RECURRENT POINTS FOR ITERATED FUNCTION SYSTEMS

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In this paper, we introduce the definitions of ω-limit point, periodic point, almost periodic point, chain recurrent point and non-wandering point for iterated function systems. Then we mainly focus on the properties of the above recurrent points, for example whether the recurrent point sets of the above recurrent points are empty, invariable, iterable or not. We find that some properties of continuous self-maps on the compact metric space still hold for iterated function systems, and some don't hold. Also, we present the relationships between these various kinds of recurrent point sets.

Keywords: ω-limit point, periodic point, almost periodic point, chain recurrent point, non-wandering point, iterated function systems.

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1. **Introduction**

Throughout this paper, let $\mathbb{N} = \{0, 1, 2 \cdots\}$ and $\mathbb{Z}^+ = \{1, 2, \cdots\}$. A *dynamical ststem* is a pair (X, f), where X is a compact metric space with metric d and $f : X \to X$ is a continuous map. The asymptotic properties of orbits are the core content in the study of dynamical systems. And the main task is to study the evolution of a single state in the dynamical systems. Therefore, it is very valuable to study the recurrent points. Many research about the recurrent points is referred to [4, 8, 12] and references therein.

Iterated function systems[2, 3] and φ-contractive parent-child possibly infinite iterated function systems[7] are widely used for fractal structures, the image compression, the image processing and dynamic systems. And iterated function systems, due to their relatively simple structure and wide applicability, have become foundational tools in the study of fractal geometry and dynamical systems, making them easier to use for analyzing and promoting fundamental theoretical research. Let f_0, f_1 be two continuous self-maps on X. Then $\mathcal{F} = \{X; f_0, f_1\}$ is called an iterated function systems (IFS). The topic of iterated function system is currently an intensely studied area of dynamical properties, with papers from many authors at this point. See [11, 13]

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for shadowing property, [5, 6] for sensitivity and transitivity, [1] for chaos and [10] for attractor, etc..

For any subset Y of X, let

$$\mathcal{F}(Y) = f_0(Y) \bigcup f_1(Y).$$

And for any $k \in \mathbb{Z}^+$, let

$$\mathcal{F}^k = \{X; f_{\alpha_{k-1}} \circ \cdots \circ f_{\alpha_1} \circ f_{\alpha_0} | \alpha_0, \alpha_1, \cdots, \alpha_{k-1} \in \{0,1\}\}.$$

An orbit of $x \in X$ under the iterated function system IFS $\mathcal{F}$ is a sequence $\{f_\alpha^n(x)\}_{n=0}^\infty$, where

$$\alpha = \alpha_0\alpha_1\alpha_2 \cdots \in \Sigma := \{s = s_0 s_1 s_2 \cdots | s_i \in \{0,1\}\},$$

and

$$f_\alpha^n(x) = f_{\alpha_n} \circ \cdots \circ f_{\alpha_1} \circ f_{\alpha_0}(x), \forall n \in \mathbb{Z}^+,$$

$f_\alpha^0(x) = x$. Set

$$orb_\alpha(x) = \{f_\alpha^n(x)\}_{n=0}^\infty,$$
$$orb_\alpha(x, \mathcal{F}^n) = \{f_\alpha^{kn}(x)\}_{k=0}^\infty,$$
$$orb_\alpha(f_\alpha^i(x)) = \{f_\alpha^{i+n}(x)\}_{n=0}^\infty,$$
$$orb_\alpha(f_\alpha^i(x), \mathcal{F}^n) = \{f_\alpha^{i+kn}(x)\}_{k=0}^\infty.$$

Inspired by this, we generalize the definitions of the sets of ω limit points, periodic points, almost periodic points, chain recurrent points, recurrent points, non-wandering points and strong non-wandering points to our new setting, and mainly study their properties whether the recurrent point sets of the above recurrent points are empty, invariable, closed, iterable or not, in the next section. Then we present the relationships between these kinds of recurrent points set in Section 3. The results of this paper is numerous. A summary of what we have investigated and why is given in the conclusion.

2. **Preliminaries and Basic Concepts**

Let (X, d) be a compact metric space and f_0, f_1 be two continuous maps on X. The iterated function system IFS $\mathcal{F}$ is the action of the semi-group generated by $\{f_0, f_1\}$ on X.

2.1. ω limit point

Definition 2.1. *Let $\alpha \in \Sigma$. We say that $y \in X$ is a* ω-limit point *of $x \in X$ under its orbit $orb_\alpha(x)$, if there exists a sequence $\{n_i\} \subset \mathbb{N}$ such that $\lim\limits_{n_i \to \infty} f_\alpha^{n_i}(x) = y$. We use $\omega(x, \mathcal{F}, orb_\alpha(x))$ to denote the set of the ω-limit points of x under its orbit $orb_\alpha(x)$. And set*

$$\omega(x, \mathcal{F}) = \bigcup_{\alpha \in \Sigma} \omega(x, \mathcal{F}, orb_\alpha(x)).$$

Since X is compact, the following proposition is obvious.

