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DUALITY FOR A CLASS OF NONSMOOTH SEMI-INFINITE MULTIOBJECTIVE FRACTIONAL OPTIMIZATION PROBLEMS

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In this paper, we continue the effort of Singh et al. [V. Singh, A. Jayswal, I. Stancu-Minasian and A.M. Rusu-Stancu, Isolated and proper efficiencies for semi-infinite multiobjective fractional problems, U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 3, 2021, pp. 111-124] to discuss duality results for a nonsmooth semi-infinite multiobjective fractional optimization problem with infinite number of inequality constraints by employing some advanced tools of variational analysis and generalized differentiation. We propose a Mond-Weir dual problem and prove weak/strong duality theorems for local properly efficient solutions under generalized convexity. In order to justify the significance of obtained results we consider a numerical example.

Keywords: Limiting subdifferential, generalized convex function, strongly isolated solution, positively properly efficient solution, efficient solution, semi-infinite multiobjective fractional programming, optimality conditions.

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1. Introduction

Many realistic problems that arise practically in various fields of science, business, and technology deal with several conflicting ratios of objective functions, which have to be optimized simultaneously. Such problems are known as vector fractional problems or multiobjective fractional problems. The difficulty of multiobjective fractional programming lies in the fact that the ratio of objectives of vector fractional problems are in conflict with each other and an improvement of one objective may lead to the reduction of other objectives. A multiobjective fractional model provides the mathematical framework to deal with such situations. The available literature on optimality conditions and various types of duality for multiobjective fractional programming problems is very rich (see, for example, several monographs on multiobjective fractional programming which have been published in recent past (c.f., [1, 2, 3, 17, 18, 20, 27, 28, 29, 31])).

The specialty of a multiobjective fractional optimization problem is that its objective functions are generally not convex functions. Indeed under all the more restrictive concavity/convexity assumptions, multiobjective fractional optimization problems are generally nonconvex ones. While, the (approximate) extremal principle [22], which plays a central role in variational analysis and generalized differentiation, has been well-recognized as a variational counterpart of the separation theorem for nonconvex sets. Subsequently, utilizing the

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extremal principle and other advanced techniques of variational analysis and generalized differentiation to prove optimality conditions appears to be appropriate for nonconvex and nonsmooth optimization problems.

A semi-infinite multiobjective fractional problem is the simultaneously minimization of finitely many scalar ratio objective functions subject to an arbitrary (possibly infinite) set of constraint functions. Fundamental theoretical aspects and a wide range of applications of both scalar and multiobjective semi-infinite fractional programming problems have been studied intensively by many researchers (see, for example, [1, 7, 13, 16, 19] and others).

During the most recent two decades, there has been a vastly fast evolution in subdifferential calculus of nonsmooth analysis which is well-recognized for its numerous applications to optimization theory. The Mordukhovich subdifferential is a highly vital concept in nonsmooth analysis and closely related to optimality conditions of locally Lipschitzian functions of optimization theory (see, [15, 24, 32]). The Mordukhovich subdifferential is a closed subset of the Clarke subdifferential and these subdifferentials are in general nonconvex sets, unlike the well-known Clarke subdifferentials. Therefore, keeping the importance of optimization problems and its wide applications, the explanations of the optimality conditions and calculus rules in terms of Mordukhovich subdifferentials provide more sharp results than those given in terms of the Clarke generalized gradient (see e.g., [22]). Chuong and Kim [6] derived optimality conditions and duality relations that are expressed in terms of limiting/Mordukhovich subdifferentials for nonsmooth multiobjective fractional programming problems.

In this paper, inspired by the earlier works, we use the limiting/ Mordukhovich subdifferentials given in [22, 23] to demonstrate several duality theorems under the assumption of generalized convexity. Although many discussions have been done on this topic, it still remains a very interesting and demanding area of research. There are several approaches developed in the literature, see [14, 12, 20, 18, 22, 23] and the references therein.

We now moving forward to discuss the contents of this paper. Section 2 consists of some basic definitions and background material. In Section 3, we turn to an investigation of the notion of duality for (local) positively properly efficient solutions in a nonsmooth semi-infinite multiobjective fractional optimization problem. Here, we propose a Mond-Weir type dual problem and prove weak and strong duality theorems. Finally, the paper is concluded in Section 4.

