A GENERALIZATION OF COMPLETE AND ELEMENTARY SYMMETRIC FUNCTIONS - PART I

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In this paper, we consider the generating functions of the complete and elementary symmetric functions and provide a new generalization of these classical symmetric functions. Some classical relationships involving the complete and elementary symmetric functions are reformulated in a more general context.

Keywords: symmetric functions, complete homogeneous symmetric functions, elementary symmetric functions, power sum symmetric functions

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1. Introduction

A formal power series in the variables $x_1, x_2, \ldots, x_n$ is called symmetric if it is invariant under any permutation of the variables. These symmetric formal power series are traditionally called symmetric functions. A symmetric function is homogeneous of degree k if every monomial in it has total degree k. Symmetric functions are ubiquitous in mathematics and mathematical physics. For example, they appear in elementary algebra (e.g. Viete's Theorem), representation theories of symmetric groups and general linear groups over $\mathbb{C}$ or finite fields. They are also important objects to study in algebraic combinatorics.

A partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n, i.e.,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n \qquad \text{and} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0.$$

The positive integers in the sequence are called parts [1]. The multiplicity of the part i in λ, denoted by t_i, is the number of parts of λ equal to i. We denote by $l(\lambda)$ the number of parts of λ. In order to indicate that $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ or $\lambda = [1^{t_1} 2^{t_2} \ldots n^{t_n}]$ is a partition of n, we use the notation $\lambda \vdash n$.

We recall some basic facts about monomial symmetric functions. Proofs and details can be found in Macdonald's book [6]. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ is an integer partition with $k \leq n$ then, the monomial symmetric function

$$m_\lambda(x_1, x_2, \ldots, x_n) = m_{[\lambda_1, \lambda_2, \ldots, \lambda_k]}(x_1, x_2, \ldots, x_n)$$

is the sum of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ and all distinct monomials obtained from it by a permutation of variables. For instance, with $\lambda = [2, 1, 1]$ and $n = 4$, we have:

$$\begin{aligned} m_{[2,1,1]}(x_1, x_2, x_3, x_4) &= x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_4 + x_1 x_2^2 x_4 \\ &\quad + x_1 x_2 x_4^2 + x_1^2 x_3 x_4 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2. \end{aligned}$$

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The kth complete homogeneous symmetric function h_k is the sum of all monomials of total degree k in these variables, i.e.,

$$h_k(x_1,x_2,\ldots,x_n)=\sum_{\lambda\vdash k} m_\lambda(x_1,x_2,\ldots,x_n)=\sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k},$$

and the kth elementary symmetric function is defined by

$$e_k(x_1,x_2,\ldots,x_n)=m_{[1^k]}(x_1,x_2,\ldots,x_n)=\sum_{1\le i_1< i_2<\cdots< i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k},$$

where $e_0(x_1,x_2,\ldots,x_n)=h_0(x_1,x_2,\ldots,x_n)=1$. In particular, when $\lambda=[k]$, we have the kth power sum symmetric function

$$p_k(x_1,x_2,\ldots,x_n)=m_{[k]}(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{n} x_i^k,$$

with $p_0(x_1,x_2,\ldots,x_n)=n$.

The complete homogeneous symmetric functions are characterized by the following identity of formal power series in t:

$$\sum_{k=0}^{\infty} h_k(x_1,x_2,\ldots,x_n)t^k=\prod_{i=1}^{n}(1-x_it)^{-1}. \tag{1}$$

Analogously, for the elementary symmetric functions we have:

$$\sum_{k=0}^{\infty} e_k(x_1,x_2,\ldots,x_n)t^k=\prod_{i=1}^{n}(1+x_it). \tag{2}$$

Inspired by these generating functions, we introduce the generalized symmetric functions $H_k^{(s)}(x_1,x_2,\ldots,x_n)$ and $E_k^{(s)}(x_1,x_2,\ldots,x_n)$ as follows:

$$\sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k=\prod_{i=1}^{n}\left(1-x_it+\cdots+(-x_it)^s\right)^{-1} \tag{3}$$

and

$$\sum_{k=0}^{\infty} E_k^{(s)}(x_1,x_2,\ldots,x_n)t^k=\prod_{i=1}^{n}\left(1+x_it+\cdots+(x_it)^s\right), \tag{4}$$

where s is a positive integer.