Proposition 2.1. *Let $x \in X$ and $\alpha \in \Sigma$. Then $\omega(x, \mathcal{F}, orb_\alpha(x)) \neq \emptyset$ and $\omega(x, \mathcal{F}) \neq \emptyset$.*

Theorem 2.1. *Let $x \in X$ and $\alpha \in \Sigma$. Then $\omega(x, \mathcal{F}, orb_\alpha(x)) = \overline{\omega(x, \mathcal{F}, orb_\alpha(x))}$.*

Proof. It is obvious that $\omega(x, \mathcal{F}, orb_\alpha(x)) \subset \overline{\omega(x, \mathcal{F}, orb_\alpha(x))}$. Next we only need to show that

$$\overline{\omega(x, \mathcal{F}, orb_\alpha(x))} \subset \omega(x, \mathcal{F}, orb_\alpha(x)).$$

Let $y \in \overline{\omega(x, \mathcal{F}, orb_\alpha(x))}$. Then for any $\varepsilon > 0$ there exists $y' \in \omega(x, \mathcal{F}, orb_\alpha(x))$ such that $d(y, y') < \frac{\varepsilon}{2}$. And there is a sequence $\{n_i\} \subset \mathbb{N}$ such that $\lim\limits_{n_i \to \infty} f_\alpha^{n_i}(x) = y'$. It results that $\lim\limits_{n_i \to \infty} f_\alpha^{n_i}(x) = y$. So, $y \in \omega(x, \mathcal{F}, orb_\alpha(x))$. That is,

$$\overline{\omega(x, \mathcal{F}, orb_\alpha(x))} \subset \omega(x, \mathcal{F}, orb_\alpha(x)).$$

□

Likewise, we can get the following result.

Theorem 2.2. *Let $x \in X$. Then $\omega(x, \mathcal{F}) = \overline{\omega(x, \mathcal{F})}$.*

Theorem 2.3. *Let $x \in X$. Then, $\mathcal{F}(\omega(x, \mathcal{F})) = \omega(x, \mathcal{F})$.*

Proof. Firstly, we show that

$$\mathcal{F}(\omega(x, \mathcal{F})) \subset \omega(x, \mathcal{F}).$$

For any given $z \in f_0(\omega(x, \mathcal{F}))$, there exists $y \in \omega(x, \mathcal{F})$ such that $f_0(y) = z$. Since $y \in \omega(x, \mathcal{F})$, there exist $\alpha \in \Sigma$ and a sequence $\{n_i\} \subset \mathbb{N}$ such that

$$\lim_{n_i \to \infty} f_\alpha^{n_i}(x) = y.$$

By the continuity of f_0, we have

$$\lim_{n_i \to \infty} f_0(f_\alpha^{n_i}(x)) = f_0(\lim_{n_i \to \infty} f_\alpha^{n_i}(x)) = f_0(y).$$

So, $f_0(y) \in \omega(x, \mathcal{F})$. It is similar for $z \in f_1(\omega(x, \mathcal{F}))$.

Next we show that

$$\omega(x, \mathcal{F}) \subset \mathcal{F}(\omega(x, \mathcal{F})).$$

Let $z \in \omega(x, \mathcal{F})$. Then there exist $\alpha \in \Sigma$ and a sequence $\{n_i\} \subset \mathbb{N}$ such that

$$\lim_{n_i \to \infty} f_\alpha^{n_i}(x) = z.$$

Let $\{f_\alpha^{n_{i_j}-1}(x)\}_{j=0}^\infty$ be a convergent subsequence of $\{f_\alpha^{n_i-1}(x)\}_{i=0}^\infty$ with either $\alpha_{n_{i_j}} = 0, \forall j \geq 0$ or $\alpha_{n_{i_j}} = 1, \forall j \geq 0$. Without loss of generality, let

$$\alpha_{n_{i_j}} = 0, \forall j \geq 0.$$

Set $\lim\limits_{j \to \infty} f_\alpha^{n_{i_j}-1}(x) = y$. Then $y \in \omega(x, \mathcal{F})$. By the continuity of f_0,

$$f_0(y) = f_0(\lim_{j \to \infty} f_\alpha^{n_{i_j}-1}(x)) = \lim_{j \to \infty} f_\alpha^{n_{i_j}}(x) = \lim_{i \to \infty} f_\alpha^{n_i}(x) = z.$$

Hence, $z \in f_0(\omega(x, \mathcal{F}))$. □

By Definition 2.1, we can get the following theorem.

Theorem 2.4. *Let $x \in X$ and $\alpha \in \Sigma$. Then for any $i > 0$,*

$$\omega(f_\alpha^i(x), \mathcal{F}, orb_\alpha(f_\alpha^i(x))) \subset \omega(x, \mathcal{F}, orb_\alpha(x)).$$

Proof. Let $i > 0$ and let $y \in \omega(f_\alpha^i(x), \mathcal{F}, orb_\alpha(f_\alpha^i(x)))$. Then there exists a sequence $\{n_j\}$ of positive integers such that

$$\lim_{n_j \to \infty} f_\alpha^{n_j+i}(x) = \lim_{n_j \to \infty} f_\alpha^{n_j}(f_\alpha^i(x)) = y.$$

Then $y \in \omega(x, \mathcal{F}, orb_\alpha(x))$. So, $\omega(f_\alpha^i(x), \mathcal{F}, orb_\alpha(f_\alpha^i(x))) \subset \omega(x, \mathcal{F}, orb_\alpha(x))$. □

Theorem 2.5. *Let $x \in X$ and $\alpha \in \Sigma$. Then for any $n > 0$,*

$$\omega(x, \mathcal{F}, orb_\alpha(x)) = \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n)).$$

Proof. Firstly, we show that $\omega(x, \mathcal{F}, orb_\alpha(x)) \supset \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n))$. For any given $y \in \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n))$, there exists $0 \le i \le n-1$ such that

$$y \in \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n)).$$

Then there exists a sequence $\{k_j\}$ of positive integers such that $\lim_{k_j \to \infty} f_\alpha^{i+k_j n}(x) = y$. So, $y \in \omega(x, \mathcal{F}, orb_\alpha(x))$.

