NEW CRITERIA FOR H-MATRICES AND SPECTRAL DISTRIBUTION

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In this paper, based on the numerical relationship between row and column sums, an equivalent representation for double α_1-matrices is given by partition of the row and column index sets. As its application, we obtain a subclass of H-matrices and the corresponding (Cassini-type) spectral distribution theorem. And then, we provide a numerical example to illustrates the effectiveness of the new results. Finally, two extended criteria for H-matrices are given.

Keywords: double α_1-matrices; H-matrices; irreducible; nonzero elements chain; spectral distribution

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1. Introductions

In the literature on iterative methods of solving linear systems, (nonsingular) H-matrices are widely used because they display many applications when discretizing certain nonlinear parabolic equations in solving the linear complementary problem. However, it is rather difficult in practice to determine whether a matrix is an H-matrix or not.

The eigenvalue localization problem is very closely related to H-matrix theory. It's well known that the Geršgorin theorem for eigenvalue inclusion domain is equivalent to the theorem of non-singularity of strictly diagonally dominant matrices. Strictly doubly diagonally dominant matrices are generalizations of strictly diagonally dominant matrices. The Brauer theorem on strictly doubly diagonally dominant matrices, which gives rise to the Cassini ovals, resembles the Geršgorin theorem on strictly diagonally dominant matrices. Both the Geršgorin discs and the Cassini ovals are classical but effective tools for locating the eigenvalues (spectrum) of a square matrix. Similarly, the

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statements about non-singularity of some subclasses of H-matrices produce new theorems for eigenvalue inclusion domain. In this paper, we propose a subclass of H-matrices and the corresponding (Cassini-type) spectral distribution theorem.

Throughout the paper, we denote $\mathbb{C}^{n\times n}$ the set of all $n \times n$ complex matrices. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ and $\mathcal{M} = \{(i, j) : i \neq j; i, j \in \mathcal{N}\}$. Meanwhile, for $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, for the sake of simplicity, we denote

$$r_i \triangleq r_i(A) = \sum_{j\neq i}^{n} |a_{ij}|;\ c_i \triangleq c_i(A) = \sum_{j\neq i}^{n} |a_{ji}|, (i, j \in \mathcal{N}). \tag{1}$$

According to [1] and [2], suppose $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, satisfies $| a_{ii} |> r_i$ $(i \in \mathcal{N})$, then A is called a strictly (row) diagonally dominant matrix and denoted by $A \in D$. According to [3], suppose $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, satisfies $| a_{ii}a_{jj} |> r_ir_j$ $\big((i, j) \in \mathcal{M}\big)$, then A is said to be a strictly doubly (row) diagonally dominant matrix and denoted by $A \in \tilde{D}$. If there exists a positive diagonal matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ such that AX is a strictly diagonally dominant matrix, then A is called a generalized strictly diagonally dominant matrix and denoted by $A \in D^*$.

Next, the comparison matrix of A, which is denoted by $\mu(A) = [m_{ij}]$, is defined by

$$m_{ij} = \begin{cases} |a_{ij}|, i = j; \\ -|a_{ij}|, i \neq j. \end{cases}$$

If $A = \mu(A)$, and the eigenvalues of A have positive real parts, we call A a (nonsingular) M-matrix. We say that A is an H-matrix if $\mu(A)$ is an M-matrix. In other terms, an H-matrix can be described as matrix with $A \in D^*$ (See [4] and [5]). Thus, all diagonal entries of A are non-zero. So, we always need the assumption that $a_{ii} \neq 0$ for all $i \in \mathcal{N}$.

2. **Review of known results**

For a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, the set of all eigenvalues is called spectrum of the matrix A and denoted by $\sigma(A)$. We denote

$$\Gamma_i(A) = \{z \in \mathbb{C} :| z - a_{ii} |\leqslant r_i, i \in \mathcal{N}\};$$

$$\overline{\Gamma}_i(A) = \{z \in \mathbb{C} :| z - a_{ii} |\leqslant \min\{r_i, c_i\}, i \in \mathcal{N}\};$$

$$\mathcal{H} = \{i : r_i > c_i, i \in \mathcal{N}\},\ \mathcal{L} = \{i : r_i < c_i, i \in \mathcal{N}\},$$

$$\widetilde{\Gamma}_{ij}(A) = \{z \in \mathbb{C} :| z-a_{ii} | (c_j-r_j)+ | z-a_{jj} | (r_i-c_i) \leqslant c_jr_i-c_ir_j, i \in \mathcal{H}, j \in \mathcal{L}\};$$

$$\mathcal{K}_{ij}(A) = \{z \in \mathbb{C} :| z - a_{ii} || z - a_{jj} |\leqslant r_ir_j, (i, j) \in \mathcal{M}\};$$

$$\widetilde{\mathcal{K}}^1_{ij}(A) = \{z \in \mathbb{C} :| z - a_{ii} || z - a_{jj} |\leqslant \alpha r_ir_j + (1 - \alpha)c_ic_j, (i, j) \in \mathcal{M}\};$$

and

$$\widetilde{\mathcal{K}}^2_{ij}(A) = \{z \in \mathbb{C} :| z - a_{ii} || z - a_{jj} |\leqslant (r_ir_j)^{\alpha}(c_ic_j)^{1-\alpha}, (i, j) \in \mathcal{M}\}.$$

The following classical result in matrix theory is well known.