Clearly, by setting $s=1$ in (3) and (4), we obtain the generating functions for complete and elementary symmetric functions. In addition, by (4) we easily deduce that

$$E_k^{(s)}(x_1,x_2,\ldots,x_n)=\sum_{\substack{\lambda\vdash k\\ \lambda_1\le s}} m_\lambda(x_1,x_2,\ldots,x_n). \tag{5}$$

Moreover, considering that

$$E_k^{(k)}(x_1,x_2,\ldots,x_n)=h_k(x_1,x_2,\ldots,x_n), \tag{6}$$

the generalized symmetric functions $E_k^{(s)}$ can be seen as another generalization of the complete homogenous symmetric function h_k. To illustrate (5), we have

$$E_5^{(3)}(x_1,x_2,x_3)=m_{[2,2,1]}(x_1,x_2,x_3)+m_{[3,1,1]}(x_1,x_2,x_3)+m_{[3,2]}(x_1,x_2,x_3)$$

The symmetric functions $E_k^{(s)}(x_1,x_2,\ldots,x_n)$ are not essentially a new generalization of the elementary symmetric functions $e_k(x_1,x_2,\ldots,x_n)$. An equivalent definition of these symmetric functions already exists in a paper published in 2018 by Bazeniar et *al.* [2]:

$$E_k^{(s)}(x_1,x_2,\dots,x_n)=\sum_{\substack{\lambda\vdash k\\ 0\le\lambda_1,\lambda_2,\dots,\lambda_n\le s}} x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}, \tag{7}$$

where $E_0^{(s)}(x_1,x_2,\dots,x_n)=1$ and $E_k^{(s)}(x_1,x_2,\dots,x_n)=0$ unless $0\le k\le sn$. Moreover, the authors proved that the symmetric functions $E_k^{(s)}(x_1,x_2,\dots,x_n)$ satisfy the following recurrence relation.

$$E_k^{(s)}(x_1,x_2,\dots,x_n)=\sum_{j=0}^{s} x_n^j E_{k-j}^{(s)}(x_1,x_2,\dots,x_{n-1}). \tag{8}$$

A similar result can be derived for the symmetric function $H_k^{(s)}(x_1,x_2,\dots,x_n)$, namely,

$$H_k^{(s)}(x_1,x_2,\dots,x_{n-1})=\sum_{j=0}^{s}(-1)^j x_n^j H_{k-j}^{(s)}(x_1,x_2,\dots,x_n). \tag{9}$$

Very recently, Fu and Mei [3] and Grinberg [4] independently introduced the generalized symmetric functions $E_k^{(s)}$. Grinberg denoted these functions by $G(s,k)$ and called them *Petrie symmetric functions* while Fu and Mei used the notation $h_k^{[s]}$ and referred to them as *truncated homogeneous symmetric functions*.

In this paper, motivated by these results, we investigate the properties of the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$. We collect some classical relationships involving complete, elementary and power sum symmetric functions and provide generalizations for them.

2. **Newton identities revisited**

There is a fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^j e_j(x_1,x_2,\dots,x_n)h_{k-j}(x_1,x_2,\dots,x_n)=\delta_{0,k}, \tag{10}$$

where $\delta_{i,j}$ is the Kronecker delta. This relation is valid for all $k>0$, and any number of variables n. We have the following generalization of this identity.

