ON GENERALIZED ALMOST θ- CONTRACTIONS WITH AN APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

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In this study, combining the concept of α-admissibility with θ-contraction and almost contraction concepts in the setting of complete metric spaces, we present some existence theorems of fixed point for new contractions type. An example is furnished to demonstrate the usability of our outcomes; moreover, we apply our main results to prove the existence of the solutions for a boundary value problem of fractional differential equations with integral boundary conditions.

Keywords: fixed point, θ-contraction, almost contraction, α-admissible, fractional differential equations.

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1. Introduction and preliminaries

In recent years, the fixed point theory has become one of the most important tools to solve some problems in diverse fields as non linear analysis, physics, biology and game theory. Banach provided the first fixed point theorem in metric spaces, which was generalized in different directions, and one of these generalizations was given by Berinde [7], where he introduced the concept of almost contraction as a generalization to the weak contraction, in sense of Berinde. Afterward, many results have been obtained for example, see [5, 6, 26, 27]. Recently, Babu et al. [3] introduced a new type of contractive condition called "condition (B)", and they proved the existence of a fixed point for this class of mappings, in the same way, Ćirić et al.[13] introduced the concept of almost generalized contractive condition and established some fixed point results in ordered metric spaces.

Samet et al. [25] introduced a new concept called α-admissible and they obtained some fixed point results for $\alpha - \psi$-contractive mappings, later some results have been given by using such concept; see, for instance [4, 20, 24]. Recently, Jleli and Samet [17] introduced a new contractions type called θ-contraction to prove the existence of fixed points for such contractions. It is worth mentioning here that a contraction in the sense of Banach is a particular of θ contraction, while there are some θ-contractions that are not Banach contraction. After that, several authors studied different variations of θ-contraction; for example, see [1, 14, 15, 22, 28].

In this work, we combine the notion of α admissible mappings with θ- contraction and Berinde type contraction concepts to introduce a new type of contractions and related fixed point results in complete metric spaces. We also deduce the existence of fixed points in partially ordered metric spaces and in complete metric spaces endowed with a graph by using our main results. Finally, we provide an example and an application to the existence of the

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solutions for a boundary value problem of fractional differential equations to illustrate the importance of the obtained results.

Definition 1.1. [7] *Let (X, d) be a metric space. A mapping $T : X \to X$ is called an almost $((\delta\text{-}L)$ weak) contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

Definition 1.2. [3] *A self mapping T on a metric space (X, d) is said satisfies the condition (B), if there exist $\delta \geq 0$ and $L \geq 0$ such that for all $x, y \in X$ we have*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).$$

Definition 1.3. [10] *A self-mapping T on metric space (X, d) is said a strong almost contraction of Ćiric type, if there exist $\delta \geq 0$ and $L \geq 0$ such that for all $x, y \in X$ we have*

$$d(Tx, Ty) \leq \delta M(x, y) + Ld(y, Tx),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$.

Definition 1.4. [25] *Let X be a nonempty set and $T : X \to X$,*
$\alpha : X \times X \to [0, \infty)$ be two mappings. T is α-admissible if for $x, y \in X$ with $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$.

Definition 1.5. [17]*Let Θ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ such that:*

(θ_1) : *θ is non decreasing,*
(θ_2) : *for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,*
(θ_3) : *there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t) - 1}{t^r} = l$.*

Example 1.1.
(1) $\theta_1(t) = e^t$.
(2) $\theta_2(t) = e^{te^t}$.
(3) $\theta_3(t) = e^{\sqrt{x}}$.
(4) $\theta_4(t) = e^{\sqrt{t}e^t}$.

Throughout this paper, we will denote by Φ the set of all continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying :

(1) : ψ is nondecreasing ,
(2) : $\sum_{i=1}^{\infty} \psi^n(t) < \infty$, for all $t \in [0, +\infty)$.

Clearly, if $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in [0, +\infty)$.

Definition 1.6. [7] *Let (X, d) be a metric space. A map $T : X \to X$ is said to be a weak ψ-contraction if there exist $L \geq 0$ and $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)) + Ld(y, Tx),$$

for all $x, y \in X$.

2. Main results

Definition 2.1. *Let (X,d) be a metric space and $\alpha : X\times X \to \mathbb{R}$. A mapping $T : X \to X$ is called a generalized almost (α,ψ,θ) contraction, if there exist a function $\theta\in\Theta$, $\psi\in\Psi$, $L\geq 0$ and $k:(0,\infty)\to[0,1)$ satisfies $\lim_{t\to s^+}\sup k(t)<1$ for all $s\in(0,\infty)$ such that*

$$\alpha(x,y)\theta(d(Tx,Ty)) \leq [\theta(\psi(M(x,y)) + LN(x,y)]^{k(M(x,y))}, \tag{1}$$

for all $x,y\in X$ with $d(Tx,Ty)>0$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}$$

and $N(x,y) = min\{d(x,Ty), d(y,Tx)\}$.

If $\alpha(x,y)=1$ for all $x,y\in X$, then T is called a generalized almost (ψ,θ) contraction.

