies the relation (15), $\forall t \in \mathbb{Z}^m$.

Let $s \le t_0$. We apply Theorem 2.1 for the restriction of $x(\cdot)$ to the set $\big\{t \in \mathbb{Z}^m \,\big|\, t \ge s\big\}$. It follows

$$x(t) = G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)}(x(s)), \quad \forall t \ge s.$$

For $t = t_0$, we obtain $x_0 = G_1^{(t_0^1-s^1)} \circ G_2^{(t_0^2-s^2)} \circ \ldots \circ G_m^{(t_0^m-s^m)}(x(s))$. Since the functions G_α are bijective, it follows $x(s) = G_1^{(s^1-t_0^1)} \circ \ldots \circ G_m^{(s^m-t_0^m)}(x_0)$.

Consequently, for any $t \ge s$, we have

$$x(t) = G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)}(x(s)) =$$
$$= G_1^{(t^1-s^1)} \circ G_2^{(t^2-s^2)} \circ \ldots \circ G_m^{(t^m-s^m)} \circ G_1^{(s^1-t_0^1)} \circ G_2^{(s^2-t_0^2)} \circ \ldots \circ G_m^{(s^m-t_0^m)}(x_0)$$
$$= G_1^{(t^1-t_0^1)} \circ G_2^{(t^2-t_0^2)} \circ \ldots \circ G_m^{(t^m-t_0^m)}(x_0).$$

We showed that, for any $s \le t_0$ and any $t \ge s$, the m-sequence $x(t)$ satisfies the relation (15). Since $\displaystyle\bigcup_{s\in\mathbb{Z}^m,\, s\le t_0} \big\{t \in \mathbb{Z}^m \,\big|\, t \ge s\big\} = \mathbb{Z}^m$, it follows that the relation (15) holds for any $t \in \mathbb{Z}^m$. □

Theorem 2.2. *Let M be a nonempty set. For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function $G_\alpha\colon M \to M$ and we associate the recurrence equation*

$$x(t + 1_\alpha) = G_\alpha(x(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{16}$$

The following statements are equivalent:

i) For any $\alpha \in \{1, 2, \ldots, m\}$, the functions G_α are bijective and

$$G_\alpha \circ G_\beta = G_\beta \circ G_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}. \tag{17}$$

ii) There exists $t_0 \in \mathbb{Z}^m$ such that $\forall \alpha_0 \in \{1, 2, \ldots, m\}$, $\forall x_0 \in M$, there exists a unique m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \ge t_0 - 1_{\alpha_0}\big\} \to M$, which, for any $t \ge t_0 - 1_{\alpha_0}$, satisfies the relations (16)*, and the condition $x(t_0) = x_0$.*

iii) There exist the points $t_0, t_1 \in \mathbb{Z}^m$, with $t_1^\alpha < t_0^\alpha$, $\forall \alpha$, such that, for each $x_0 \in M$, there exists a unique m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \ge t_1\big\} \to M$, which, for each $t \ge t_1$ satisfies the relations (16)*, and also the condition $x(t_0) = x_0$.*

iv) For each $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \le t_0$, and for any $x_0 \in M$, there exists a unique m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \ge t_1\big\} \to M$, which, for any $t \ge t_1$, satisfies the relations (16)*, and also the condition $x(t_0) = x_0$.*

v) There exists $t_0 \in \mathbb{Z}^m$*, such that, for any* $x_0 \in M$*, there exists a unique* m*-sequence* $x\colon \mathbb{Z}^m \to M$*, which, for any* $t \in \mathbb{Z}^m$ *satisfies the relations* (16)*, and* $x(t_0) = x_0$*.*
vi) For each pair $(t_0, x_0) \in \mathbb{Z}^m \times M$*, there exists a unique* m*-sequence* $x\colon \mathbb{Z}^m \to M$*, which, for any* $t \in \mathbb{Z}^m$ *satisfies the relations* (16)*, and* $x(t_0) = x_0$*.*

Proof. *ii)* $\Longrightarrow$ *i)*: The relations (17) and the surjectivity of functions G_α follow from Proposition 2.1, *a*), and the injectivity of the functions G_α follow from Proposition 2.2.

i) $\Longrightarrow$ *vi)*: It follows from Proposition 2.3.

vi) $\Longrightarrow$ *iv)*: Considering the restrictions of the functions $x(t)$ to $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_{\alpha_0}\right\}$ (for each α_0), from Proposition 2.1, *a*), it follows that the relations (17) hold and that the functions G_α are surjective.