Next we show that $\omega(x, \mathcal{F}, orb_\alpha(x)) \subset \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n))$. For any given $y \in \omega(x, \mathcal{F}, orb_\alpha(x))$, there exists a sequence $\{n_j\}$ of positive integers such that

$$\lim_{n_j \to \infty} f_\alpha^{n_j}(x) = y.$$

Then there exist a subsequence $\{n_{j_k}\}$ of $\{n_j\}$, a sequence $\{q_k\}$ of positive integers and $0 \le r < n$ such that $n_{j_k} = nq_k + r$. Thus,

$$\lim_{k \to \infty} f_\alpha^{nq_k+r}(x) = \lim_{k \to \infty} f_\alpha^{n_{j_k}}(x) = y.$$

So, $y \in \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n))$. □

By Theorem 2.5, we can get the following theorem.

Theorem 2.6. *Let $x \in X$ and $n > 0$. Then*

$$\omega(x, \mathcal{F}) = \bigcup_{\alpha \in \Sigma} \bigcup_{i=0}^{n-1} \omega(f_\alpha^i(x), \mathcal{F}^n, orb_\alpha(f_\alpha^i(x), \mathcal{F}^n)).$$

2.2. Periodic point

Definition 2.2. *We say that $x \in X$ is a* periodic point *of IFS $\mathcal{F}$ if there exsits $\alpha \in \Sigma$ and $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$ we have $f_\alpha^{n+i}(x) = f_\alpha^i(x)$. The period of x is the smallest number $n \in \mathbb{Z}^+$ satisfying the above equality for all integers $i \in \mathbb{N}$. If $n = 1$, we say that x is a fixed point of IFS $\mathcal{F}$. We use $P(\mathcal{F})$ and $Fix(\mathcal{F})$ to denote the sets of the periodic points and fixed points of IFS $\mathcal{F}$, respectively.*

The following example demonstrates that the property that fixed point set and periodic point set are invariable for continuous self-maps doesn't hold for iterated function systems.

Example 2.1. *Consider two continuous maps f_0, f_1 on Σ as follows:*

$$f_0(s_0s_1\cdots) = 0s_0s_1\cdots,\ f_1(s_0s_1\cdots) = 1s_0s_1\cdots, \forall s_0s_1\cdots \in \Sigma.$$

We show that $\mathcal{F}(Fix(\mathcal{F})) \nsubseteq Fix(\mathcal{F})$ and $\mathcal{F}(P(\mathcal{F})) \nsubseteq P(\mathcal{F})$. For this, select $x = 0\cdots 0 \cdots \in \Sigma$. It is easy to show that $f_0(x) = x \in Fix(\mathcal{F}) \subset P(\mathcal{F})$. While, $f_1(x) \nsubseteq Fix(\mathcal{F})$. And for any $\alpha \in \Sigma$ and any $n > 0$, $f_\alpha^n(f_1(x)) \neq f_1(x)$. So, $f_1(x) \nsubseteq P(\mathcal{F})$.

Now we show that the periodic point is retentive under iteration of IFS $\mathcal{F}$, but the fixed point is not.

Theorem 2.7. *For any $n > 0$, $P(\mathcal{F}) = P(\mathcal{F}^n)$.*

Proof. Firstly, we prove that $P(\mathcal{F}) \subset P(\mathcal{F}^n)$. For any given $x \in P(\mathcal{F})$, there exists $\alpha \in \Sigma$ such that there exists $m \in \mathbb{N}$ satisfying

$$f_\alpha^{k+m}(x) = f_\alpha^k(x), \forall k \in \mathbb{N}.$$

Then, $f_\alpha^{n(k+m)}(x) = f_\alpha^{nk}(x), \forall k \in \mathbb{N}$. So, $x \in P(\mathcal{F}^n)$.

Next, we prove that $P(\mathcal{F}) \supset P(\mathcal{F}^n)$. Let $x \in P(\mathcal{F}^n)$. Then there exist $\alpha \in \Sigma$ and $m > 0$ such that $f_\alpha^{mn}(x) = x$. Select $\beta = \alpha_0\alpha_1\cdots\alpha_{mn}\alpha_0\alpha_1\cdots\alpha_{mn}\alpha_0\cdots$, then

$$f_\beta^{mn+i}(x) = f_\beta^i(x), \forall i \geq 0.$$

So, $x \in P(\mathcal{F})$. □

For fixed point of IFS $\mathcal{F}$, it is easy to show that the corresponding conclusion is different.

Theorem 2.8. *For any $n > 0$, $Fix(\mathcal{F}^n) \nsubseteq Fix(\mathcal{F})$ and $Fix(\mathcal{F}) \subset Fix(\mathcal{F}^n)$.*

2.3. Almost periodic point

Definition 2.3. *We say that $x \in X$ is an* almost periodic point *of IFS $\mathcal{F}$ if there exists $\alpha \in \Sigma$ and for any $\varepsilon > 0$ there exists $N > 0$ such that for any $q \geq 0$, there exists $r \in \mathbb{N}$, $q < r \leq q + N$ satisfying*

$$d(f_\alpha^{r+i}(x), f_\alpha^i(x)) < \varepsilon, \forall i \in \mathbb{N}.$$

We use $AP(\mathcal{F})$ to denote the set of almost periodic points of IFS $\mathcal{F}$.

Firstly, we show that $AP(\mathcal{F})$ is conditional invariable for IFS $\mathcal{F}$.

Theorem 2.9. *If* $f_0 \circ f_1 = f_1 \circ f_0$, *then* $\mathcal{F}(AP(\mathcal{F})) \subset AP(\mathcal{F})$.