Theorem 2.1. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and $A \in D$; then it is an H-matrix.

The alternative formulation of the above result is as follows (see [6]):

Theorem 2.2. (The Geršgorin Theorem) For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and if λ is an eigenvalue of A; then there is a positive integer $k \in \mathcal{N}$, such that

$$| \lambda - a_{kk} | \leqslant r_k, \tag{2}$$

or, equivalently, $\lambda \in \Gamma_k(A)$.

For each $\lambda \in \sigma(A)$, $\sigma(A) \subseteq \Gamma(A) = \bigcup_{i\in\mathcal{N}} \Gamma_i(A)$.

The following well-known result was found by Ostrowski and rediscovered by Brauer (see [7]).

Theorem 2.3. (The Brauer Theorem) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and $A \in \tilde{D}$; then it is an H-matrix.

Obviously, the Brauer Theorem can be reformulated in the following equivalent way (see [8]).

Theorem 2.4. (The Cassini Ovals Theorem) For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and if λ is an eigenvalue of A; then there is a pair of positive integers $(i, j) \in \mathcal{M}$, such that

$$| \lambda - a_{ii} || \lambda - a_{jj} | \leqslant r_i r_j, \tag{3}$$

or, equivalently, $\lambda \in \mathcal{K}_{ij}(A)$.

For each $\lambda \in \sigma(A)$, $\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{(i,j)\in\mathcal{M}} \mathcal{K}_{ij}(A)$.

Concerning nonsingularity of matrices, there are two well known results that combine the information about a matrix and its transpose, where A^T is the transpose of A (see [9] and [10]).

Theorem 2.5. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be such that

$$| a_{ii}a_{jj} | > (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}, \big((i, j) \in \mathcal{M}\big) \tag{4}$$

holding for some $\alpha \in [0, 1]$; then it is a nonsingular matrix, moreover it is an H-matrix.

Theorem 2.6. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be such that

$$| a_{ii}a_{jj} | > \alpha r_i r_j + (1 - \alpha)c_i c_j, \big((i, j) \in \mathcal{M}\big) \tag{5}$$

holding for some $\alpha \in [0, 1]$; then it is a nonsingular matrix, moreover it is an H-matrix.

The matrices that fulfill the condition (4) are known as (strictly) double α_2-matrices, while (strictly) double α_1-matrices are the matrices that fulfill the condition (5). The two classes of matrices are two subclasses of H-matrices which generalize the Brauer Theorem property. As before, the corresponding theorems in the field of eigenvalue localization are as follows.

Theorem 2.7. ([9]) For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and if λ is an eigenvalue of A; then for each $\alpha \in [0, 1]$ there is a pair of positive integers $(i, j) \in \mathcal{M}$, such that

$$| \lambda - a_{ii} || \lambda - a_{jj} | \leqslant (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}, \tag{6}$$

or, equivalently, $\lambda \in \widetilde{\mathcal{K}}^2_{ij}(A)$.

For each $\lambda \in \sigma(A)$, $\sigma(A) \subseteq \mathcal{K}_2(A) = \bigcap\limits_{0\leqslant\alpha\leqslant1} \bigcup\limits_{(i,j)\in\mathcal{M}} \widetilde{\mathcal{K}}^2_{ij}(A)$.

By the generalized arithmetic-geometric mean inequality (see [11]) which is as follows:

$$\alpha\tau + (1-\alpha)\sigma \geqslant \tau^{\alpha}\sigma^{1-\alpha}, \tag{7}$$

where $\sigma, \tau \geqslant 0$, $\alpha \in [0, 1]$, with equality holding for $\tau = \sigma$ or $\alpha = 0$, else or $\alpha = 1$, we easily get the following theorem.

Theorem 2.8. For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and if λ is an eigenvalue of A; then for each $\alpha \in [0, 1]$ there is a pair of positive integers $(i, j) \in \mathcal{M}$, such that

$$| \lambda - a_{ii} || \lambda - a_{jj} | \leqslant \alpha r_i r_j + (1-\alpha)c_i c_j, \tag{8}$$

or, equivalently, $\lambda \in \widetilde{\mathcal{K}}^1_{ij}(A)$.

For each $\lambda \in \sigma(A)$, $\sigma(A) \subseteq \mathcal{K}_1(A) = \bigcap\limits_{0\leqslant\alpha\leqslant1} \bigcup\limits_{(i,j)\in\mathcal{M}} \widetilde{\mathcal{K}}^1_{ij}(A)$.