Let $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \leq t_0$, and $x_0 \in M$. There exists a unique m-sequence $\tilde{x}\colon \mathbb{Z}^m \to M$ such that $\tilde{x}(t_0) = x_0$ and the relations (16) are true, $\forall t \in \mathbb{Z}^m$.

To prove the existence, it is sufficient to select x as the restriction of $\tilde{x}$ to $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\}$.

Let $y\colon \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\} \to M$, be a function such that $y(t_0) = x_0$ and for which the relations (16) hold, $\forall t \geq t_1$. We shall prove that the functions x and y are equal.

From Proposition 2.1, *c*), there exists $\tilde{y}\colon \mathbb{Z}^m \to M$ such that $\tilde{y}(t_1) = y(t_1)$ and for which the relations (16) hold, $\forall t \in \mathbb{Z}^m$. From Theorem 2.1, it follows that y and the restriction of $\tilde{y}$ to $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\}$ coincide. Since $t_0 \geq t_1$, we have $\tilde{y}(t_0) = y(t_0) = x_0$. It follows that the functions $\tilde{x}$ and $\tilde{y}$ coincide. Consequently, for each $t \geq t_1$, we have: $y(t) = \tilde{y}(t) = \tilde{x}(t) = x(t)$.

iv) $\Longrightarrow$ *ii)* is an obvious implication.

We have proved that the statements *i)*, *ii)*, *iv)*, *vi)* are equivalent.

i) $\Longrightarrow$ *iii)*: We have *i)* $\Longleftrightarrow$ *iv)*, and *iv)* $\Longrightarrow$ *iii)* is obvious.

iii) $\Longrightarrow$ *i)*: For each α, we have $t_1^\alpha < t_0^\alpha$, i.e., $t_0^\alpha - 1 \geq t_1^\alpha$. Hence, for all α, $t_0 - 1_\alpha \geq t_1$. Considering the restrictions of the functions x to $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0 - 1_{\alpha_0}\right\}$ (for each α_0), from Proposition 2.1, *a*), it follows that the relations (17) are true and the functions G_α are surjective.

Let $\alpha_0 \in \{1, 2, \ldots, m\}$. We shall prove that G_{α_0} is injective.

Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$. According to Proposition 2.1, *c*), there exist the functions $y, z\colon \mathbb{Z}^m \to M$ for which the relations (16) hold, $\forall t \in \mathbb{Z}^m$, and $y(t_0 - 1_{\alpha_0}) = p$, $z(t_0 - 1_{\alpha_0}) = q$. Let $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

$$y(t_0) = G_{\alpha_0}(y(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(p) = x_0,$$
$$z(t_0) = G_{\alpha_0}(z(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(q) = x_0.$$

Applying the uniqueness property for the restrictions of the functions y and z to the set $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\}$, we obtain $y(t) = z(t)$, $\forall t \geq t_1$.

Since $t_0 - 1_{\alpha_0} \geq t_1$, it follows $y(t_0 - 1_{\alpha_0}) = z(t_0 - 1_{\alpha_0})$, i.e., $p = q$.

i) $\Longrightarrow$ *v)*: We have *i)* $\Longleftrightarrow$ *vi)*, and *vi)* $\Longrightarrow$ *v)* is obvious.

v) $\Longrightarrow$ *i*): For each α, we have $t_1^\alpha < t_0^\alpha$, i.e., $t_0^\alpha - 1 \geq t_1^\alpha$. Hence $t_0 - 1_\alpha \geq t_1$, $\forall \alpha$. Considering the restrictions of the functions x to $\{t \in \mathbb{Z}^m \mid t \geq t_0 - 1_{\alpha_0}\}$ (for each α_0), by Proposition 2.1, *a*), it follows that the relations (17) hold and that the functions G_α are surjective.

Let $\alpha_0 \in \{1, 2, \ldots, m\}$. We shall prove that G_{α_0} is injective.

Let $p, q \in M$ such that $G_{\alpha_0}(p) = G_{\alpha_0}(q)$. According to Proposition 2.1, *c*), there exist the m-sequences $y, z\colon \mathbb{Z}^m \to M$ for which the relations (16) are true, $\forall t \in \mathbb{Z}^m$, and $y(t_0 - 1_{\alpha_0}) = p$, $z(t_0 - 1_{\alpha_0}) = q$. Let $x_0 = G_{\alpha_0}(p) = G_{\alpha_0}(q)$.