2. Preliminaries

The aim of this section is to provide some basic concepts and auxiliary results that will be used often throughout the paper.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Unless otherwise stated, all the spaces considered in the paper are Banach whose norms are always denoted by $\|.\|$ and X^* is dual of a given space X. The canonical pairing between X and X^* is denoted by $\langle .,. \rangle$ and $S^\circ = \{x^* \in X^* : \langle x^*, x \rangle \leq 0,\ \forall x \in S\}$ is the polar cone of a set $S \subset X$. As usual, the notation cl$S$ and int$S$ represent the closure and respectively, the interior of S. Now, we recall the following Definitions 2.1-2.4 from Mordukhovich [22].

Definition 2.1 (Mordukhovich [22])**.** *Let $F : X \rightrightarrows X^*$ be a multifunction. Then the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^* is given by*

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) = \{x^* \in X^* : \exists \text{ sequences } x_n \to \bar{x} \text{ and } x_n^* \stackrel{w^*}{\to} x^*$$

$$\text{with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N}\},$$

where the notation $\overset{w^*}{\to}$ *indicates the convergence in the weak** *topology of* X^* *and* $\mathbb{N}$ *denotes the set of all natural numbers.*

Definition 2.2 (Mordukhovich [22]). *For a given* $\epsilon \geq 0$ *and a set* S*, the collection of* ϵ*-normals to* S *at* $\bar{x} \in S$ *is defined by*

$$\widehat{N}_\epsilon(\bar{x}, S) = \{x^* \in X^* : \operatorname{Lim\,sup}_{x \overset{S}{\to} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \epsilon\}, \tag{2.1}$$

where $x \overset{S}{\to} \bar{x}$ *means that* $x \to \bar{x}$ *with* $x \in S$.

In the above definition, for all $\epsilon \geq 0$, if $\bar{x} \notin S$, we write $\widehat{N}(\bar{x}, S) = \emptyset$. If we suppose $\epsilon = 0$ in (**2**.1), then the set $\widehat{N}_0(\bar{x}, S)$ is called the Fréchet normal cone to S at $\bar{x}$.

Definition 2.3 (Mordukhovich [22]). *The limiting/Mordukhovich normal cone to* S *at* $\bar{x} \in S$*, denoted by* $N(\bar{x}, S)$*, is obtained from* $\widehat{N}_\epsilon(x, S)$ *by taking the sequential Painlevé-Kuratowski upper limits as*

$$N(\bar{x}, S) = \operatorname{Lim\,sup}_{x \overset{S}{\to} \bar{x}, \epsilon \downarrow 0} \widehat{N}_\epsilon(x, S) \tag{2.2}$$

If $\bar{x} \notin S$, we put $N(\bar{x}, S) = \emptyset$. Note that, if S is (locally) closed around $\bar{x}$, i.e., there is a neighborhood $U \subset X$ of $\bar{x}$ such that $S \cap \mathrm{cl}U$ is closed then one can put $\epsilon = 0$ in (**2**.2) (see Mordukhovich [22], Theorem 1.6).

Definition 2.4 (Mordukhovich [22]). *The limiting/Mordukhovich subdifferential and the Fréchet subdifferentials of an extended real-valued function* $\psi : X \to \bar{\mathbb{R}} = [-\infty, \infty]$*, at* $\bar{x} \in X$ *with* $|\psi(\bar{x})| < \infty$ *are respectively defined by*

$$\partial\psi(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N((\bar{x}, \psi(\bar{x})), \mathrm{epi}\psi)\} \tag{2.3}$$

and

$$\widehat{\partial}\psi(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in \widehat{N}((\bar{x}, \psi(\bar{x})), \mathrm{epi}\psi)\}, \tag{2.4}$$

where $\mathrm{epi}\psi = \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq \psi(x)\}$.

If $|\psi(\bar{x})| = \infty$, one puts $\partial\psi(\bar{x}) = \widehat{\partial}\psi(\bar{x}) = \emptyset$. It is clear from Mordukhovich [22] that, if ψ is a convex function, then above-defined limiting/ Mordukhovich subdifferential and the Fréchet subdifferentials coincide with the subdifferential in the sense of convex analysis (cf. Rockafellar [25]).

In the sequel of the paper, assume that S is a nonempty locally closed subset of X, and let J be an arbitrary (possibly infinite) index set.