Theorem 2.1. *Let k, n and s be positive integers and let $x_1,x_2,\dots,x_n$ be independent variables. Then*

$$\sum_{j=0}^{k}(-1)^j E_j^{(s)}(x_1,x_2,\dots,x_n)H_{k-j}^{(s)}(x_1,x_2,\dots,x_n)=\delta_{0,n}. \tag{11}$$

Proof. By (3) and (4), we see that

$$\left(\sum_{k=0}^{\infty}(-1)^k E_k^{(s)}(x_1,x_2,\dots,x_n)t^k\right)\left(\sum_{k=0}^{\infty}H_k^{(s)}(x_1,x_2,\dots,x_n)t^k\right)=1.$$

Considering the well known Cauchy product of two power series, we obtain

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j E_j^{(s)}(x_1,x_2,\dots,x_n)H_{k-j}^{(s)}(x_1,x_2,\dots,x_n)\right)t^k=1.$$

This concludes the proof. □

Theorem 2.1 and [7, Theorem 1] allow us to derive two symmetric identities for the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$.

Corollary 2.1. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. The symmetric functions $H_k^{(s)} = H_k^{(s)}(x_1, x_2, \ldots, x_n)$ and $E_k^{(s)} = E_k^{(s)}(x_1, x_2, \ldots, x_n)$ are related by*

$$H_k^{(s)} = \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1, t_2, \ldots, t_k} \prod_{i=1}^{k} \left(E_i^{(s)}\right)^{t_i}$$

and

$$E_k^{(s)} = \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1, t_2, \ldots, t_k} \prod_{i=1}^{k} \left(H_i^{(s)}\right)^{t_i}.$$

The problem of expressing power sum symmetric polynomials in terms of elementary symmetric polynomials and vice-versa and the problem of expressing power sum symmetric polynomials in terms of complete symmetric polynomials and vice-versa were solved a long time ago. The relations, called Newton's identities

$$ke_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j}(x_1, x_2, \ldots, x_n) p_j(x_1, x_2, \ldots, x_n) \tag{12}$$

or

$$kh_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} h_{k-j}(x_1, x_2, \ldots, x_n) p_j(x_1, x_2, \ldots, x_n) \tag{13}$$

are well known. Recently, Merca [8] proved that the complete, elementary and power sum symmetric functions are related by

$$p_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} (-1)^{j-1} j e_j(x_1, x_2, \ldots, x_n) h_{k-j}(x_1, x_2, \ldots, x_n) \tag{14}$$

and derived new relationships between complete and elementary symmetric functions:

$$2ke_k = \sum_{k_1+k_2+k_3=k} (-1)^{k_3} (k_1+k_2) e_{k_1} e_{k_2} h_{k_3}, \tag{15}$$

and

$$kh_k = \sum_{k_1+k_2+k_3=k} (-1)^{k_3-1} k_3 h_{k_1} h_{k_2} e_{k_3}, \tag{16}$$

where k_1, k_2, k_3 are nonnegative integers.

In order to provide the generalizations of (12)-(16), we consider the symmetric function $P_k^{(s)}$ defined as

$$P_k^{(s)}(x_1, x_2, \ldots, x_n) = c_k^{(s)} p_k(x_1, x_2, \ldots, x_n),$$

where

$$c_k^{(s)} = \begin{cases} (-1)^k \cdot s, & \text{if } k \equiv 0 \bmod (s+1), \\ (-1)^{k-1}, & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

(1) $kE_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} (-1)^{j-1} P_j^{(s)}(x_1, x_2, \ldots, x_n) E_{k-j}^{(s)}(x_1, x_2, \ldots, x_n);$

(2) $kH_k^{(s)}(x_1,x_2,\dots,x_n)=\sum_{j=1}^{k}P_j^{(s)}(x_1,x_2,\dots,x_n)H_{k-j}^{(s)}(x_1,x_2,\dots,x_n);$

(3) $P_k^{(s)}(x_1,x_2,\dots,x_n)=\sum_{j=1}^{k}(-1)^{j-1}jE_j^{(s)}(x_1,x_2,\dots,x_n)H_{k-j}^{(s)}(x_1,x_2,\dots,x_n).$