$y(t_0) = G_{\alpha_0}(y(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(p) = x_0$,

$z(t_0) = G_{\alpha_0}(z(t_0 - 1_{\alpha_0})) = G_{\alpha_0}(q) = x_0$.

From uniqueness, we obtain $y(t) = z(t)$, $\forall t \in \mathbb{Z}^m$; for $t = t_0 - 1_{\alpha_0}$ it follows $y(t_0 - 1_{\alpha_0}) = z(t_0 - 1_{\alpha_0})$, i.e., $p = q$. □

3. Non-autonomous discrete multitime multiple recurrence

Let $t_1 \in \mathbb{Z}^m$. Consider the functions $F_\alpha\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, which define the recurrence equation

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{18}$$

Let $\widetilde{M} = \{s \in \mathbb{Z}^m \mid s \geq t_1\} \times M$ and let $G_\alpha\colon \widetilde{M} \to \widetilde{M}$,

$G_\alpha(s, x) = \big(s + 1_\alpha, F_\alpha(s, x)\big), \quad \forall (s, x) \in \widetilde{M}$.

The functions G_α define the recurrence

$$\big(s(t + 1_\alpha), x(t + 1_\alpha)\big) = \big(s(t) + 1_\alpha, F_\alpha(s(t), x(t))\big), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{19}$$

which is equivalent to

$$\begin{cases} x(t + 1_\alpha) = F_\alpha(s(t), x(t)) \\ s(t + 1_\alpha) = s(t) + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \ldots, m\}. \tag{20}$$

The unknown function is $\big(s(\cdot), x(\cdot)\big)$. Denoting $y = (s, x)$, the recurrence (19) can be rewritten in the form

$$y(t + 1_\alpha) = G_\alpha(y(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{21}$$

with the unknown function $y(\cdot) = \big(s(\cdot), x(\cdot)\big)$.

Lemma 3.1. *a*) *Let $t_0, t_1, s_0 \in \mathbb{Z}^m$, with $t_0 \geq t_1$.*

Then the m-sequence $s\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \mathbb{Z}^m$ satisfies, for each $t \geq t_1$,

$$s(t + 1_\alpha) = s(t) + 1_\alpha, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{22}$$

and the condition $s(t_0) = s_0$,

if and only if $s(t) = t - t_0 + s_0$, $\forall t \geq t_1$.

b) *Let $t_0, s_0 \in \mathbb{Z}^m$. The function $s\colon \mathbb{Z}^m \to \mathbb{Z}^m$ satisfies, for each $t \in \mathbb{Z}^m$, the relations* (22) *and the condition $s(t_0) = s_0$ if and only if $s(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$.*

Proof. Let $\tilde{s}\colon \mathbb{Z}^m \to \mathbb{Z}^m$, $\tilde{s}(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$. The function $\tilde{s}$ satisfies, for any $t \in \mathbb{Z}^m$, the relations (22) and $\tilde{s}(t_0) = s_0$.

For each α, we consider the function

$$\widetilde{G}_\alpha\colon \mathbb{Z}^m \to \mathbb{Z}^m, \quad \widetilde{G}_\alpha(s) = s + 1_\alpha,\ \forall s \in \mathbb{Z}^m.$$

The relations (22) are equivalent to $s(t+1_\alpha) = \widetilde{G}_\alpha(s(t))$, $\forall \alpha \in \{1,\dots,m\}$.

One observes that $\widetilde{G}_\alpha \circ \widetilde{G}_\beta(s) = \widetilde{G}_\beta \circ \widetilde{G}_\alpha(s) = s + 1_\alpha + 1_\beta$, $\forall s \in \mathbb{Z}^m$.

For any α, the function $\widetilde{G}_\alpha$ is bijective. Its inverse is $(\widetilde{G}_\alpha)^{-1}(s) = s - 1_\alpha$.

According to Theorem 2.2, *iv*), there exists a unique function $s\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \mathbb{Z}^m$ which satisfies the recurrence (22), $\forall t \geq t_1$, and the condition $s(t_0) = t_0$. By uniqueness, it follows that s coincides with the restriction of the function $\tilde{s}$ to the set $\{t \in \mathbb{Z}^m \mid t \geq t_1\}$; hence $s(t) = t - t_0 + s_0$, $\forall t \geq t_1$.