The problem to be considered in the present analysis is the following semi-infinite multiobjective fractional programming problem of the form:

$$\text{(P)} \qquad \min_{\mathbb{R}^p_+} \left\{\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) : x \in \mathbb{F}\right\}.$$

Here, the constraint set is defined by

$$\mathbb{F} = \{x \in S : h_j(x) \leq 0, \ j \in J\}, \tag{2.5}$$

and the functions f_i, g_i, $i = 1, ..., p$, and h_j, $j \in J$, are locally Lipschitz on X. For the purpose of convenience, we assume further that $g_i(x) > 0$, $i = 1, ..., p$, for all $x \in S$, and that $f_i(\bar{x}) \leq 0$, $i = 1, ..., p$, for the reference point $\bar{x} \in S$. Hereafter, we use the notation $h_J = (h_j)_{j \in J}$ and $\theta = (\theta_1, \theta_2, ..., \theta_p)$, where $\theta_i = \frac{f_i}{g_i}$, $i = 1, ..., p$.

By keeping in view, the definition local positively properly efficient solution in multiobjective optimization, given by Göpfert et al. [14, p. 110], we present the following definitions.

Definition 2.5. *(i) A point $\bar{x} \in \mathbb{F}$ is called a local efficient solution of problem (P) iff there exists a neighborhood U of $\bar{x}$ such that*

$$\forall x \in U \cap \mathbb{F}, \qquad \theta(x) - \theta(\bar{x}) \notin -\mathbb{R}^p_+ \setminus \{0\}.$$

(ii) A point $\bar{x} \in \mathbb{F}$ is called a local positively properly efficient solution of problem (P) iff there exist a neighborhood U of $\bar{x}$ and $\lambda \in \operatorname{int}\mathbb{R}^p_+$ such that

$$\forall x \in U \cap \mathbb{F}, \qquad \langle \lambda, \theta(x) \rangle \geq \langle \lambda, \theta(\bar{x}) \rangle .$$

The set of local efficient solutions and local positively properly efficient solutions of problem (P) are denoted by loc$E$(P) and locE^p(P) respectively. If $U = X$, one has the concepts of efficient solution and positively properly efficient solution for problem (P), and in this case we denote these solution sets by E(P) and E^p(P) respectively.

It is known (see e.g., [9, 10]) that for our framework the inclusions

$$\text{loc}E^p(\text{P}) \subset \text{loc}E(\text{P})$$

are always valid, and the converse inclusions do not hold in general.

Definition 2.6 (Chuong and Yao[7]). *Let $\bar{x} \in \mathbb{F}$. We say that the limiting constraint qualification (LCQ) is satisfied at $\bar{x}$ iff*

$$N(\bar{x}, \mathbb{F}) \subset \bigcup_{\mu \in \Lambda(\bar{x})} \left[\sum_{j \in J} \mu_j \partial h_j(\bar{x}) \right] + N(\bar{x}, S).$$

Definition 2.7 (Chuong [4]). *We say that (θ, h_J) is generalized convex on S at $\bar{x} \in S$ if for any $x \in S$, $u_i^* \in \partial f_i(\bar{x})$, $v_i^* \in \partial g_i(\bar{x})$, $i = 1, ..., p$, and $\xi_j^* \in \partial h_j(\bar{x})$, $j \in J$ there exists $\omega \in N(\bar{x}, S)^\circ$ such that*

$$f_i(x) - f_i(\bar{x}) \geq \langle u_i^*, \omega \rangle, \ i = 1, ..., p,$$

$$g_i(x) - g_i(\bar{x}) \geq \langle v_i^*, \omega \rangle, \ i = 1, ..., p,$$

$$h_j(x) - h_j(\bar{x}) \geq \langle \xi_j^*, \omega \rangle, \ j \in J.$$

3. Mond-Weir type duality for proper efficiency

In this section, we present a Mond-Weir dual problem of problem (P) and establish weak and strong duality theorems under generalized convexity assumptions.

The following necessary condition for local positively properly efficient in semi-infinite multiobjective fractional problem (P) under the fulfillment of the (LCQ) defined in Definition **2**.6, derived by Singh et al. [[26], Theorem 3.3], will be required in the proof of the strong duality theorem.