Proof. For $\omega_{j,s}=e^{2j\pi i/s}$ with $j=1,2,\dots,s-1$, we can see that

$$1-t+\cdots+(-t)^{s-1}=(-1)^{s-1}\prod_{j=1}^{s-1}(\omega_{j,s}+t)=\prod_{j=1}^{s-1}(1+\omega_{j,s}t)\,,$$

where we take into account that $\prod_{j=1}^{s-1}\omega_{j,s}=(-1)^{s-1}$ and $\omega_{j,s}=1/\omega_{s-j,s}$. On one hand, we have

$$\begin{aligned}
\frac{d}{dt}\ln\prod_{i=1}^{n}\left(1-x_it+\cdots+(-x_it)^s\right)^{-1}&=\frac{d}{dt}\ln\prod_{i=1}^{n}\prod_{j=1}^{s}(1+\omega_{j,s+1}x_it)^{-1} \qquad (17)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{s}\frac{d}{dt}\ln(1+\omega_{j,s+1}x_it)^{-1}=-\sum_{i=1}^{n}\sum_{j=1}^{s}\frac{\omega_{j,s+1}x_i}{1+\omega_{j,s+1}x_it}\\
&=-\sum_{j=1}^{s}\sum_{i=1}^{n}\left(\omega_{j,s+1}x_i-(\omega_{j,s+1}x_i)^2t+(\omega_{j,s+1}x_i)^3t^2-\cdots\right)\\
&=\sum_{k=1}^{\infty}(-1)^k\left(\sum_{j=1}^{s}\omega_{j,s+1}^k\right)\left(\sum_{i=1}^{n}x_i^k\right)t^{k-1}\\
&=\sum_{k=1}^{\infty}(-1)^kp_k\left(\omega_{1,s+1},\omega_{2,s+1},\dots,\omega_{s,s+1}\right)p_k(x_1,x_2,\dots,x_n)t^{k-1}\\
&=\sum_{k=1}^{\infty}P_k^{(s)}(x_1,x_2,\dots,x_n)t^{k-1},
\end{aligned}$$

where we have invoked that

$$p_k(\omega_{1,s+1},\omega_{2,s+1},\dots,\omega_{s,s+1})=\begin{cases}s, & \text{if } k\equiv 0 \bmod (s+1),\\ -1, & \text{otherwise.}\end{cases}$$

On the other hand, we can write

$$\begin{aligned}
\sum_{k=1}^{\infty}P_k^{(s)}(x_1,x_2,\dots,x_n)t^{k-1}&=\frac{d}{dt}\ln\left(\sum_{k=0}^{\infty}(-1)^kE_k^{(s)}(x_1,x_2,\dots,x_n)t^k\right)^{-1}\\
&=-\left(\sum_{k=1}^{\infty}(-1)^kkE_k^{(s)}(x_1,\dots,x_n)t^{k-1}\right)\left(\sum_{k=0}^{\infty}(-1)^kE_k^{(s)}(x_1,\dots,x_n)t^k\right)^{-1}. \qquad (18)
\end{aligned}$$

By this identity, with t replaced by $-t$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty}kE_k^{(s)}(x_1,x_2,\dots,x_n)t^{k-1}\\
&\quad=\left(\sum_{k=0}^{\infty}E_k^{(s)}(x_1,x_2,\dots,x_n)t^k\right)\left(\sum_{k=1}^{\infty}(-1)^{k-1}P_k^{(s)}(x_1,x_2,\dots,x_n)t^{k-1}\right)
\end{aligned}$$

and the first identity follows.