Recently, in [12], L. Cvetković, etc gave the following eigenvalue inclusion region by the necessary and sufficient condition of α_1-matrices (see [13]).

Theorem 2.9. ([12]) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$; then

$$\sigma(A) \subseteq \mathcal{A}_1(A) = \overline{\Gamma}(A) \bigcup \widetilde{\Gamma}(A), \tag{9}$$

where $\overline{\Gamma}(A) = \bigcup\limits_{i\in\mathcal{N}} \overline{\Gamma}_i(A)$ and $\widetilde{\Gamma}(A) = \bigcup\limits_{i\in\mathcal{H}, j\in\mathcal{L}} \widetilde{\Gamma}_{ij}(A)$.

Example 3.9 of [10] shows that $\mathcal{K}_1(A) \subseteq \mathcal{A}_1(A)$.

Remark 2.1. Obviously, for Theorems 2.7 and 2.8, there exists a problem: the methods for the field of eigenvalue localization need to take an arbitrary parameter $\alpha \in [0, 1]$ into account, and it seems hard and complicated to decide the optimum value of α. In this work, to overcome this drawback, we propose a new improved version, which is always convergent for a double α_1-matrix.

3. Criterion for identifying H-matrices and spectral distribution

Remark 3.1. We begin this section by making a remark that for a double α_1-matrix, if $\alpha = 0$, then $A^T \in \tilde{D}$; and if $\alpha = 1$, then $A \in \tilde{D}$. Consequently, in either case, A is an H-matrix (see [14]). So, we consider the case $\alpha \in (0, 1)$ only when we discuss double α_1-matrices in the following discussion.

Throughout the next paper, we will use the following notations.

$$\mathcal{M}_1 = \big\{(i,j) \in \mathcal{M} : r_i r_j <\mid a_{ii}a_{jj} \mid< c_i c_j\big\};$$

$$\mathcal{M}_2 = \big\{(i,j) \in \mathcal{M} : c_i c_j <\mid a_{ii}a_{jj} \mid< r_i r_j\big\};$$

$$\mathcal{M}_3 = \big\{(i,j) \in \mathcal{M} :\mid a_{ii}a_{jj} \mid\geqslant c_i c_j > r_i r_j\big\};$$

$$\mathcal{M}_4 = \big\{(i,j) \in \mathcal{M} :\mid a_{ii}a_{jj} \mid\geqslant r_i r_j > c_i c_j\big\};$$

$$\mathcal{M}_5 = \big\{(i,j) \in \mathcal{M} :\mid a_{ii}a_{jj} \mid> r_i r_j = c_i c_j\big\};$$

$$\mathcal{M}_0 = \big\{(i,j) \in \mathcal{M} :\mid a_{ii}a_{jj} \mid\leqslant c_i c_j, \mid a_{ii}a_{jj} \mid\leqslant r_i r_j\big\}.$$

Obviously, $\mathcal{M} = \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5 \bigcup \mathcal{M}_0$.

We also state and prove some lemmas before we present our main results of this section.

Lemma 3.1. Consider the function $f(t) = at + b(1-t)$, for any $t \in (0, 1)$. Then

(1) if $a > b > 0$, we get that $f(t)$ is a monotonically increasing function;
(2) if $b > a > 0$, we get that $f(t)$ is a monotonically decreasing function.

Proof. Since $f'(t) = a - b$, this conclusion is obvious. □

Lemma 3.2. For any $\varepsilon > 0$ ($\varepsilon = o(x), x \to 0$), we define

$$\mathcal{E} = \big\{z \in \mathbb{C} :\mid z-a \mid\mid z-b \mid\leqslant c+\varepsilon\big\} \tag{10}$$

and

$$\mathcal{F} = \big\{z \in \mathbb{C} :\mid z-a \mid\mid z-b \mid\leqslant c\big\}. \tag{11}$$

Then $\mathcal{E} = \mathcal{F}$.

Proof. "$\mathcal{F} \subset \mathcal{E}$" is obvious. We need only to check "$\mathcal{F} \supset \mathcal{E}$". Suppose $z \in \mathcal{E}$, $z \notin \mathcal{F}$; then $\mid z-a \mid\mid z-b \mid> c$. Choose $\varepsilon_0 = \frac{1}{2}(\mid z-a \mid\mid z-b \mid -c)$; then $\mid z-a \mid\mid z-b \mid= c+2\varepsilon_0 > c+\varepsilon_0$, however, $z \in \mathcal{E}$, a contradiction. Hence, $z \in \mathcal{F}$. Therefore we have $\mathcal{E} = \mathcal{F}$. □

Now, we will give an equivalent representation for double α_1-matrices.