Proof. Let $y \in \mathcal{F}(AP(\mathcal{F}))$. Without loss of generality, suppose that $y \in f_0(AP(\mathcal{F}))$. Let $\varepsilon > 0$. Since f_0 is continuous, there exists $\delta > 0$ such that for any $x, y \in X$, $d(x, y) < \delta$ implies $d(f_0(x), f_0(y)) < \varepsilon$. Then there exists $x \in AP(\mathcal{F})$ such that $f_0(x) = y$. And there exist $\alpha \in \Sigma$ and $N > 0$ for any $q \geq 0$ there exists r, $q < r \leq q + N$ satisfying

$$d(f_\alpha^{r+i}(x), f_\alpha^i(x)) < \delta, \forall i \geq 0.$$

By the continuity of f_0, we have

$$d(f_0 \circ f_\alpha^{r+i}(x), f_0 \circ f_\alpha^i(x)) = d(f_\alpha^{r+i}(f_0(x)), f_\alpha^i(f_0(x))) < \varepsilon, \forall i \geq 0.$$

So, $y \in AP(\mathcal{F})$. □

The Example 2.1 in which $f_0 \circ f_1 \neq f_1 \circ f_0$ demonstrates that the condition "$f_0 \circ f_1 = f_1 \circ f_0$" in Theorem 2.9 cannot be removed. For this, we select $x = 0 \cdots 0 \cdots \in AP(\mathcal{F})$. Next we use the proof by contradiction to show that $y = f_1(x) \notin AP(\mathcal{F})$.

Suppose that $y \in AP(\mathcal{F})$. Then there exists $\alpha \in \Sigma$ for $\frac{1}{3}$, there is $N_1 > 0$ such that for any $k \geq 1$, there exists r_k, $(k-1)N_1 < r_k \leq kN_1$ satisfying $d(f_\alpha^{r_k}(y), y) < \frac{1}{3}$. Hence,

$$\alpha_{r_k} = 0 \ and \ \alpha_{r_k - 1} = 1, \forall k \geq 1. \tag{1}$$

What's more, for $\frac{1}{4N_1} > 0$, there exists $N_2 > 0$ such that there is r, $0 < r \leq N_2$ satisfying

$$d(f_\alpha^r(y), y) < \frac{1}{4N_1}.$$

Then $r > 3N_1$ and $\alpha_r = 1$, $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 0$, which is in contradiction with (1). So, $y \notin AP(\mathcal{F})$

The following result states that the almost periodic point is retentive under iteration of IFS $\mathcal{F}$.

Theorem 2.10. *For any* $n > 0$, $AP(\mathcal{F}) = AP(\mathcal{F}^n)$.

Proof. Firstly, we prove that $AP(\mathcal{F}) \subset AP(\mathcal{F}^n)$. For any given $x \in AP(\mathcal{F})$ there is $\alpha \in \Sigma$ such that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that for any $q \geq 0$, there is r, $q < r \leq q + N_1$ satisfying $d(f_\alpha^{r+i}(x), f_\alpha^i(x)) < \varepsilon, \forall i \geq 0$. Hence, for any $n > 0$, we have

$$d(f_\alpha^{n(r+j)}(x), f_\alpha^{jn}(x)) < \varepsilon, \forall j \geq 0.$$

So, $x \in AP(\mathcal{F}^n)$.

Next, we show that $AP(\mathcal{F}) \supset AP(\mathcal{F}^n)$. For any given $x \in AP(\mathcal{F}^n)$, there is $\alpha \in \Sigma$ such that for any $\varepsilon > 0$, there exists $N_2 > 0$ such that for any $q \geq 0$, there is r, $q < r \leq q + N_2$ satisfying

$$d(f_\alpha^{k(r+i)}(x), f_\alpha^{ki}(x)) < \delta, \forall i \geq 0,$$

where δ satisfies that for any $x, y \in X$, $d(x,y) < \delta$ implies $d(f_\beta^s(x), f_\beta^s(y)) < \varepsilon$, $\forall 0 < s < k, \forall \beta \in \Sigma$. Set $N = kN_2$. Then for any $q \geq 0$, there exists l, $0 \leq l < k$ such that $q - l = 0(mod\ k)$. Let $q' = q - l$. Then $q + N \geq q + N - l = q' + kN_2$. Therefore, there exists r, $\frac{q'}{k} < r \leq \frac{q'}{k} + N_2$ such that

$$d(f_\alpha^{kr+ki}(x), f_\alpha^{ki}(x)) < \delta, \forall i \geq 0.$$

By the uniform continuity of $f_\beta^s, \forall 0 < s < k, \forall \beta \in \Sigma$, $d(f_\alpha^{kr+ki+s}(x), f_\alpha^{ki+s}(x)) < \varepsilon$, $\forall i \geq 0, \forall 0 < s < k$. That is,

$$d(f_\alpha^{kr+i}(x), f_\alpha^{i}(x)) < \varepsilon, \forall i \geq 0.$$

Note that $q < rk \leq q + N$. So, $x \in AP(\mathcal{F})$. □

2.4. Chain recurrent point

Definition 2.4. *We say that $x \in X$ is a* chain recurrent point *of IFS $\mathcal{F}$ if for any $\varepsilon > 0$, there exists a finite number of points $x_0, x_1, \cdots, x_m \in X$ such that $x_0 = x_m = x$ and*

$$\inf_{k=\overline{0,1}} d(f_k(x_i), x_{i+1}) < \varepsilon, i = \overline{0, m-1}.$$

We use $CR(\mathcal{F})$ to denote the set of the chain recurrent points of IFS $\mathcal{F}$.

We mainly study that $CR(\mathcal{F})$ is closed but not invariable for IFS $\mathcal{F}$.

Theorem 2.11. *$CR(\mathcal{F}) = \overline{CR(\mathcal{F})}$, However, $\mathcal{F}(CR(\mathcal{F})) \nsubseteq CR(\mathcal{F})$.*

Proof. Firstly we show that $CR(\mathcal{F}) = \overline{CR(\mathcal{F})}$. For this, suppose that x is a cluster point of $CR(\mathcal{F})$. Then we only need to prove that $x \in CR(\mathcal{F})$.