To prove the second identity, we consider

$$\begin{aligned}\sum_{k=1}^{\infty} & P_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1} \\ &= \frac{d}{dt}\ln\prod_{i=1}^{n}\left(1-x_it+\cdots+(-x_it)^s\right)^{-1} = \frac{d}{dt}\ln\sum_{k=0}^{\infty}H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k \\ &= \left(\sum_{k=1}^{\infty}kH_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1}\right)\left(\sum_{k=0}^{\infty}H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right)^{-1}.\end{aligned}$$

Rewriting (18) as

$$\begin{aligned}\sum_{k=1}^{\infty} & P_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1} \\ &= \left(\sum_{k=1}^{\infty}(-1)^{k-1}kE_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1}\right)\left(\sum_{k=0}^{\infty}H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right),\end{aligned}$$

we derive the last identity and the theorem is proved. □

Corollary 2.2. *Let k, n and s be positive integers and let $x_1,x_2,\ldots,x_n$ be independent variables. The symmetric functions $E_k^{(s)}=E_k^{(s)}(x_1,x_2,\ldots,x_n)$ and $H_k^{(s)}=H_k^{(s)}(x_1,x_2,\ldots,x_n)$ are related by*

$$2kE_k^{(s)} = \sum_{k_1+k_2+k_3=k}(-1)^{k_3}(k_1+k_2)E_{k_1}^{(s)}E_{k_2}^{(s)}H_{k_3}^{(s)}$$

and

$$kH_k^{(s)} = \sum_{k_1+k_2+k_3=k}(-1)^{k_3-1}k_3H_{k_1}^{(s)}H_{k_2}^{(s)}E_{k_3}^{(s)},$$

where k_1,k_2,k_3 are nonnegative integers.

It is well-known that the power sum symmetric functions p_k can be expressed in terms of elementary symmetric functions e_k using Girard-Newton-Waring formula [5, eq. 8], namely

$$p_k = \sum_{t_1+2t_2+\ldots+kt_k=k}\frac{(-1)^{k+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}e_1^{t_1}e_2^{t_2}\cdots e_k^{t_k}.$$

There is a very similar result which combines the power sum symmetric functions p_k and the complete homogeneous symmetric functions h_k, i.e.,

$$p_k = \sum_{t_1+2t_2+\ldots+kt_k=k}\frac{(-1)^{1+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}h_1^{t_1}h_2^{t_2}\cdots h_k^{t_k}.$$

The following two theorems provide generalizations of these relations.

Theorem 2.3. *Let k, n and s be positive integers and let $x_1,x_2,\ldots,x_n$ be independent variables. The power sum symmetric function $p_k=p_k(x_1,x_2,\ldots,x_n)$ and the generalized symmetric functions $E_k^{(s)}=E_k^{(s)}(x_1,x_2,\ldots,x_n)$ and $H_k^{(s)}=H_k^{(s)}(x_1,x_2,\ldots,x_n)$ are related by*

$$p_k = \frac{\displaystyle\sum_{t_1+2t_2+\cdots+kt_k=k}\frac{(-1)^{t_1+t_2+\cdots+t_k}}{t_1+t_2+\cdots+t_k}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}\prod_{i=1}^{k}\left(E_i^{(s)}\right)^{t_i}}{\displaystyle\sum_{t_1+2t_2+\cdots+st_s=k}\frac{(-1)^{t_1+t_2+\cdots+t_s}}{t_1+t_2+\cdots+t_s}\binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s}}$$

and

$$p_k = \frac{\displaystyle\sum_{t_1+2t_2+\cdots+kt_k=k} \frac{(-1)^{1+t_1+t_2+\cdots+t_k}}{t_1+t_2+\cdots+t_k}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}\prod_{i=1}^{k}\left(H_i^{(s)}\right)^{t_i}}{\displaystyle\sum_{t_1+2t_2+\cdots+st_s=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_s}}{t_1+t_2+\cdots+t_s}\binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s}}.$$