According to Theorem 2.2, *vi*), it follows that there exists a unique function $\sigma\colon \mathbb{Z}^m \to \mathbb{Z}^m$ which satisfies the recurrence (22), $\forall t \in \mathbb{Z}^m$, and the condition $\sigma(t_0) = t_0$. From uniqueness, it follows that $\sigma = \tilde{s}$; hence $\sigma(t) = t - t_0 + s_0$, $\forall t \in \mathbb{Z}^m$. ☐

In the above conditions, the following result is obtained.

Proposition 3.1. *a) For $\alpha, \beta \in \{1, 2, \dots, m\}$, we have*

$$F_\alpha(t + 1_\beta, F_\beta(t,x)) = F_\beta(t + 1_\alpha, F_\alpha(t,x)), \quad \forall t \geq t_1,\ \forall x \in M \tag{23}$$

if and only if $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$.
b) Let $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$ and $x_0 \in M$.

If $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to M$ satisfies the recurrence (18), $\forall t \geq t_1$, and the condition $x(t_0) = x_0$, then the m-sequence

$$y\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \widetilde{M}, \quad y(t) = (t, x(t)), \quad \forall t \geq t_1,$$

satisfies the recurrence (21), $\forall t \geq t_1$, and the condition $y(t_0) = (t_0, x_0)$.

Conversely, if $y(\cdot) = (s(\cdot), x(\cdot))\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to \widetilde{M}$ satisfies the recurrence (21), $\forall t \geq t_1$, and the condition $y(t_0) = (t_0, x_0)$, then $s(t) = t, \forall t \geq t_1$ and $x(\cdot)$ satisfies the recurrence (18), $\forall t \geq t_1$, and the condition $x(t_0) = x_0$.

Proof. *a*) For any $(s,x) \in \widetilde{M}$, we have: $G_\alpha \circ G_\beta(s,x) = G_\beta \circ G_\alpha(s,x) \iff$
$\iff \big(s+1_\beta+1_\alpha, F_\alpha(s+1_\beta, F_\beta(s,x))\big) = \big(s+1_\alpha+1_\beta, F_\beta(s+1_\alpha, F_\alpha(s,x))\big)$
$\iff F_\alpha(s + 1_\beta, F_\beta(s,x)) = F_\beta(s + 1_\alpha, F_\alpha(s,x))$.

b) Let $x(\cdot)$ be a solution of the recurrence (18), with $x(t_0) = x_0$. We have to show that the m-sequence $y(t) = (t, x(t))$ satisfies the relations (20); since, for that $y(\cdot)$ we have $s(t) = t$, the relations (20) become

$$\begin{cases} x(t+1_\alpha) = F_\alpha(t, x(t)) \\ t + 1_\alpha = t + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \dots, m\}. \tag{24}$$

The second relation in (24) is obvious, and the first is true because the m-sequence $x(\cdot)$ is a solution of the recurrence (18).

The relation $y(t_0) = (t_0, x_0)$ is obvious.

Conversely, let $y(\cdot) = \big(s(\cdot), x(\cdot)\big)$ be a solution of the recurrence (21), with $y(t_0) = (t_0, x_0)$. Hence $s(\cdot)$ and $x(\cdot)$ satisfies the relations (20) and the condition $s(t_0) = t_0$, $x(t_0) = x_0$.

Since $s(t + 1_\alpha) = s(t) + 1_\alpha$, $\forall t \geq t_1$, $\forall \alpha$, and $s(t_0) = t_0$, from Lemma 3.1 it follows that $s(t) = t$, $\forall t \geq t_1$.

Hence, the first relation in (20) becomes $x(t+1_\alpha) = F_\alpha(t, x(t))$, i.e., $x(\cdot)$ is solution of the recurrence (18). □

Theorem 3.1. *Let M be an arbitrary nonempty set and $t_0 \in \mathbb{Z}^m$. We consider the functions $F_\alpha\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, such that*

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \tag{25}$$

$$\forall t \geq t_0,\ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

Then, for any $x_0 \in M$, there exists a unique m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$ which satisfies

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{26}$$

and the condition $x(t_0) = x_0$.

Proof. Let $\widetilde{M} = \big\{s \in \mathbb{Z}^m \,\big|\, s \geq t_0\big\} \times M$ and let $G_\alpha\colon \widetilde{M} \to \widetilde{M}$,

$$G_\alpha(s, x) = \big(s + 1_\alpha, F_\alpha(s, x)\big), \quad \forall (s, x) \in \widetilde{M}.$$

We apply Proposition 3.1 (for $t_1 = t_0$); according to step a), it follows that $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$, $\forall \alpha \in \{1, 2, \ldots, m\}$. From Theorem 2.1, b), it follows that there exists a unique m-sequence $y(\cdot) = \big(s(\cdot), x(\cdot)\big)\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to \widetilde{M}$ which satisfies

$$y(t + 1_\alpha) = G_\alpha(y(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{27}$$

and the condition $y(t_0) = (t_0, x_0)$. From Proposition 3.1, b), it follows that $x(\cdot)$ satisfies the relations (26) and the initial condition $x(t_0) = x_0$.