Let $\mathbb{R}^{(J)}_+$ be the collection of all the functions $\mu : J \to \mathbb{R}$ taking positive values μ_j only at finitely many points of J, and equal to zero at other points. The set of active constraint multipliers at $\bar{x} \in S$ is defined by

$$\Lambda(\bar{x}) = \{\mu \in \mathbb{R}^{(J)}_+ : \mu_j h_j(\bar{x}) = 0, \ \forall j \in J\}. \qquad (\mathbf{3}.1)$$

Theorem 3.1. *Let the (LCQ) be satisfied at $\bar{x} \in \mathbb{F}$. If $\bar{x} \in$ locE^p(P), then there exist $\lambda = (\lambda_1, ..., \lambda_p) \in \operatorname{int}\mathbb{R}^p_+$ and $\mu \in \Lambda(\bar{x})$ such that*

$$0 \in \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})} \partial g_i(\bar{x}) \right) + \sum_{j \in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x}, S).$$

Let $y \in X$, $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ with $\sum_{i=1}^{p} \lambda_i = 1$, and $\mu \in \mathbb{R}^{(J)}_+$. In connection with semi-infinite multiobjective fractional optimization problem (P), we consider a semi-infinite multiobjective fractional Mond-Weir [21] dual problem of the form:

$$\text{(D)} \qquad \max_{\mathbb{R}^p_+} \left\{ \tilde{\theta}(y, \lambda, \mu) = \left(\frac{f_1(y)}{g_1(y)}, ..., \frac{f_p(y)}{g_p(y)} \right) : (y, \lambda, \mu) \in \mathbb{W} \right\}.$$

Here, the constraint set is defined by

$$\mathbb{W} = \left\{ (y, \lambda, \mu) \in S \times \mathrm{int}\mathbb{R}^p_+ \times \mathbb{R}^{(J)}_+ : 0 \in \sum_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(\partial f_i(y) - \frac{f_i(y)}{g_i(y)} \partial g_i(y) \right) \right. \\ \left. + \sum_{j \in J} \mu_j \partial h_j(y) + N(y, S), \sum_{j \in J} \mu_j h_j(y) \geq 0 \right\}. \tag{3.2}$$

We need to notice that a (local) efficient solution (respectively, (local) positively properly efficient solution) of the dual problem (D) is defined similarly as in Definition 2.5 by replacing $-\mathbb{R}^p_+$ (respectively, $\mathrm{int}\mathbb{R}^p_+$) by $\mathbb{R}^p_+$ (respectively, $-\mathrm{int}\mathbb{R}^p_+$). Also, the set of efficient solutions (respectively, positively properly efficient solutions) of problem (D) is denoted by $E(\mathbb{W})$ (respectively, $E^p(\mathbb{W})$).

In what follows, we use the following notation for convenience:

$$a \preceq b \Leftrightarrow a - b \in -\mathbb{R}^p_+ \setminus \{0\}, \qquad a \npreceq b \text{ is the negation of } a \preceq b.$$

Now, we establish weak and strong duality relations between (P) and (D).

Theorem 3.2 (Weak Duality)**.** *Let $x \in \mathbb{F}$ and $(y, \lambda, \mu) \in \mathbb{W}$. Assume that (θ, h_J) is generalized convex on S at y. Then $\theta(x) \npreceq \tilde{\theta}(y, \lambda, \mu)$.*

Proof. Since $(y, \lambda, \mu) \in \mathbb{W}$, there exist $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ with $\sum_{i=1}^{p} \lambda_i = 1$, and $\mu \in \mathbb{R}^{(J)}_+$, $u_i^* \in \partial f_i(y)$, $v_i^* \in \partial g_i(y)$, $i = 1, ..., p$, and $\xi_j^* \in \partial h_j(y)$, $j \in J$ such that

$$-\left[\sum_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(u_i^* - \frac{f_i(y)}{g_i(y)} v_i^* \right) + \sum_{j \in J} \mu_j \xi_j^* \right] \in N(y, S), \tag{3.3}$$

$$\sum_{j \in J} \mu_j h_j(y) \geq 0. \tag{3.4}$$

Suppose to the contrary that

$$\theta(x) \preceq \tilde{\theta}(y, \lambda, \mu).$$

Due to $\lambda \in \mathrm{int}\mathbb{R}^p_+$ the above inequality imply

$$\left\langle \lambda, \theta(x) - \tilde{\theta}(y, \lambda, \mu) \right\rangle < 0.$$