Proof. Considering (4) and the logarithmic series

$$\ln(1+t)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n}, \qquad |t|<1,$$

we can write

$$\begin{aligned}
&\ln\left(\sum_{k=0}^{\infty}E_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right) = \ln\prod_{i=1}^{n}\left(1+x_it+\cdots+(x_it)^s\right)\\
&=\sum_{i=1}^{n}\ln\left(1+x_it+\cdots+(x_it)^s\right) = \sum_{i=1}^{n}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\sum_{j=1}^{s}(x_it)^j\right)^m\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{i=1}^{n}\sum_{k=m}^{sm}\sum_{\substack{t_1+t_2+\cdots+t_s=m\\ t_2+2t_2+\cdots st_s=k}}\binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s}(x_it)^k\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{k=m}^{sm}\sum_{\substack{t_1+t_2+\cdots+t_s=m\\ t_2+2t_2+\cdots st_s=k}}\binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s}p_k(x_1,x_2,\ldots,x_n)t^k\\
&=\sum_{k=1}^{\infty}\sum_{t_1+2t_2+\cdots+st_s=k}\frac{(-1)^{1+t_1+t_2+\cdots+t_s}}{t_1+t_2+\cdots+t_s}\binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s}p_k(x_1,x_2,\ldots,x_n)t^k.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\ln\left(1+\sum_{k=1}^{\infty}E_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right) \qquad (19)\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\sum_{k=1}^{\infty}E_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right)^m\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{k=1}^{\infty}\sum_{\substack{t_1+t_2+\cdots+t_k=m\\ t_1+2t_2+\cdots+kt_k=k}}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}\prod_{i=1}^{k}\left(E_i^{(s)}\right)^{t_i}t^k\\
&=\sum_{k=1}^{\infty}\sum_{t_1+2t_2+\cdots+kt_k=k}\frac{(-1)^{1+t_1+t_2+\cdots+t_k}}{t_1+t_2+\cdots+t_k}\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}\prod_{i=1}^{k}\left(E_i^{(s)}\right)^{t_i}t^k.
\end{aligned}$$

and the first identity follows easily. In a similar way, considering (3) we can prove the second identity. We obtain

$$\begin{aligned}
&\ln\left(\sum_{k=0}^{\infty}H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right)\\
&=\ln\prod_{i=1}^{n}\left(1+(-x_it)+\cdots+(-x_it)^s\right)^{-1} = -\sum_{i=1}^{n}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\sum_{j=1}^{s}(-x_it)^j\right)^m
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{i=1}^{n} \sum_{k=m}^{sm} \sum_{\substack{t_1+t_2+\cdots+t_s=m\\ t_2+2t_2+\cdots st_s=k}} \binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s} (-x_i t)^k$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{k=m}^{sm} \sum_{\substack{t_1+t_2+\cdots+t_s=m\\ t_2+2t_2+\cdots st_s=k}} \binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s} p_k(x_1,x_2,\ldots,x_n)(-t)^k$$

$$= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+st_s=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_s}}{t_1+t_2+\cdots+t_s} \binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s} p_k(x_1,x_2,\ldots,x_n) t^k$$

and

$$\ln\left(1+\sum_{k=1}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right) \tag{20}$$

$$= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{(-1)^{1+t_1+t_2+\cdots+t_k}}{t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \prod_{i=1}^{k} \left(H_i^{(s)}\right)^{t_i} t^k.$$

The proof is finished. □

Theorem 2.4. *Let k, n and s be positive integers and let $x_1,x_2,\ldots,x_n$ be independent variables. The symmetric functions $E_k^{(s)} = E_k^{(s)}(x_1,x_2,\ldots,x_n)$, $H_k^{(s)} = H_k^{(s)}(x_1,x_2,\ldots,x_n)$ and $P_k^{(s)} = P_k^{(s)}(x_1,x_2,\ldots,x_n)$ are related by*

$$P_k^{(s)} = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{(-1)^{1+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \prod_{i=1}^{k} \left(H_i^{(s)}\right)^{t_i}$$

and

$$P_k^{(s)} = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \prod_{i=1}^{k} \left(E_i^{(s)}\right)^{t_i}.$$