Uniqueness of $x(\cdot)$: let $\tilde{x}\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$ an m-sequence which satisfies the relations (26) and the condition $\tilde{x}(t_0) = x_0$. From Proposition 3.1, b), it follows that the m-sequence

$$\tilde{y}\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to \widetilde{M}, \quad \tilde{y}(t) = (t, \tilde{x}(t)), \quad \forall t \geq t_0,$$

satisfies the relations (27) and the condition $\tilde{y}(t_0) = (t_0, x_0)$.

From the uniqueness property of the solution of the recurrence (27) (Theorem 2.1, b)), it follows that the functions y and $\tilde{y}$ coincide; hence $\big(s(t), x(t)\big) = \big(t, \tilde{x}(t)\big)$, $\forall t \geq t_0$; we obtain $x(t) = \tilde{x}(t)$, $\forall t \geq t_0$. □

Proposition 3.2. *Let $\alpha_0 \in \{1, 2, \ldots, m\}$, $t_0 \in \mathbb{Z}^m$.*
a) Let $F\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \times M \to M$. If, for any $x_0 \in M$, there exists at least one m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$, which satisfies

$$x(t + 1_{\alpha_0}) = F(t, x(t)), \quad \forall t \geq t_0 - 1_{\alpha_0}, \tag{28}$$

and the condition $x(t_0) = x_0$, then the function $F(t_0 - 1_{\alpha_0}, \cdot)$ is surjective.

b) Suppose that for the functions $F_\alpha\colon \{t\in\mathbb{Z}^m \,|\, t\ge t_0-1_{\alpha_0}\}\times M\to M$, *the relations* (25) *hold,* $\forall t\ge t_0-1_{\alpha_0}$, $\forall x\in M$, $\forall\alpha,\beta\in\{1,2,\dots,m\}$. *If, for any* $x_0\in M$, *there exists at most one* m*-sequence* $x\colon \{t\in\mathbb{Z}^m \,|\, t\ge t_0-1_{\alpha_0}\}\to M$, *which satisfies*

$$x(t+1_\alpha)=F_\alpha(t,x(t)),\quad \forall t\ge t_0-1_{\alpha_0},\ \forall\alpha\in\{1,2,\dots,m\}, \tag{29}$$

and the condition $x(t_0)=x_0$, *then the function* $F_{\alpha_0}(t_0-1_{\alpha_0},\cdot)$ *is injective.*

Proof. *a*) Let $z\in M$. There exists an m-sequence $x(\cdot)$ which satisfies (28) and the initial condition $x(t_0)=z$. For $t=t_0-1_{\alpha_0}$, one obtains $z=F(t_0-1_{\alpha_0},x(t_0-1_{\alpha_0}))$. Since z is arbitrary, it follows that $F(t_0-1_{\alpha_0},\cdot)$ is surjective.

b) Let $p,q\in M$ such that $F_{\alpha_0}(t_0-1_{\alpha_0},p)=F_{\alpha_0}(t_0-1_{\alpha_0},q)$. We can apply Theorem 3.1. There exist the functions $x,\tilde{x}\colon \{t\in\mathbb{Z}^m \,|\, t\ge t_0-1_{\alpha_0}\}\to M$ for which the relations (29) are true, and $x(t_0-1_{\alpha_0})=p$, $\tilde{x}(t_0-1_{\alpha_0})=q$.

Let $x_0=F_{\alpha_0}(t_0-1_{\alpha_0},p)=F_{\alpha_0}(t_0-1_{\alpha_0},q)$. Then

$$x(t_0)=F_{\alpha_0}(t_0-1_{\alpha_0},x(t_0-1_{\alpha_0}))=F_{\alpha_0}(t_0-1_{\alpha_0},p)=x_0,$$
$$\tilde{x}(t_0)=F_{\alpha_0}(t_0-1_{\alpha_0},\tilde{x}(t_0-1_{\alpha_0}))=F_{\alpha_0}(t_0-1_{\alpha_0},q)=x_0.$$

It follows that x and $\tilde{x}$ coincide; hence $x(t_0-1_{\alpha_0})=\tilde{x}(t_0-1_{\alpha_0})$, i.e., $p=q$. □

The next result can be proven without difficulty.