For any given $\varepsilon > 0$, since f_0 and f_1 are continuous, there exists $0 < \delta < \frac{\varepsilon}{2}$ such that for any $z \in X$ with $d(x,z) < \delta$, $d(f_k(x), f_k(z)) < \frac{\varepsilon}{2}$, $k = \overline{0,1}$. Since x is a cluster point of $CR(\mathcal{F})$, there exists $y \in CR(\mathcal{F})$ with $x \neq y$ such that $d(x,y) < \delta$. Since $y \in CR(\mathcal{F})$, there exists a finite number of points $y_0, y_1, \cdots, y_m \in X$ such that $y_0 = y_m = y$ and

$$\inf_{k=\overline{0,1}} d(f_k(y_i), y_{i+1}) = d(f_{n_i}(y_i), y_{i+1}) < \frac{\varepsilon}{2}, i = \overline{0, m-1}.$$

Let $x_0 = x$, $x_1 = y_1$, $\cdots$, $x_{m-1} = y_{m-1}$, $x_m = x$, then $d(f_{n_i}(x_i), x_{i+1}) < \varepsilon, i = \overline{0, m-1}$. Therefore, $\inf\limits_{k=\overline{0,1}} d(f_k(x_i), x_{i+1}) < \varepsilon, i = \overline{0, m-1}$. Hence, $x \in CR(\mathcal{F})$.

Next we illustrate that $\mathcal{F}(CR(\mathcal{F})) \nsubseteq CR(\mathcal{F})$.

Example 2.2. *Consider two maps f_0, f_1 on $\mathbb{Z}^+$ as follows:*

$$f_0(1) = 1, f_0(n) = n+1, n = 2, 3, \cdots \text{ and } f_1(n) = n+1, n = 1, 2, \cdots.$$

Since 1 is a fixed point of f_0, $1 \in CR(\mathcal{F})$. But $f_1(1) = 2 \notin CR(\mathcal{F})$. □

Interestingly, we can get the result that $CR(\mathcal{F})$ is invariable for IFS $\mathcal{F}$ by strengthening the condition.

Theorem 2.12. *Suppose that for any $x \in CR(\mathcal{F})$ and any $\varepsilon > 0$, there exists a finite number of points $x_0, x_1, \cdots, x_m$ with $x_0 = x_m = x$ such that*

$$\sup_{k=\overline{0,1}} d(f_k(x_0), x_1) < \varepsilon \text{ and } \inf_{k=\overline{0,1}} d(f_k(x_i), x_{i+1}) < \varepsilon, i = \overline{1, m-1}.$$

Then, $\mathcal{F}(CR(\mathcal{F})) \subset CR(\mathcal{F})$.

Proof. For any $y \in \mathcal{F}(CR(\mathcal{F}))$, there exists $x \in CR(\mathcal{F})$ such that $\phi(x) = y$, where $\phi \in \{f_0, f_1\}$. For any $\varepsilon > 0$, let $0 < \delta < \frac{\varepsilon}{2}$ such that for any $x, y \in X$, $d(x, y) < \delta$ implies $d(f_0(x), f_0(y)) < \frac{\varepsilon}{2}$ and $d(f_1(x), f_1(y)) < \frac{\varepsilon}{2}$. Since $x \in CR(\mathcal{F})$, there exists a finite number of points $x_0, x_1, \cdots, x_m \in X$ with $x_0 = x_m = x$ such that

$$\inf_{k=\overline{0,1}} d(f_k(x_i), x_{i+1}) = d(f_{n_i}(x_i), x_{i+1}) < \delta, i = \overline{1, m-1},$$

and $\sup_{k=\overline{0,1}} d(f_k(x_0), x_1) < \delta$. Then, $d(\phi(x), x_1) < \delta$. Let $y_0 = \phi(x)$, $y_1 = x_2$, $y_2 = x_3$, $\cdots$, $y_{m-1} = x_m$, $y_m = \phi(x)$. Then,

$$\inf_{k=\overline{0,1}} d(f_k(y_0), y_1) \leq \inf_{k=\overline{0,1}} \{d(f_k \circ \phi(x), f_{n_1}(x_1)) + d(f_{n_1}(x_1), x_2)\} < \varepsilon,$$

$$\inf_{k=\overline{0,1}} d(f_k(y_i), y_{i+1}) \leq \inf_{k=\overline{0,1}} d(f_k(x_{i+1}), x_{i+2}) < \varepsilon, i = \overline{1, m-2},$$

and

$$\inf_{k=\overline{0,1}} d(f_k(y_{m-1}), y_m) \leq \inf_{k=\overline{0,1}} d(f_k(x_m), \phi(x)) = 0.$$

So, $y \in CR(\mathcal{F})$. □

2.5. Recurrent point

Definition 2.5. *We say that $x \in X$ is a* recurrent point *of IFS $\mathcal{F}$ if there exist $\alpha \in \Sigma$ and a sequence $\{n_i\} \subset \mathbb{N}$ such that*

$$\lim_{n_i \to \infty} f_\alpha^{n_i}(x) = x.$$

We use $R(\mathcal{F})$ to denote the set of the recurrent points of IFS $\mathcal{F}$.

Firstly, we show that $R(\mathcal{F})$ is conditional invariable for IFS $\mathcal{F}$.