This is equivalent to the following inequality

$$\sum_{i=1}^{p} \lambda_i \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(y)}{g_i(y)} \right] < 0. \tag{3.5}$$

By the definition of the polar cone and the generalized convexity of (θ, h_J) on S at y, it follows from (**3**.3) that for each $x \in S$, there is $\omega \in N(y,S)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(\langle u_i^*, \omega \rangle - \frac{f_i(y)}{g_i(y)} \langle v_i^*, \omega \rangle \right) + \sum_{j \in J} \mu_j \langle \xi_j^*, \omega \rangle \\ &\leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left[f_i(x) - f_i(y) - \frac{f_i(y)}{g_i(y)} (g_i(x) - g_i(y)) \right] + \sum_{j \in J} \mu_j (h_j(x) - h_j(y)) \\ &= \sum_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(f_i(x) - \frac{f_i(y)}{g_i(y)} g_i(x) \right) + \sum_{j \in J} \mu_j (h_j(x) - h_j(y)). \end{aligned}$$

Hence, $0 \leq \sum\limits_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(f_i(x) - \frac{f_i(y)}{g_i(y)} g_i(x) \right) + \sum\limits_{j \in J} \mu_j (h_j(x) - h_j(y))$. From (**3**.4) and the fact that $x \in \mathbb{F}$, the above inequality yields $0 \leq \sum\limits_{i=1}^{p} \frac{\lambda_i}{g_i(y)} \left(f_i(x) - \frac{f_i(y)}{g_i(y)} g_i(x) \right)$, or equivalently

$$0 \leq \sum_{i=1}^{p} \lambda_i \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(y)}{g_i(y)} \right],$$

which contradicts (**3**.5). This completes the proof. □

The example below shows that the generalized convex property of (θ, h_J) on S imposed in the above theorem cannot be omitted.

Example 3.1. *Let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)} \right)$, where*

$$f_1(x) = \min\{0, x_1^3\}, f_2(x) = \min\{0, x_2^3\},$$
$$g_1(x) = g_2(x) = x_1^2 + x_2^2 + 1, \ x = (x_1, x_2) \in \mathbb{R}^2,$$

and let $h_j : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$h_j(x) = -j(|x_1| + |x_2|), \ x = (x_1, x_2) \in \mathbb{R}^2, \ j \in J = (0, \infty).$$

Consider problem (P) *with $p = 2$, and $S = [-2, 0] \times [-2, 0] \subset \mathbb{R}^2$. Then $\mathbb{F} = S$, and let us select $\bar{x} = (\bar{x}_1, \bar{x}_2) = \{(-1, -1)\} \in \mathbb{F}$. Now, consider the dual problem* (D)*. By choosing $\bar{y} = \{(0, 0)\} \in S$, $\bar{\lambda} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\bar{\mu} = 0$, it holds that $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in \mathbb{W}$ and that*

$$\theta(\bar{x}) = \left(-\frac{1}{3}, -\frac{1}{3} \right) \leq (0, 0) = \tilde{\theta}(\bar{y}, \bar{\lambda}, \bar{\mu}),$$

showing that conclusion of Theorem **3**.*2 fails to hold. The reason is that (θ, h_J) is not generalized convex on S at $\bar{y}$.*

The next theorem presents a strong duality relation between the primal problem (P) and the dual problem (D). This theorem shows that the absence of duality gap holds for a Mond-Weir dual if some requirements are satisfied.

Theorem 3.3 (Strong Duality)**.** *If $\bar{x} \in \mathrm{loc}E^p(\mathrm{P})$, and the* (LCQ) *is satisfied at $\bar{x}$, then there exist $(\bar{\lambda}, \bar{\mu}) \in \mathrm{int}\mathbb{R}_+^p \times \mathbb{R}_+^{(J)}$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{W}$ and $\theta(\bar{x}) = \tilde{\theta}(\bar{x}, \bar{\lambda}, \bar{\mu})$. Furthermore, if (θ, h_J) is assumed to be generalized convex on S at any $y \in S$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in E(\mathbb{W})$.*