Theorem 3.1. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, $\mathcal{M}_0 = \emptyset$; then A is a double α_1-matrix if and only if the following condition holds

$$\frac{\mid a_{ss}a_{tt} \mid -r_s r_t}{c_s c_t - r_s r_t} + \frac{\mid a_{ii}a_{jj} \mid -c_i c_j}{r_i r_j - c_i c_j} > 1, \big((s,t) \in \mathcal{M}_1, (i,j) \in \mathcal{M}_2\big). \tag{12}$$

Proof. First, let us assume that A is a double α_1-matrix; then $\mathcal{M}_0 = \emptyset$. Therefore, for any $(s,t) \in \mathcal{M}_1$, there exists some $\alpha \in (0,1)$, such that

$$| a_{ss}a_{tt} |> \alpha r_s r_t + (1-\alpha)c_s c_t = c_s c_t - \alpha(c_s c_t - r_s r_t),$$

i.e., $\alpha > \dfrac{c_s c_t - | a_{ss}a_{tt} |}{c_s c_t - r_s r_t}$, that is

$$1-\alpha < \frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t}. \tag{13}$$

Similarly, for any $(i,j) \in \mathcal{M}_2$, we obtain that

$$| a_{ii}a_{jj} |> \alpha r_i r_j + (1-\alpha)c_i c_j = c_i c_j + \alpha(r_i r_j - c_i c_j),$$

i.e., $\alpha < \dfrac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j}$, which combined with (13), implies

$$\frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} + \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} > 1, \big((s,t) \in \mathcal{M}_1, (i,j) \in \mathcal{M}_2\big).$$

Conversely, assume the condition (12) holds. For any $(s,t) \in \mathcal{M}_1$, it directly implies $0 < \dfrac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} < 1$, that is

$$0 < 1 - \frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} < 1.$$

Similarly, for any $(i,j) \in \mathcal{M}_2$, we obtain

$$0 < \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} < 1.$$

The strict inequality of (12) ensures that there exists a parameter α, for any $(s,t) \in \mathcal{M}_1$, $(i,j) \in \mathcal{M}_2$, such that

$$0 < \frac{c_s c_t - | a_{ss}a_{tt} |}{c_s c_t - r_s r_t} = 1 - \frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} < \alpha < \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} < 1. \tag{14}$$

Starting from the inequality $\dfrac{c_s c_t - | a_{ss}a_{tt} |}{c_s c_t - r_s r_t} < \alpha$ of (14), for any $(s,t) \in \mathcal{M}_1$, we get

$$| a_{ss}a_{tt} |> \alpha r_s r_t + (1-\alpha)c_s c_t.$$

In the same way, from the inequality $\alpha < \dfrac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j}$ of (14), for any $(i,j) \in \mathcal{M}_2$, we get

$$| a_{ii}a_{jj} |> \alpha r_i r_j + (1-\alpha)c_i c_j.$$

Moreover, for any $(l,m) \in \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5$, and any $\alpha \in (0,1)$, it is obvious to see that

$$| a_{ll}a_{mm} |> \alpha r_l r_m + (1-\alpha)c_l c_m.$$

Recalling that $\mathcal{M}_0 = \emptyset$, for any $(i,j) \in \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5 = \mathcal{M}$ there exists a parameter α, such that

$$| a_{ii}a_{jj} |> \alpha r_i r_j + (1-\alpha)c_i c_j.$$

Since $\alpha \in (0,1)$, this concludes the proof. $\square$

As its application, a new practical criterion for H-matrices is obtained.

Theorem 3.2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, $\mathcal{M}_0 = \emptyset$, which for any $(s,t) \in \mathcal{M}_1$, $(i,j) \in \mathcal{M}_2$ satisfies

$$\frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} + \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} > 1. \tag{15}$$

Then A is an H-matrix.

Proof. By Theorem 3.1, it is clear that A is a double α_1-matrix, and further using Theorem 2.6, we conclude that A is an H-matrix. $\square$

Having the result of Theorem 3.2, we are ready to give the corresponding (Cassini-type) spectral distribution.

Theorem 3.3. For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, if A is a double α_1-matrix; then

$$\sigma(A) \subseteq \mathcal{G}(A) = \left(\bigcup_{(s,t)\in\mathcal{M}_1 \bigcup \mathcal{M}_3} \mathcal{G}_{st} \right) \bigcup \left(\bigcup_{(i,j)\in\mathcal{M}_2 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5} \mathcal{G}_{ij} \right), \tag{16}$$

where

$$\mathcal{G}_{st} = \left\{ z \in \mathbb{C} : | z - a_{ss} || z - a_{tt} | \leqslant \left(\min_{(k,l)\in\mathcal{M}_2} \frac{| a_{kk}a_{ll} | - c_k c_l}{r_k r_l - c_k c_l} \right) r_s r_t + \left(\max_{(k,l)\in\mathcal{M}_2} \frac{r_k r_l - | a_{kk}a_{ll} |}{r_k r_l - c_k c_l} \right) c_s c_t \right\}, \big((s,t) \in \mathcal{M}_1 \bigcup \mathcal{M}_3\big),$$

and

$$\mathcal{G}_{ij} = \left\{ z \in \mathbb{C} : | z - a_{ii} || z - a_{jj} | \leqslant \left(\max_{(p,q)\in\mathcal{M}_1} \frac{c_p c_q - | a_{pp}a_{qq} |}{c_p c_q - r_p r_q} \right) r_i r_j + \left(\min_{(p,q)\in\mathcal{M}_1} \frac{| a_{pp}a_{qq} | - r_p r_q}{c_p c_q - r_p r_q} \right) c_i c_j \right\}, \big((i,j) \in \mathcal{M}_2 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5\big).$$