Theorem 2.13. *If $f_0 \circ f_1 = f_1 \circ f_0$, then $\mathcal{F}(R(\mathcal{F})) \subset R(\mathcal{F})$.*

Proof. We will show that $f_0(R(\mathcal{F})) \subset R(\mathcal{F})$. It is similar for $f_1(x)$. For any given $x \in R(\mathcal{F})$, there exist $\alpha \in \Sigma$ and a sequence $\{n_i\}$ of positive integers such that

$$\lim_{n_i \to \infty} f_\alpha^{n_i}(x) = x.$$

Since f_0 is continuous,

$$f_0(\lim_{n_i \to \infty} f_\alpha^{n_i}(x)) = \lim_{n_i \to \infty} f_0(f_\alpha^{n_i}(x)) = f_0(x).$$

So, $f_0(x) \in R(\mathcal{F})$. □

The following example demonstrates that the condition "$f_0 \circ f_1 = f_1 \circ f_0$" in Theorem 2.13 cann't be removed.

Example 2.3. *Consider two maps f_0, f_1 on $\{-1, 0, 1\}$ as follows:*

$$f_0(-1) = -1, f_0(0) = 1, f_0(1) = -1$$

and

$$f_1(-1) = -1, f_1(0) = 0, f_1(1) = -1.$$

We have that $f_1 \circ f_0(0) \neq f_0 \circ f_1(0)$, $f_0 \circ f_1 \neq f_1 \circ f_0$. And it is easy to show that $0 \in R(\mathcal{F})$, but $f_0(0) \notin R(\mathcal{F})$.

Now we show that the recurrent point is retentive under iteration of IFS $\mathcal{F}$.

Theorem 2.14. *For any $n > 0$, $R(\mathcal{F}) = R(\mathcal{F}^n)$.*

Proof. Firstly, we prove $R(\mathcal{F}) \subset R(\mathcal{F}^n)$. Let $x \in R(\mathcal{F})$ and U_0 be an open neighborhood of x. Then there exist $\alpha \in \Sigma$ and a sequence $\{n_i\}$ of positive integers such that $\lim\limits_{n_i \to \infty} f_\alpha^{n_i}(x) = x$. In addition, there exist a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ and an integer r, $0 \leq r < n$ such that for each j there exists $q_j \in \mathbb{N}$ satisfying $n_{i_j} = nq_j + r$. Let $m_j = n_{i_j}$, then

$$\lim_{m_j \to \infty} f_\alpha^{m_j}(x) = x.$$

Thus, there exists m_{j_1} such that $f_\alpha^{m_{j_1}}(x) \in U_0$. By the continuity of $f_\alpha^{m_{j_1}}$, there exists an open neighborhood U_1 of x satisfying $U_1 \subset U_0$ and $f_\alpha^{m_{j_1}}(U_1) \subset U_0$. And there exists m_{j_2} such that

$$f_\alpha^{m_{j_2}}(x) \in U_1,$$

$\cdots$. By induction, we can get n open neighborhoods $U_0, U_1, \cdots, U_{n-1}$ of x and n integers $m_{j_1}, m_{j_2}, \cdots, m_{j_n}$ satisfying
(1) $U_{n-1} \subset U_{n-2} \subset \cdots \subset U_0$,
(2) $f_\alpha^{m_{j_k}}(U_k) \subset U_{k-1}$, $k = \overline{1, n-1}$,
(3) $f_\alpha^{m_{j_k}}(x) \in U_{k-1}$, $k = \overline{1, n-1}$.
Therefore, $f_\alpha^{m_{j_1}} \circ f_\alpha^{m_{j_2}} \circ \cdots \circ f_\alpha^{m_{j_n}}(x) \in U_0$. Since $m_{j_1} + m_{j_2} + \cdots + m_{j_n} = 0($ mod $n)$, $x \in R(\mathcal{F}^n)$.

Next, we prove that $R(\mathcal{F}) \supset R(\mathcal{F}^n)$. For any given $x \in R(\mathcal{F}^n)$, $x \in \omega(x, \mathcal{F}^n, orb_\alpha(x, \mathcal{F}^n))$. Then, it is easy to show that $x \in \omega(x, \mathcal{F}, orb_\alpha(x))$. So, $x \in R(\mathcal{F})$. □

2.6. Non-wandering point

Definition 2.6. *We say that $x \in X$ is a* non-wandering point *of IFS $\mathcal{F}$ if for any $\varepsilon > 0$ there exist $\alpha \in \Sigma$, $y_0 \in X$ and $n > 0$ such that*

$$d(x, y_0) < \varepsilon \text{ and } d(f_\alpha^n(y_0), x) < \varepsilon.$$

We use $\Omega(\mathcal{F})$ to denote the set of the non-wandering points of IFS $\mathcal{F}$.

Firstly, we study if $\Omega(\mathcal{F})$ is a closed set.

Theorem 2.15. $\Omega(\mathcal{F}) = \overline{\Omega(\mathcal{F})}$.

Proof. Let $\varepsilon > 0$ and let x be a cluster point of $\Omega(\mathcal{F})$. Then there exists $x' \in \Omega(\mathcal{F})$ such that $d(x, x') < \frac{\varepsilon}{2}$. Since $x' \in \Omega(\mathcal{F})$, there exist $y_0 \in X$, $\alpha \in \Sigma$ and $n > 0$ such that

$$d(x', y_0) < \frac{\varepsilon}{2} \text{ and } d(f_\alpha^n(y_0), x') < \frac{\varepsilon}{2}.$$

Therefore,

$$d(x, y_0) \leq d(x, x') + d(x', y_0) < \varepsilon \text{ and } d(f_\alpha^n(y_0), x) \leq d(f_\alpha^n(y_0), x') + d(x', x) < \varepsilon.$$

So, $x \in \Omega(\mathcal{F})$. □

Now we illustrate that $\Omega(\mathcal{F})$ is neither invariable nor iterable for IFS $\mathcal{F}$, respectively.