Proof. According to (17), (19) and (20), we have

$$\sum_{k=1}^{\infty} P_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1} = \frac{d}{dt} \ln \prod_{i=1}^{n} \left(1-x_i t+\cdots+(-x_i t)^s\right)^{-1}$$

$$= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{(-1)^{1+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \prod_{i=1}^{k} \left(H_i^{(s)}\right)^{t_i} t^{k-1}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} P_k^{(s)}(x_1,x_2,\ldots,x_n)t^{k-1} = \frac{d}{dt} \ln\left(\sum_{k=0}^{\infty} E_k^{(s)}(x_1,x_2,\ldots,x_n)t^k\right)$$

$$= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{(-1)^{1+t_1+t_2+\cdots+t_k}\cdot k}{t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \prod_{i=1}^{k} \left(E_i^{(s)}\right)^{t_i} t^{k-1}.$$

These conclude the proof. □

As a consequence of Theorems 2.3 and 2.4, we remark the following family of identities.

Corollary 2.3. *Let k and s be positive integers. Then*

$$\sum_{t_1+2t_2+\cdots+st_s=k} \frac{(-1)^{t_1+t_2+\cdots+t_s}\cdot k}{t_1+t_2+\cdots+t_s} \binom{t_1+t_2+\cdots+t_s}{t_1,t_2,\ldots,t_s} = \begin{cases} s, & \text{if } k \equiv 0 \bmod (s+1),\\ -1, & \text{otherwise.} \end{cases}$$

It is well-known that the complete and elementary symmetric functions can be expressed in terms of the power sum symmetric functions, i.e.,

$$h_k = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{1}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k} \tag{21}$$

and

$$e_k = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_k}}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k}. \tag{22}$$

The following result provides a generalization of these relations.

Theorem 2.5. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. The symmetric functions $E_k^{(s)} = E_k^{(s)}(x_1, x_2, \ldots, x_n)$, $H_k^{(s)} = H_k^{(s)}(x_1, x_2, \ldots, x_n)$ and $P_k^{(s)} = P_k^{(s)}(x_1, x_2, \ldots, x_n)$ are related by*

$$H_k^{(s)} = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{1}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} \prod_{i=1}^{k} \left(P_i^{(s)}\right)^{t_i}$$

and

$$E_k^{(s)} = \sum_{t_1+2t_2+\ldots+kt_k=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_k}}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} \prod_{i=1}^{k} \left(P_i^{(s)}\right)^{t_i}.$$

Proof. In order to prove this identity, we take into account the following two relations:

$$\ln\left(\sum_{k=0}^{\infty} H_k^{(s)} t^k\right) = \sum_{k=1}^{\infty} \frac{t^k}{k} P_k^{(s)} \quad \text{and} \quad \ln\left(\sum_{k=0}^{\infty} E_k^{(s)} t^k\right) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^k}{k} P_k^{(s)}$$

Considering the exponential series $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, $|z| < 1$, we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k^{(s)} t^k &= \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k!} P_k^{(s)}\right) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} P_k^{(s)}\right)^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=1}^{\infty} \sum_{\substack{t_1+t_2+\cdots t_k=m \\ t_1+2t_2+\cdots+kt_k=k}} \binom{t_1+t_2+\cdots+t_k}{t_1, t_2, \cdots, t_k} \prod_{i=1}^{k} \left(\frac{t^i}{i} P_i^{(s)}\right)^{t_i} \\
&= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{1}{(t_1+t_2+\cdots+t_k)!} \binom{t_1+t_2+\cdots+t_k}{t_1, t_2, \cdots, t_k} \prod_{i=1}^{k} \left(\frac{1}{i} P_i^{(s)}\right)^{t_i} t^k \\
&= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{1}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} \prod_{i=1}^{k} \left(P_i^{(s)}\right)^{t_i} t^k
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} E_k^{(s)} t^k &= \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^k}{k!} P_k^{(s)}\right) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^k}{k!} P_k^{(s)}\right)^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=1}^{\infty} \sum_{\substack{t_1+t_2+\cdots t_k=m \\ t_1+2t_2+\cdots+kt_k=k}} \binom{t_1+t_2+\cdots+t_k}{t_1, t_2, \cdots, t_k} \prod_{i=1}^{k} \left((-1)^{i-1} \frac{t^i}{i} P_i^{(s)}\right)^{t_i} \\
&= \sum_{k=1}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \frac{(-1)^{k+t_1+t_2+\cdots+t_k}}{1^{t_1}t_1!2^{t_2}t_2!\cdots k^{t_k}t_k!} \prod_{i=1}^{k} \left(P_i^{(s)}\right)^{t_i} t^k.
\end{aligned}$$