Proof. First, let us assume that A is a double α_1-matrix; then $\mathcal{M}_0 = \emptyset$. From the proof of Theorem 3.1, for any parameter α with

$$0 < \frac{c_s c_t - | a_{ss}a_{tt} |}{c_s c_t - r_s r_t} = 1 - \frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} < \alpha < \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} < 1,$$

i.e., for any $\alpha \in I = \left(\max_{(s,t)\in\mathcal{M}_1} \frac{c_s c_t - | a_{ss}a_{tt} |}{c_s c_t - r_s r_t}, \min_{(i,j)\in\mathcal{M}_2} \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} \right) \subset (0,1)$, we obtain

$$| a_{ii}a_{jj} |> \alpha r_i r_j + (1-\alpha)c_i c_j, \big((i,j) \in \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5 = \mathcal{M}\big).$$

By Theorem 2.8, for any eigenvalue λ of A, there is a pair of positive integers $(i,j)\in\mathcal{M}$, such that

$$|\lambda-a_{ii}||\lambda-a_{jj}|\leqslant \alpha r_ir_j+(1-\alpha)c_ic_j,(\alpha\in I).$$

We consider the following five cases.

Case 1. If $(s,t)\in\mathcal{M}_1$, then $c_sc_t>r_sr_t>0$. So, the function $f(t)=r_sr_tt+c_sc_t(1-t)$ is monotonically decreasing by Lemma 3.1. Thus, let $\varepsilon_1=\dfrac{\varepsilon}{c_sc_t-r_sr_t}>0$, where $\varepsilon>0$ $\big(\varepsilon=o(x),x\to 0\big)$; we get

$$\begin{aligned}|\lambda-a_{ss}||\lambda-a_{tt}|\leqslant&\left(\min_{(k,l)\in\mathcal{M}_2}\frac{|a_{kk}a_{ll}|-c_kc_l}{r_kr_l-c_kc_l}-\varepsilon_1\right)r_sr_t+\\&\left[1-\left(\min_{(k,l)\in\mathcal{M}_2}\frac{|a_{kk}a_{ll}|-c_kc_l}{r_kr_l-c_kc_l}-\varepsilon_1\right)\right]c_sc_t\\=&\left(\min_{(k,l)\in\mathcal{M}_2}\frac{|a_{kk}a_{ll}|-c_kc_l}{r_kr_l-c_kc_l}-\varepsilon_1\right)r_sr_t+\\&\left[1-\left(1-\max_{(k,l)\in\mathcal{M}_2}\frac{r_kr_l-|a_{kk}a_{ll}|}{r_kr_l-c_kc_l}\right)+\varepsilon_1\right]c_sc_t\\=&\left(\min_{(k,l)\in\mathcal{M}_2}\frac{|a_{kk}a_{ll}|-c_kc_l}{r_kr_l-c_kc_l}\right)r_sr_t+\\&\left(\max_{(k,l)\in\mathcal{M}_2}\frac{r_kr_l-|a_{kk}a_{ll}|}{r_kr_l-c_kc_l}\right)c_sc_t+\varepsilon,\end{aligned}$$

i.e.,

$$\lambda\in\left\{z\in\mathbb{C}:|z-a_{ss}||z-a_{tt}|\leqslant\begin{array}{l}\left(\min\limits_{(k,l)\in\mathcal{M}_2}\dfrac{|a_{kk}a_{ll}|-c_kc_l}{r_kr_l-c_kc_l}\right)r_sr_t+\\\left(\max\limits_{(k,l)\in\mathcal{M}_2}\dfrac{r_kr_l-|a_{kk}a_{ll}|}{r_kr_l-c_kc_l}\right)c_sc_t+\varepsilon\end{array}\right\}.$$

Now, by applying Lemma 3.2, we obtain that $\lambda\in\mathcal{G}_{st}$, $(s,t)\in\mathcal{M}_1$.