Lemma 3.2. *Let* $\beta\in\{1,2,\dots,m\}$ *and* $F\colon\mathbb{Z}^m\times M\to M$.

Let $G\colon\mathbb{Z}^m\times M\to\mathbb{Z}^m\times M$, $G(t,x)=\big(t+1_\beta,F(t,x)\big)$, $\forall(t,x)\in\mathbb{Z}^m\times M$.

a) The function G *is injective if and only if, for any* $t\in\mathbb{Z}^m$, *the function* $F(t,\cdot)$ *is injective.*

b) The function G *is surjective if and only if, for any* $t\in\mathbb{Z}^m$, *the function* $F(t,\cdot)$ *is surjective.*

Theorem 3.2. *Let* M *be a nonempty set. For each* $\alpha\in\{1,2,\dots,m\}$, *we consider the function* $F_\alpha\colon\mathbb{Z}^m\times M\to M$, *to which we associate the recurrence equation*

$$x(t+1_\alpha)=F_\alpha(t,x(t)),\quad \forall\alpha\in\{1,2,\dots,m\}. \tag{30}$$

The following statements are equivalent:

i) For any $\alpha\in\{1,2,\dots,m\}$ *and any* $t\in\mathbb{Z}^m$, *the functions* $F_\alpha(t,\cdot)$ *are bijective and*

$$F_\alpha(t+1_\beta,F_\beta(t,x))=F_\beta(t+1_\alpha,F_\alpha(t,x)), \tag{31}$$
$$\forall(t,x)\in\mathbb{Z}^m\times M,\ \forall\alpha,\beta\in\{1,2,\dots,m\}.$$

ii) For any pair $(t_0,x_0)\in\mathbb{Z}^m\times M$, *and any index* $\alpha_0\in\{1,2,\dots,m\}$, *there exists a unique* m*-sequence* $x\colon\{t\in\mathbb{Z}^m \,|\, t\ge t_0-1_{\alpha_0}\}\to M$, *which, for each* $t\ge t_0-1_{\alpha_0}$, *satisfies the relations* (30), *and also the condition* $x(t_0)=x_0$.

iii) For any $t_0,t_1\in\mathbb{Z}^m$, *with* $t_1\le t_0$, *and any* $x_0\in M$, *there exists an* m*-sequence* $x\colon\{t\in\mathbb{Z}^m \,|\, t\ge t_1\}\to M$, *which, for any* $t\ge t_1$, *satisfies the relations* (30), *and also the condition* $x(t_0)=x_0$.

iv) For any $(t_0,x_0)\in\mathbb{Z}^m\times M$, *there exists a unique* m*-sequence* $x\colon\mathbb{Z}^m\to M$, *which, for any* $t\in\mathbb{Z}^m$, *satisfies the relation* (30), *and also* $x(t_0)=x_0$.

Proof. *ii*) $\Longrightarrow$ *i*): Let $t_1 \in \mathbb{Z}^m$. For any $t_0 \geq t_1$ and any $x_0 \in M$, there exists a unique m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0 - 1_{\alpha_0}\big\} \to M$, which, for any $t \geq t_0 - 1_{\alpha_0}$, satisfies the relations (30), and also the condition $x(t_0) = x_0$.

The restriction of the function $x(\cdot)$ to $\big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\}$ satisfies, for any $t \geq t_0$, the recurrence (30), and also the condition $x(t_0) = x_0$.

From Proposition 1.1 it follows that the relations (31) hold, for any $t \geq t_1$. Since t_1 is arbitrary, we deduce that the relations (31) are true, for any $t \in \mathbb{Z}^m$.

The surjectivity of functions $F_\alpha(t,\cdot)$ follows from Proposition 3.2, *a*).

The injectivity of functions $F_\alpha(t,\cdot)$ follows from Proposition 3.2, *b*).

i) $\Longrightarrow$ *iv*): For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function

$G_\alpha\colon \mathbb{Z}^m \times M \to \mathbb{Z}^m \times M, \quad G_\alpha(t,x) = \big(t+1_\alpha, F(t,x)\big), \ \forall (t,x) \in \mathbb{Z}^m \times M.$

Similar to the proof of Proposition 3.1, it is shown that the relations (31) are true, for any $(t,x) \in \mathbb{Z}^m \times M$, if and only if $G_\alpha \circ G_\beta = G_\beta \circ G_\alpha$.