Proof. By assumption, $\bar{x} \in \mathrm{loc}E^p(\mathrm{P})$, and the (LCQ) is satisfied at $\bar{x}$. Then, there exist $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}_+^p$ and $\mu \in \Lambda(\bar{x})$ such that necessary conditions (Theorem **3**.1) are fulfilled at $\bar{x}$. Thus, we have

$$0 \in \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})} \partial g_i(\bar{x}) \right) + \sum_{j \in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x}, S). \qquad (\mathbf{3}.6)$$

Setting

$$\bar{\lambda}_i = \frac{\lambda_i}{\sum\limits_{i=1}^{p} \lambda_i},\ i = 1, ..., p,\ \bar{\mu}_j = \frac{\mu_j}{\sum\limits_{i=1}^{p} \lambda_i},\ j \in J,$$

it is easy to see that $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_p) \in \text{int}\mathbb{R}^p_+$, $\sum\limits_{i=1}^{p} \bar{\lambda}_i = 1$, and $\bar{\mu} = (\bar{\mu}_j)_{j\in J} \in \mathbb{R}^{(J)}_+$.

Observe that the assertion in (**3**.6) is also valid when λ_i's and μ_j's are replaced by $\bar{\lambda}_i$'s and $\bar{\mu}_j$'s, respectively. In addition, due to the fact that $\mu \in \Lambda(\bar{x})$, $\mu_j h_j(\bar{x}) = 0$, for all $j \in J$, and thus, $\sum\limits_{j\in J} \mu_j h_j(\bar{x}) = 0$. So, we conclude that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{W}$. Obviously

$\theta(\bar{x}) = \tilde{\theta}(\bar{x}, \bar{\lambda}, \bar{\mu})$.

Now, by assumption that (θ, h_J) is generalized convex on S at any $y \in S$, thus invoking the weak duality result in Theorem 3.2, we obtain

$$\tilde{\theta}(\bar{x}, \bar{\lambda}, \bar{\mu}) = \theta(\bar{x}) \not\preceq \tilde{\theta}(y, \lambda, \mu),$$

for any $(y, \lambda, \mu) \in \mathbb{W}$. It means that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in E(\mathbb{W})$. This complets the proof. □

4. Conclusion

In the present work, we have proposed a Mond-Weir dual problem for a nonsmooth semi-infinite multiobjective fractional optimization problem, and examined weak and strong duality relations under the generalized convex assumptions. We will extend the results established in the paper to a larger class of nonsmooth variational and nonsmooth control multiobjective optimization problems. This will orient the future research of the authors.

REFERENCES

[1] *T. Antczak*, Sufficient optimality conditions for semi-infinite multiobjective fractional programming under (ϕ, ρ)-V-invexity and generalized (ϕ, ρ)-V-invexity, Filomat, **30**(2016), No. 14, 3649-3665.

[2] *T. Antczak and K. Shukla*, Higher order duality for a new class of nonconvex semi-infinite multiobjective fractional programming with support functions, J. Appl. Anal. Comput., **10**(2020), No. 6, 2806-2825.

[3] *I. K. Argyros and R. U. Verma*, Role of Hanson-Antczak-Zalmai-V-invexities in semiinfinte multiobjective fractional programming and parametric duality models, Panam. Math. J., **30**(2020), No. 1, 1-2.

[4] *T. D. Chuong*, Nondifferentiable fractional semi-infinite multiobjective optimization problems, Oper. Res. Lett., **44**(2016), No. 2, 260-266.

[5] *T. D. Chuong, N. Q. Huy and J. C. Yao*, Stability of semi-infinite vector optimization problems under functional perturbations, J. Glob. Optim., **45**(2009), No. 4, 583-595.

[6] *T. D. Chuong and D. S. Kim*, A class of nonsmooth fractional multiobjective optimization problems, Ann. Oper. Res., **244**(2016), No. 2, 367-383.

[7] *T. D. Chuong and J. C. Yao*, Isolated and proper efficiencies in semi-infinite vector optimization problems, J. Optim. Theory Appl., **162**(2014), No. 2, 447-462.

[8] *N. Dinh, B.S. Mordukhovich, and T. T. A. Nghia*, Qualification and optimality conditions for DC programs with infinite constraints, Acta Math. Vietnam., **34**(2009), No. 1, 125-155.

[9] *A. M. Geoffrion*, Proper efficiency and the theory of vector maximization, J. Math. Anal. Appl., **22**(1968), No. 3, 618-630.