Case 2. If $(i,j)\in\mathcal{M}_2$, then $r_ir_j>c_ic_j>0$. So, the function $f(t)=r_ir_jt+c_ic_j(1-t)$ is monotonically increasing by Lemma 3.1. Thus, let $\varepsilon_2=\dfrac{\varepsilon}{r_ir_j-c_ic_j}>0$, where $\varepsilon>0$ $\big(\varepsilon=o(x),x\to 0\big)$; we get

$$\begin{aligned}|\lambda-a_{ii}||\lambda-a_{jj}|\leqslant&\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}+\varepsilon_2\right)r_ir_j+\\&\left[1-\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}+\varepsilon_2\right)\right]c_ic_j\\=&\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}+\varepsilon_2\right)r_ir_j+\end{aligned}$$

$$\left[1-\left(1-\min_{(p,q)\in\mathcal{M}_1}\frac{|a_{pp}a_{qq}|-r_pr_q}{c_pc_q-r_pr_q}\right)-\varepsilon_2\right]c_ic_j$$

$$=\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}\right)r_ir_j+$$

$$\left(\min_{(p,q)\in\mathcal{M}_1}\frac{|a_{pp}a_{qq}|-r_pr_q}{c_pc_q-r_pr_q}\right)c_ic_j+\varepsilon,$$

i.e.,

$$\lambda\in\left\{z\in\mathbb{C}:|z-a_{ii}||z-a_{jj}|\leqslant\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}\right)r_ir_j+\left(\min_{(p,q)\in\mathcal{M}_1}\frac{|a_{pp}a_{qq}|-r_pr_q}{c_pc_q-r_pr_q}\right)c_ic_j+\varepsilon\right\}.$$

Lemma 3.2 obviously implies that $\lambda\in\mathcal{G}_{ij}$, $(i,j)\in\mathcal{M}_2$.

Case 3. If $(s,t)\in\mathcal{M}_3$, similarly to the proof of *Case* 1, we can show $\lambda\in\mathcal{G}_{st}$, $(s,t)\in\mathcal{M}_3$.

Case 4. If $(i,j)\in\mathcal{M}_4$, similarly to the proof of *Case* 2, we get $\lambda\in\mathcal{G}_{ij}$, $(i,j)\in\mathcal{M}_4$.

Case 5. If $(i,j)\in\mathcal{M}_5$, then $r_ir_j=c_ic_j<|a_{ii}a_{jj}|$; thus it is obvious that

$$|a_{ii}a_{jj}|>\alpha r_ir_j+(1-\alpha)c_ic_j,(\alpha\in I).$$

Especially, let $\alpha=\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}$; then by Theorem 2.8, we get

$$|\lambda-a_{ii}||\lambda-a_{jj}|\leqslant\left(\max_{(p,q)\in\mathcal{M}_1}\frac{c_pc_q-|a_{pp}a_{qq}|}{c_pc_q-r_pr_q}\right)r_ir_j+\left(\min_{(p,q)\in\mathcal{M}_1}\frac{|a_{pp}a_{qq}|-r_pr_q}{c_pc_q-r_pr_q}\right)c_ic_j.$$

Therefore, $\lambda\in\mathcal{G}_{ij}$, $(i,j)\in\mathcal{M}_5$.

Note that A is a double α_1-matrix, and that $\mathcal{M}=\mathcal{M}_1\bigcup\mathcal{M}_2\bigcup\mathcal{M}_3\bigcup\mathcal{M}_4\bigcup\mathcal{M}_5$, from which we conclude

$$\lambda\in\mathcal{G}(A)=\left(\bigcup_{(s,t)\in\mathcal{M}_1\bigcup\mathcal{M}_3}\mathcal{G}_{st}\right)\bigcup\left(\bigcup_{(i,j)\in\mathcal{M}_2\bigcup\mathcal{M}_4\bigcup\mathcal{M}_5}\mathcal{G}_{ij}\right).$$

The proof is completed. □

Finally, we provide a numerical example which illustrates the effectiveness and advantage of the new criteria.

Example 3.1. Let

$$A = \begin{pmatrix} 3.3 & 0.5 & 1.5 \\ 2 & 2.5 & 1 \\ 1.6 & 1 & 2.5\mathrm{i} \end{pmatrix}.$$

Then we have

$$\begin{gathered} | a_{11} |= 3.3, | a_{22} |= 2.5, | a_{33} |= 2.5; \\ r_1 = 2, r_2 = 3, r_3 = 2.6; \\ c_1 = 3.6, c_2 = 1.5, c_3 = 2.5. \end{gathered}$$

But, we notice $| a_{33} |= 2.5 = c_3 < r_3 = 2.6$. The condition satisfies neither Theorem 1 in [15] nor Theorem 4 or 5 in [12], so we obtain that A is not an α_1-matrix or an α_2-matrix (see [13]). Hence, we can't get that A is an H-matrix.

According to the notations of this paper, we have

$$\mathcal{M}_1 = \{(1,3)\}, \mathcal{M}_2 = \{(2,3)\}, \mathcal{M}_4 = \{(1,2)\}, \mathcal{M}_3 = \mathcal{M}_5 = \mathcal{M}_0 = \emptyset.$$

By calculation, we obtain

$$\frac{| a_{11}a_{33} | - r_1 r_3}{c_1 c_3 - r_1 r_3} + \frac{| a_{22}a_{33} | - c_2 c_3}{r_2 r_3 - c_2 c_3} \approx 0.8026 + 0.6173 = 1.4199 > 1.$$

Therefore, A satisfies the condition of Theorem 3.2; then A is an H-matrix.