From Lemma 3.2, we deduce that, for any $\alpha \in \{1, 2, \ldots, m\}$, the function G_α is bijective.

Let $(t_0, x_0) \in \mathbb{Z}^m \times M$. According to Theorem 2.2, *vi*), there exists a unique m-sequence $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$, which, for any $t \in \mathbb{Z}^m$, satisfies the relations

$$y(t+1_\alpha) = G_\alpha(y(t)), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{32}$$

and $y(t_0) = (t_0, x_0)$, which are equivalent to

$$\begin{cases} x(t+1_\alpha) = F_\alpha(s(t), x(t)) \\ s(t+1_\alpha) = s(t) + 1_\alpha \end{cases}, \quad \forall \alpha \in \{1, 2, \ldots, m\} \tag{33}$$

and $s(t_0) = t_0$, $x(t_0) = x_0$.

From Lemma 3.1, we obtain $s(t) = t$, $\forall t \in \mathbb{Z}^m$. Replacing in the first relation of (33), it follows that the m-sequence $x\colon \mathbb{Z}^m \to M$ satisfies the relations (30), $\forall t \in \mathbb{Z}^m$.

Uniqueness of $x(\cdot)$: let $\tilde{x}\colon \mathbb{Z}^m \to M$ be a function which satisfies the relations (30), $\forall t \in \mathbb{Z}^m$, and the condition $\tilde{x}(t_0) = x_0$. Easily we find that the function $\tilde{y}\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M, \quad \tilde{y}(t) = (t, \tilde{x}(t)), \quad \forall t \in \mathbb{Z}^m$,
satisfies the relations (32), $\forall t \in \mathbb{Z}^m$, and the condition $\tilde{y}(t_0) = (t_0, x_0)$.

From the uniqueness property of solutions of the recurrence (32) (according to Theorem 2.2, *vi*)) it follows that the functions y and $\tilde{y}$ coincide; hence $\big(t, x(t)\big) = \big(t, \tilde{x}(t)\big)$, $\forall t \in \mathbb{Z}^m$; we obtain $x(t) = \tilde{x}(t)$, $\forall t \in \mathbb{Z}^m$.

iv) $\Longrightarrow$ *iii*): For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the function G_α defined as in the proof of the implication *i*) $\Longrightarrow$ *iv*).

Let $t_0 \in \mathbb{Z}^m$ and $(s_0, x_0) \in \mathbb{Z}^m \times M$. We shall show that there exists a unique m-sequence $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$, which, for any $t \in \mathbb{Z}^m$, satisfies the relations (32), and $y(t_0) = (s_0, x_0)$.

There exists a unique m-sequence $\tilde{x}\colon \mathbb{Z}^m \to M$ which, for any $t \in \mathbb{Z}^m$, satisfies the relations (30), and also the initial condition $\tilde{x}(s_0) = x_0$.
Let $s\colon \mathbb{Z}^m \to \mathbb{Z}^m$, $x\colon \mathbb{Z}^m \to M$, $s(t) = t - t_0 + s_0$, $x(t) = \tilde{x}(t - t_0 + s_0)$, $\forall t \in \mathbb{Z}^m$.

Easily we find that the m-sequence $y(\cdot) = (s(\cdot), x(\cdot))\colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$ satisfies, for any $t \in \mathbb{Z}^m$, the recurrence (32) and $y(t_0) = (s_0, x_0)$.

The uniqueness of $y(\cdot)$: Let $\tilde{y}(\cdot) = (\sigma(\cdot), z(\cdot)) \colon \mathbb{Z}^m \to \mathbb{Z}^m \times M$ which satisfies, for any $t \in \mathbb{Z}^m$, the recurrence (32) and the condition $\tilde{y}(t_0) = (s_0, x_0)$. Hence, for any $t \in \mathbb{Z}^m$, the functions $\sigma(\cdot)$, $z(\cdot)$ satisfy the recurrence (33) and $\sigma(t_0) = s_0$, $z(t_0) = x_0$. From Lemma 3.1, it follows $\sigma(t) = t - t_0 + s_0 = s(t)$. One observes that the m-sequence $\tilde{z} \colon \mathbb{Z}^m \to M$, $\tilde{z}(t) = z(t + t_0 - s_0)$, $\forall t \in \mathbb{Z}^m$, satisfies the recurrence (30), $\forall t \in \mathbb{Z}^m$, and $\tilde{z}(s_0) = x_0$. It follows that the functions $\tilde{x}$ and $\tilde{z}$ coincide. Hence, we have $\tilde{z}(t - t_0 + s_0) = \tilde{x}(t - t_0 + s_0)$, i.e., $z(t) = x(t)$. Since $\sigma(\cdot) = s(\cdot)$ and $z(\cdot) = x(\cdot)$, we have $\tilde{y}(\cdot) = y(\cdot)$.