Example 2.4. *Consider two maps f_0, f_1 same as in Example 2.3.*

For any $\varepsilon > 0$, $d(0, 0) = 0$ and $d(f_1(0), 0) = d(0, 0) = 0$. So, $0 \in \Omega(\mathcal{F})$. What's more, there is only $1 \in \{-1, 0, 1\}$ satisfying $d(f_0(0), 1) < \frac{1}{2}$. While for any $\alpha \in \Sigma$ and any $n > 0$, $f_\alpha^n(f_0(0)) = -1$. So, $f_0(0) \notin \Omega(\mathcal{F})$. That is $\mathcal{F}(\Omega(\mathcal{F})) \nsubseteq \Omega(\mathcal{F})$.

Example 2.5. *Consider the constant map $f_0(x) = 1$ and f_1 defined as the map f in Example 1.4.1 of* [9].

For the sake of convenience, we write again the map f in Example 1.4.1 of [9] as follows. Let f be a continuous map on $[0, 1]$ with

$$f(a) = c, f(b) = 1, f(c) = d, f(d) = c, f(1) = a,$$

where $0 < a < b < c < d < 1$. What's more, f is strictly monotonically increasing on $[0, b]$ and is linear on $[b, c]$, $[c, d]$, $[c, 1]$. Then $a \in [0, 1]$ is a non-wandering point of f. Thus, $a \in \Omega(\mathcal{F})$. However, for any small enough $\varepsilon > 0$ and any $\alpha \in \Sigma$, $f_\alpha^2([a - \varepsilon, a + \varepsilon]) \subset [c, 1]$, and $[c, 1]$ is an invaiable set for IFS $\mathcal{F}^2$. So, $a \notin \Omega(\mathcal{F}^2)$. That is $\Omega(\mathcal{F}) \nsubseteq \Omega(\mathcal{F}^2)$.

2.7. Strong non-wandering point

Definition 2.7. *We say that $x \in X$ is a* strong non-wandering point *of IFS $\mathcal{F}$ if for any open neighborhood V of x there exist $\alpha \in \Sigma$ and $y \in X$ such that*

$$\limsup_{N \to \infty} \frac{1}{N} \sharp\{i | f_\alpha^i(y) \in V, 0 \leq i \leq N - 1\} > 0,$$

where $\sharp A$ is the cardinal number of A. We use $S\Omega(\mathcal{F})$ to denote the set of the strong non-wandering points of IFS $\mathcal{F}$.

Now we show that $S\Omega(\mathcal{F})$ is an invariable closed set.

Theorem 2.16. (1) $S\Omega(\mathcal{F}) = \overline{S\Omega(\mathcal{F})}$,

(2) $\mathcal{F}(S\Omega(\mathcal{F})) \subset S\Omega(\mathcal{F})$.

Proof. (1) Let $\varepsilon > 0$ and let z be a cluster point of $S\Omega(\mathcal{F})$. Then there exists $x \in S\Omega(\mathcal{F})$ such that $d(x, z) < \frac{\varepsilon}{2}$. And there exist $\alpha \in \Sigma$ and $y \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{i | d(f_\alpha^i(y), x) < \frac{\varepsilon}{2}, 0 \le i \le N-1\} > 0.$$

Since for any $i \ge 0$, $d(f_\alpha^i(y), z) \le d(f_\alpha^i(y), x) + d(x, z)$,

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{i | d(f_\alpha^i(y), z) < \varepsilon, 0 \le i \le N-1\} > 0.$$

So, $z \in S\Omega(\mathcal{F})$.

(2) Let $y \in \mathcal{F}(S\Omega(\mathcal{F}))$, without the loss of generality, we can take $y = f_0(x)$, where $x \in S\Omega(\mathcal{F})$. Let $\varepsilon > 0$ and let $\delta > 0$ such that for any $x, y \in X$, $d(x, y) < \delta$ implies $d(f_0(x), f_0(y)) < \varepsilon$. Since $x \in S\Omega(\mathcal{F})$, there exist $\alpha \in \Sigma$ and $z \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{i | d(f_\alpha^i(z), x) < \delta, 0 \le i \le N-1\} > 0.$$

Furthermore, there exists $\beta \in \Sigma$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{i | d(f_\beta^i(z), y) < \varepsilon, 0 \le i \le N-1\} > 0.$$

So, $y \in S\Omega(\mathcal{F})$.

□

3. Relationships among various kinds of recurrent point

In this section we study the relationships between these various kinds of recurrent point sets. Before this, we introduce the definitions of weakly almost point and quasi-weakly almost periodic point for the IFS.

Definition 3.1. *We say that $x \in X$ is a* weakly almost periodic point *of the IFS $\mathcal{F}$ if there exsits $\alpha \in \Sigma$ and for any $\varepsilon > 0$ there exists a positive integer N_ε such that*

$$\sharp\{i | d(f_\alpha^{i+j}(x), f_\alpha^j(x)) < \varepsilon, \forall j \in \mathbb{N}, 0 \le i \le nN_\varepsilon\} \ge n, \forall n > 0.$$

We use $W(\mathcal{F})$ to denote the set of the weakly almost points of IFS $\mathcal{F}$.

Definition 3.2. *We say that $x \in X$ is a* quasi-weakly almost periodic point *of IFS $\mathcal{F}$ if there exsits $\alpha \in \Sigma$ for any $\varepsilon > 0$ there exist a positive integer N_ε and a sequence $\{n_j\} \subset \mathbb{N}$ such that*

$$\sharp\{i | d(f_\alpha^{i+k}(x), f_\alpha^k(x)) < \varepsilon, \forall k \in \mathbb{N}, 0 \le i < n_j N_\varepsilon\} \ge n_j, \forall j > 0.$$

We use $QW(\mathcal{F})$ to denote the set of the quasi-weakly almost points of IFS $\mathcal{F}$.