Calculated by MATLAB 7.0, eigenvalues of the matrix A are $\lambda_1 = 4.6226 + 0.3013\mathrm{i}$, $\lambda_2 = 1.7627 + 0.0452\mathrm{i}$ and $\lambda_3 = -0.5852 + 2.2438\mathrm{i}$.

The eigenvalue inclusion regions of Theorem 2.9 in yellow and Theorem 3.3 in blue are given, respectively, by Figs.3.1 and 3.2. By comparing of the regions in Fig.3.3, it is easy to see that the region of Theorem 3.3 is smaller than that of Theorem 2.9.

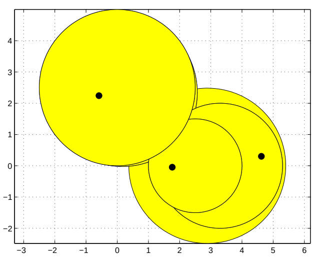

Fig.3.1

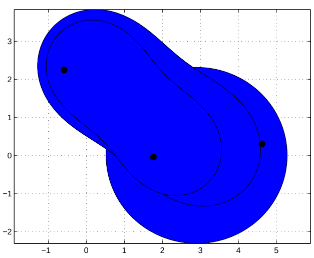

Fig.3.2

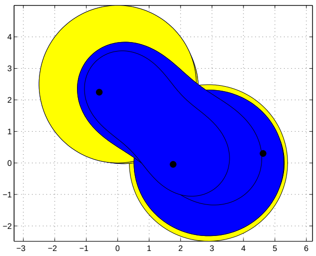

Fig.3.3

4. Criterion for identifying H-matrices via irreducibility

In this section, we will extend the previous result by letting all the considered inequalities not to be strict and $\mathcal{M}_0 \neq \emptyset$. Now, we will use the following notations: $\Psi(A)$ denotes the set of simple circuits in the directed graph of the matrix A, and if $\gamma = i_1 i_2 \dots i_k i_{k+1}$, $i_{k+1} = i_1$ is a simple circuit of length k, then the support of γ, i.e., the set $\{i_1, i_2, \dots, i_k\}$ is denoted by $\overline{\gamma}$. We will deal with irreducible matrices (see [4]). Within this class we will prove a new criterion for a matrix to be an H-matrix, which based on the following fact.

Theorem 4.1. ([16]) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geqslant 2$, be irreducible. Suppose

$$| a_{ii}a_{jj} | \geqslant (r_i r_j)^\alpha (c_i c_j)^{1-\alpha}, \big((i,j) \in \mathcal{M}\big)$$

holds for some $\alpha \in [0,1]$, with strict inequality for at least one circuit $\gamma_0 \in \Psi(A)$ and $i_0, j_0 \in \overline{\gamma}_0$, such that

$$| a_{i_0 i_0} a_{j_0 j_0} | > (r_{i_0} r_{j_0})^\alpha (c_{i_0} c_{j_0})^{1-\alpha};$$

then A is an H-matrix.

Theorem 4.2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geqslant 2$, be irreducible. For any $(p,q) \in \mathcal{M}_0 \neq \emptyset$, there is $| a_{pp}a_{qq} | = r_p r_q = c_p c_q$, and for any $(s,t) \in \mathcal{M}_1$, $(i,j) \in \mathcal{M}_2$, it satisfies

$$\frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} + \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} \geqslant 1. \tag{17}$$

If there exists at least one circuit $\gamma_0 \in \Psi(A)$ and $i_0, j_0 \in \overline{\gamma}_0$, such that

$$\frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} + \frac{| a_{i_0 i_0}a_{j_0 j_0} | - c_{i_0} c_{j_0}}{r_{i_0} r_{j_0} - c_{i_0} c_{j_0}} > 1; \tag{18}$$

then A is an H-matrix.

Proof. Similarly to the proof of Theorem 3.1, the inequality (17) implies that

$$| a_{ss}a_{tt} | \geqslant \alpha r_s r_t + (1-\alpha) c_s c_t > (r_s r_t)^\alpha (c_s c_t)^{1-\alpha}, \big((s,t) \in \mathcal{M}_1\big);$$

$$| a_{ii}a_{jj} | \geqslant \alpha r_i r_j + (1-\alpha) c_i c_j > (r_i r_j)^\alpha (c_i c_j)^{1-\alpha}, \big((i,j) \in \mathcal{M}_2\big).$$

Moreover, for any $(l, m) \in \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5$, and any $\alpha \in (0, 1)$, it is obvious that

$$| a_{ll}a_{mm} |> \alpha r_l r_m + (1 - \alpha)c_l c_m > (r_l r_m)^{\alpha}(c_l c_m)^{1-\alpha}.$$