Thus we arrive at our identities. □

At the end of this section, we remark the following recurrence relations for the generalized symmetric functions $E_k^{(s)}$ and $H_k^{(s)}$.

Theorem 2.6. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$H_k^{(s)}(x_1,x_2,\ldots,x_n) = (-x_n)^{s+1} H_{k-s-1}^{(s)}(x_1,x_2,\ldots,x_n) + H_k^{(s)}(x_1,x_2,\ldots,x_{n-1}) + x_n H_{k-1}^{(s)}(x_1,x_2,\ldots,x_{n-1})$$

and

$$E_k^{(s)}(x_1,x_2,\ldots,x_n) = x_n E_{k-1}^{(s)}(x_1,x_2,\ldots,x_n) + E_k^{(s)}(x_1,x_2,\ldots,x_{n-1}) - x_n^{s+1} E_{k-s-1}^{(s)}(x_1,x_2,\ldots,x_{n-1}).$$

Proof. Taking into account (3), we can write

$$\sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k = \frac{1}{1-x_nt+\cdots+(-x_nt)^s} \sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_{n-1})t^k$$
$$= \frac{1+x_nt}{1-(-x_nt)^{s+1}} \sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_{n-1})t^k.$$

Thus we deduce that

$$\left(1-(-x_nt)^{s+1}\right) \sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_n)t^k = (1+x_nt) \sum_{k=0}^{\infty} H_k^{(s)}(x_1,x_2,\ldots,x_{n-1})t^k.$$

Equating coefficients of t^n on each side of this identity gives the first identity. The second identity follows in a similar way considering (4). □

3. Concluding remarks

In this paper, we investigate a pair of two symmetric functions which generalize the complete and elementary symmetric functions. We show that these generalized symmetric functions satisfy many of the classical relations between complete and elementary symmetric functions. It would be very appealing to investigate combinatorial interpretations of the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$.

REFERENCES

[1] *G. E. Andrews*, The Theory of Partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] *A. Bazeniar, M. Ahmia and H. Belbachir*, Connection between bi^{s}nomial coefficients with their analogs and symmetric functions, Turk J Math. **42**(2018) 807-818.

[3] *H. Fu and Z. Mei*, Truncated homogeneous symmetric functions, Linear Multilinear Algebra. **70**(2022), No. 3, 438-448.

[4] *D. Grinberg*, Petrie symmetric functions, Algebr. Comb. **5**(2022), No. 5, 947-1013.

[5] *H. W. Gould*, The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences, Fibonacci Quart. **37**(1999), No. 2, 135-140.

[6] *I. Macdonald*, Symmetric functions and Hall polynomials, Oxford Univ Press, Oxford, 1979.

[7] *M. Merca*, A generalization of the symmetry between complete and elementary symmetric functions, Indian J. Pure Appl. Math., **45**(2014), No. 1, 75-89.

[8] *M. Merca*, New convolutions for complete and elementary symmetric functions, Integral Transforms Spec. Funct. **27**(2016), No. 12, 965-973.