Theorem 3.1. *$Fix(\mathcal{F}) \subset P(\mathcal{F}) \subset AP(\mathcal{F}) \subset W(\mathcal{F}) \subset QW(\mathcal{F}) \subset R(\mathcal{F}) \subset \Omega(\mathcal{F}) \subset CR(\mathcal{F})$.*

Proof. It is easy to know that $Fix(\mathcal{F}) \subset P(\mathcal{F}) \subset AP(\mathcal{F}) \subset W(\mathcal{F}) \subset QW(\mathcal{F}) \subset R(\mathcal{F})$ by their definitions.

Now we show that $\Omega(\mathcal{F}) \subset CR(\mathcal{F})$.

Let $x \in \Omega(\mathcal{F})$, $\varepsilon > 0$ and let $\delta > 0$ such that for any $x, y \in X$, $d(x,y) < \delta$ implies $d(f_0(x), f_0(y)) < \varepsilon$ and $d(f_1(x), f_1(y)) < \varepsilon$. Then there exist $\alpha \in \Sigma$, $y_0 \in X$ and $n > 0$ such that

$$d(x, y_0) < \delta \text{ and } d(f_\alpha^n(y_0), x) < \delta.$$

Set $x_0 = x, x_1 = f_1(y_0), x_2 = f_1(y_1), \cdots, x_{m-1} = f_1(y_{m-2})$, where $y_{i+1} = f_1(y_i), 0 \le i \le n-2$ with $x_0 = x_m = x$. Then,

$$\inf_{k=\overline{0,1}} d(f_k(x_0), x_1) = \inf_{k=\overline{0,1}} d(f_k(x), f_1(y_0)) \le d(f_1(x), f_1(y_0)) < \varepsilon,$$

$$\inf_{k=\overline{0,1}} d(f_k(x_i), x_{i+1}) = \inf_{k=\overline{0,1}} d(f_k(y_i), f_1(y_i)) = 0 < \varepsilon, i = \overline{1, m-2},$$

$$\inf_{k=\overline{0,1}} d(f_k(x_{m-1}), x_m) = \inf_{k=\overline{0,1}} \{d(f_k(y_{m-1}), y_m) + d(y_m, x))\} < \varepsilon.$$

So, $x \in CR(\mathcal{F})$. □

It is easy to get the following relationship by their definitions.

Theorem 3.2. *$AP(\mathcal{F}) \subset S\Omega(\mathcal{F}) \subset \Omega(\mathcal{F})$.*

Since (X, d) is compact, $AP(\mathcal{F}) \neq \emptyset$. Then by Theorem 3.1 and Theorem 3.2 we can get the following corollary.

Corollary 3.1. *All of $AP(\mathcal{F})$, $W(\mathcal{F})$, $QW(\mathcal{F})$, $R(\mathcal{F})$, $\Omega(\mathcal{F})$ and $CR(\mathcal{F})$ are not empty.*

Theorem 3.3. *$\overline{P(\mathcal{F})} \subset S\Omega(\mathcal{F})$.*

Proof. Let $x \in \overline{P(\mathcal{F})}$. Next we will start splitting in the following situation.

Case 1. $x \in P(\mathcal{F})$. There exist $\alpha \in \Sigma$ and $m > 0$ such that $f_\alpha^{i+m}(x) = f_\alpha^i(x)$, $\forall i \ge 0$. Let $\varepsilon > 0$. Then,

$$\frac{1}{n}\sharp\{i | d(x, f_\alpha^i(x)) < \varepsilon, 0 \le i \le n-1\} = ([\frac{n}{m}] + 1)\frac{1}{n}.$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{n}\sharp\{i | d(x, f_\alpha^i(x)) < \varepsilon, 0 \le i \le n-1\} \ge \lim_{n\to\infty} \frac{1}{n}(\frac{n}{m} - 1) = \frac{1}{m} > 0.$$

So, $x \in S\Omega(\mathcal{F})$.

Case 2. $x \notin P(\mathcal{F})$. Let $\varepsilon > 0$. By $x \in \overline{P(\mathcal{F})}$, there exists $y \in P(\mathcal{F})$ with $x \neq y$ such that $d(x, y) < \varepsilon$. And there exist $\alpha \in \Sigma$ and $m > 0$ such that $f_\alpha^{i+m}(y) = f_\alpha^i(y)$, $\forall i \ge 0$. Thus,

$$\frac{1}{n}\sharp\{i | d(x, f_\alpha^i(y)) < \varepsilon, 0 \le i \le n-1\} = ([\frac{n}{m}] + 1)\frac{1}{n}.$$

Then, it is similar with Case 1 that $x \in S\Omega(\mathcal{F})$.

□

4. Conclusion

The main properties of the sets of ω-limit points, periodic points, almost periodic points, chain recurrent points and non-wandering points for iterated function systems can be summarized in Table 1.

Table 1. The main conclusions about the various kinds of recurrent point sets.

Property	$\omega(\mathcal{F})$	$P(\mathcal{F})$	$AP(\mathcal{F})$	$CR(\mathcal{F})$	$R(\mathcal{F})$	$\Omega(\mathcal{F})$	$S\Omega(\mathcal{F})$
nonempty	$\surd$	—	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
invariable	$\surd$	$\times\star$	$\not\surd^{\star}$	$\not\surd^{\star}$	$\not\surd^{\star}$	$\times^{\star}$	$\surd$
closed	$\surd$	—	—	$\surd$	—	$\surd$	$\surd$
iterable	$\surd$	$\surd$	$\surd$	—	$\surd$	$\times$	—

(1)“$\not\surd$” means that it doesn’t hold and it holds with some additional conditions. (2)“$\star$” menas it holds for continuous slef-maps.

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