On the other hand, for any $(p, q) \in \mathcal{M}_0 \neq \emptyset$, there is $| a_{pp}a_{qq} |= r_p r_q = c_p c_q$, i.e.,

$$| a_{pp}a_{qq} |= \alpha r_p r_q + (1 - \alpha)c_p c_q = (r_p r_q)^{\alpha}(c_p c_q)^{1-\alpha}, \big(\alpha \in (0, 1)\big).$$

In a word, for any $(i, j) \in \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5 \bigcup \mathcal{M}_0 = \mathcal{M}$, there exists some $\alpha \in (0, 1)$, such that

$$| a_{ii}a_{jj} |\geqslant (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}.$$

By the inequality (18), we know that there exists at least one circuit $\gamma_0 \in \Psi(A)$ and $i_0, j_0 \in \overline{\gamma}_0$, such that

$$| a_{i_0 i_0}a_{j_0 j_0} |> (r_{i_0} r_{j_0})^{\alpha}(c_{i_0} c_{j_0})^{1-\alpha}.$$

Since A is irreducible, by Theorem 4.1, A is an H-matrix. □

5. Criterion for identifying H-matrices via nonzero elements chain

In this section, we will prove a new criterion for a matrix to be an H-matrix, which based on the following fact: a double α_2-matrix will remain to be an H-matrix if we change the irreducibility with the existence of nonzero elements chain (see [17]), more precisely with the following condition.

Theorem 5.1. ([18]) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and let

$$| a_{ii}a_{jj} |\geqslant (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}, \big((i, j) \in \mathcal{M}\big)$$

holds for some $\alpha \in (0, 1)$. For every $(i, j) \in \mathcal{M}$ with $| a_{ii}a_{jj} |= (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}$, if there exists a nonzero elements chain $a_{i_0 i_1}, a_{i_1 i_2}, \ldots, a_{i_r j_0}$, such that $i_0 = i$ or $i_0 = j$, and $j_0 \in J(A)$, where

$$J(A) = \Big\{i \in \mathcal{N} :| a_{ii}a_{jj} |> (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}, (i, j) \in \mathcal{M}\Big\}\neq \emptyset;$$

then A is an H-matrix.

Theorem 5.2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geqslant 2$, for any $(s, t) \in \mathcal{M}_1$, $(i, j) \in \mathcal{M}_2$, such that

$$\frac{| a_{ss}a_{tt} | - r_s r_t}{c_s c_t - r_s r_t} + \frac{| a_{ii}a_{jj} | - c_i c_j}{r_i r_j - c_i c_j} \geqslant 1.$$

For any $(p, q) \in \mathcal{M}_0 \neq \emptyset$, there is $| a_{pp}a_{qq} |= r_p r_q = c_p c_q$, and if there exists a nonzero elements chain $a_{p_0 p_1}, a_{p_1 p_2}, \ldots, a_{p_h q_0}$, such that $p_0 = p$ or $p_0 = q$, and $q_0 \in G(A)$, where

$$G(A) = \Big\{i \in \mathcal{N} : (i, j) \in \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5\Big\} \neq \emptyset;$$

then A is an H-matrix.

Proof. Similarly to the proof of Theorem 4.2, we can obtain that for any $(i,j) \in \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4 \bigcup \mathcal{M}_5$, there exists some $\alpha \in (0,1)$, such that

$$| a_{ii}a_{jj} | \geqslant \alpha r_i r_j + (1-\alpha)c_i c_j > (r_i r_j)^{\alpha}(c_s c_t)^{1-\alpha}.$$

By the assumption, for any $(p,q) \in \mathcal{M}_0 \neq \emptyset$, there is $| a_{pp}a_{qq} | = r_p r_q = c_p c_q$, i.e.,

$$| a_{pp}a_{qq} | = \alpha r_p r_q + (1-\alpha)c_p c_q = (r_p r_q)^{\alpha}(c_p c_q)^{1-\alpha}, \big(\alpha \in (0,1)\big)$$

and there exists a nonzero elements chain $a_{p_0 p_1}, a_{p_1 p_2}, \ldots, a_{p_h q_0}$, such that $p_0 = p$ or $p_0 = q$, and $q_0 \in \overline{J}(A)$, where

$$\overline{J}(A) = \Big\{ i \in \mathcal{N} : | a_{ii}a_{jj} | > (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}, (i,j) \in \mathcal{M} \Big\} \neq \emptyset.$$

On the base of Theorem 5.1, A is an H-matrix. □

6. **Conclusions**

An equivalent representation for double α_1-matrices is given based on the numerical relationship between row and column sums. As its application, a subclass of H-matrices and the corresponding (Cassini-type) spectral distribution theorem are obtained. In the end, the new criterion for H-matrices is extended via irreducibility and nonzero elements chain.

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