Hence, for the recurrence (32) we can apply Theorem 2.2, implication $vi) \implies iv)$. Let $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \le t_0$, and $x_0 \in M$. There exists a unique m-sequence $\tilde{x} \colon \mathbb{Z}^m \to M$, such that $\tilde{x}(t_0) = x_0$ and the relations (30) are true, $\forall t \in \mathbb{Z}^m$. It is sufficient to select x as being the restriction of $\tilde{x}$ to $\{t \in \mathbb{Z}^m \,|\, t \ge t_1\}$.

Uniqueness of the function $x(\cdot)$: Let $z \colon \{t \in \mathbb{Z}^m \,|\, t \ge t_1\} \to M$, which, for any $t \ge t_1$ satisfies the recurrence (30), and $z(t_0) = x_0$.

Let $y, \tilde{y} \colon \{t \in \mathbb{Z}^m \,|\, t \ge t_1\} \to \mathbb{Z}^m \times M$, $y(t) = (t, x(t))$, $\tilde{y}(t) = (t, z(t))$.

One observes that y and $\tilde{y}$ satisfy the recurrence (32), $\forall t \ge t_1$; we have also $y(t_0) = \tilde{y}(t_0) = (t_0, x_0)$. From Theorem 2.2, it follows that the functions y and $\tilde{y}$ coincide. We obtain $(t, x(t)) = (t, z(t))$, $\forall t \ge t_1$; hence $x(t) = z(t)$.

$iii) \implies ii)$ is an obvious implication. □

4. Recurrences based on a monoid action

Let M be a nonempty set, $(N, \cdot, e)$ be a monoid and let $\varphi \colon N \times M \to M$ be an action of the monoid N on the set M, i.e.,

$$\varphi(ab, x) = \varphi\big(a, (b, x)\big), \;\; \varphi(e, x) = x, \quad \forall a, b \in N, \forall x \in M. \tag{34}$$

For each $a \in N$, $x \in M$, we denote $\varphi(a, x) = ax$ (not to be confused with the monoid operation on N). The relations (34) become

$$(ab)x = a(bx), \;\; ex = x, \quad \forall a, b \in N, \forall x \in M.$$

We consider $a_1, a_2, \ldots, a_m \in N$, such that $a_\alpha a_\beta = a_\beta a_\alpha$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$.

For each pair $(t_0, x_0) \in \mathbb{Z}^m \times M$, the recurrence

$$x(t + 1_\alpha) = a_\alpha x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{35}$$

with the initial condition $x(t_0) = x_0$, has unique solution

$$x \colon \{t \in \mathbb{Z}^m \,|\, t \ge t_0\} \to M,$$

$$x(t) = a_1^{(t^1 - t_0^1)} a_2^{(t^2 - t_0^2)} \cdot \ldots \cdot a_m^{(t^m - t_0^m)} x_0. \tag{36}$$

This can be obtained by applying Theorem 2.1 for the functions $G_\alpha \colon M \to M$, $G_\alpha(x) = a_\alpha x$, $\forall x \in M$. We have $G_\alpha \circ G_\beta(x) = G_\beta \circ G_\alpha(x) = a_\alpha a_\beta x$. One observes that, for any $t \in \mathbb{N}^m$,

$$G_1^{(t^1)} \circ G_2^{(t^2)} \circ \ldots \circ G_m^{(t^m)}(x) = a_1^{t^1} a_2^{t^2} \cdot \ldots \cdot a_m^{t^m} x. \tag{37}$$

Suppose that for any $\alpha \in \{1, 2, \ldots, m\}$, a_α is invertible; then G_α is bijective, with the inverse $G_\alpha^{-1}(x) = a_\alpha^{-1} x$. We find that the formula (37) is true for any $t \in \mathbb{Z}^m$. There exists a unique m-sequence $\tilde{x} \colon \mathbb{Z}^m \to M$, solution of the recurrence (35), with $\tilde{x}(t_0) = x_0$; the function $\tilde{x}(\cdot)$ is a unique extension of $x(\cdot)$ and it is defined by the formula (36), but for each $t \in \mathbb{Z}^m$.

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