[10] *I. Ginchev, A. Guerraggio and M. Rocca*, From scalar to vector optimization, Appl. Math., **51**(2006), No. 1, 5-6.

[11] *A. Göpfert, C. Tammer, H. Riahi and C. Zalinescu*, Variational Methods in Partially Ordered Spaces, New York: Springer, 2003.

[12] *P. Gupta, S. K. Mishra and R. N. Mohapatra*, Duality models for multiobjective semiinfinite fractional programming problems involving type-I and related functions, Quaest. Math., **42**(2019), No. 9, 1199-1220.

[13] *A. Jayswal and V. Singh*, The characterization of efficiency and saddle point criteria for multiobjective optimization problem with vanishing constraints, Acta Math. Sci., **39**(2019), No. 2, 382-394

[14] *N. Kanzi and S. Nobakhtian*, Optimality conditions for non-smooth semi-infinite programming, Optimization, **59**(2010), No. 5, 717-727.

[15] *N. Kanzi and S. Nobakhtian*, Necessary optimality conditions for nonsmooth generalized semi-infinite programming problems, European J. Oper. Res., **205**(2010), No. 2, 253-261.

[16] *P. Q. Khanh and M. T. Nguyen*, On the Mangasarian–Fromovitz constraint qualification and Karush–Kuhn–Tucker conditions in nonsmooth semi-infinite multiobjective programming, Optim. Lett., **14**(2020), No. 8, 2055-2072.

[17] *D. S. Kim, S. J. Kim and M .H. Kim*, Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems, J. Optim. Theory Appl., **129**(2006), No. 1, 131-146.

[18] *H. Kuk, G. M. Lee and T. Tanino*, Optimality and duality for nonsmooth multiobjective fractional programming with generalized invexity, J. Math. Anal. Appl., **262**(2001), No. 1, 365-375.

[19] *X. Li and H. Miao*, Duality conditions of nonsmooth semi-infinite multi-objective fractional programming, J. Chongqing Norm. Univ. **37**(2020), No. 1, 81-85.

[20] *J. C. Liu*, Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoinvex functions, Optimization, **37**(1996), No. 1, 27-39.

[21] *B. Mond and T. Weir*, Generalized concavity and duality, In Schaible S, Ziemba WT, editors. Generalized Concavity in Optimization and Economics. New York: Academic Press, pp. 263-279 (1981)

[22] *B. S. Mordukhovich*, Variational Analysis and Generalized Differentiation. I: Basic Theory, Berlin: Springer-Verlag, Volume 330, 2006.

[23] *B. S. Mordukhovich, N. M. Nam and N. D. Yen*, Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming, Optimization, **55**(2006), 685-708.

[24] *B. S. Mordukhovich and L. Mou*, Necessary conditions for nonsmooth optimization problems with operator constraints in metric spaces, J. Convex Anal., **16**(2009), 913-937.

[25] *R. T. Rockafellar*, Convex Analysis, Princeton, NJ, Princeton University Press, 1970.

[26] *V. Singh, A. Jayswal, I. M. Stancu-Minasian and A. M. Rusu-Stancu*, Isolated and proper efficiencies for semi-infinite multiobjective fractional problems, U.P.B. Sci. Bull., Series A, **83**(2021), No. 3, 111-124.

[27] *I. M. Stancu-Minasian*, Fractional Programming: Theory Methods and Applications, Kluwer Academic Publisher: Dordrecht, 1997.

[28] *I. M. Stancu-Minasian*, A seventh bibliography of fractional programming, Adv. Model. Optim., **15**(2013), 309-386.

[29] *I. M. Stancu-Minasian*, A eighth bibliography of fractional programming, Optimization, **66**(2017), No. 3, 439-470.

[30] *I. M. Stancu-Minasian*, A ninth bibliography of fractional programming, Optimization, **68**(2019), No. 11, 2125-2169.

[31] *M. Soleimani-Damaneh*, Optimality for nonsmooth fractional multiple objective programming, Nonlinear Anal., **68**(2008), No. 10, 2873-2878.

[32] *M. Soleimani-Damaneh*, Nonsmooth optimization using Mordukhovich's subdifferential, SIAM J. Control Optim., **48**(2010), No. 5, 3403-3432.