Example 2.1. *Let $X=\{1,2,3\}$ and $d(x,y)=|x-y|$. Define $T\colon X\to X$ and $\alpha : X\times X\to [0,\infty)$ by*

$$Tx = \begin{cases} 2, & x\in\{1,2\} \\ 1, & x=3 \end{cases}$$

and $\alpha(x,y) == e^{|x-y|}$.

Taking $\theta(t)=e^t$, $\psi(t)=\frac{2}{3}t$, $L=4$ and $k(t)=e^{-\frac{1}{2}t}$. Now, we show that the contractive condition is verified.

(1) *For $x=1$ and $y=3$, we have*

$$3 \leq e^{-\frac{1}{2}}(\frac{4}{3}+4),$$

which implies

$$e^3 \leq e^{e^{-\frac{1}{2}}(\frac{4}{3}+4)}.$$

(2) *For $x=2$ and $y=3$, we have*

$$\frac{3}{2} \leq e^{-\frac{1}{2}}(\frac{4}{3}+4),$$

which implies

$$e^{\frac{3}{2}} \leq e^{e^{-\frac{1}{2}}(\frac{4}{3}+4)}.$$

Then T is a generalized almost (α,ψ,θ)- contraction.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X\to X$ be a generalized almost (α,ψ,θ) contraction, with $\theta\in\Theta$. Assume that the following conditions are satisfied:*

(1) *T is α-admissible.*
(2) *There exist $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$.*
(3) *X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n\to x\in X$ and $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, then $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.*

Then T has a fixed point.

Proof. From (2) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$, let $x_1=Tx_0$. If $x_0=x_1$, then x_0 is a fixed point. Suppose the contrary, since T is α-admissible, and by using (1) we get

$$\begin{aligned}\alpha(x_0,x_1)\theta(d(Tx_0,Tx_1)) &\leq [\theta(\psi(M(x_0,x_1)) + LN(x_0,x_1)]^{k(M(x_0,x_1))} \\ &= [\theta(\psi(M(x_0,x_1))]^{k(M(x_0,x_1))},\end{aligned}$$

but

$$\begin{aligned} M(x_0,x_1) &= \max\left\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), \frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2}\right\} \\ &= \max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_2)}{2}\right\} \\ &\leq \max\{d(x_0,x_1), d(x_1,x_2)\} \end{aligned}$$

If $d(x_0,x_1) \leq d(x_1,Tx_1)$, we get

$$d(x_1,Tx_1)) \leq \alpha(x_0,x_1)\theta(d(Tx_0,Tx_1))$$
$$\leq [\theta(\psi(d(x_1,Tx_1))]^{k(d(x_1,Tx_1))} < d(x_1,Tx_1)),$$

which is a contradiction, then we get $M(x_0,x_1) \leq d(x_0,x_1)$. On other hand

$$N(x_0,x_{11}) = \min\{d(x_0,Tx_1), d(x_1,Tx_0)\} = \min\{d(x_0,x_2), d(x_1,x_1)\} = 0.$$

Hence by (1) and from the inequality $\alpha(x_0,x_1) \geq 1$ we get

$$\theta(d(x_1,Tx_1)) \leq \alpha(x_0,x_1)\theta(d(Tx_0,Tx_1))$$
$$\leq [\theta(\psi(d(x_0,x_1))]^{k(d(x_0,x_1))}.$$

Putting $x_2 = Tx_1$, we get

$$d(x_1,x_2)) \leq \alpha(x_0,x_1)\theta(d(Tx_0,Tx_1))$$
$$\leq [\theta(\psi(d(x_0,x_1))]^{k(d(x_0,x_1))}.$$

Since T is α-admissible, we get $\alpha(x_1,x_2) \geq 1$, so (1) gives

$$\theta(d(x_2,x_3)) \leq \alpha(x_0,x_1)\theta(d(Tx_1,Tx_2))$$
$$\leq [\theta(\psi(M(x_1,x_2)) + N(x_1,x_2)]^{k(M(x_1,x_2))}.$$
$$\leq [\theta(M(x_0,x_1))]^{k(d(x_0,x_1)).k(M(x_1,x_2))}.$$

As is the first step we can check easily that $M(x_1,x_2) \leq d(x_1,x_2)$ and $N(x_1,x_2) = 0$. Then we obtain

$$\theta(d(x_2,x_3)) \leq \alpha(x_1,x_2)\theta(d(Tx_1,Tx_2))$$
$$\leq [\theta(d(x_0,x_1))]^{k(d(x_0,x_1)).k(d(x_1,x_2))}.$$

Continuing in this manner, we construct a sequence (x_n) satisfying $x_{n+1} = Tx_n$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point. Suppose the contrary, so $d(x_n,Tx_n) > 0$ and $\alpha(x_n,x_{n+1}) \geq 1$ since α-admissibility of T) for all $n \in \mathbb{N}$ and by using (1) we get

$$\alpha(x_n,x_{n+1})\theta(d(Tx_{n-1},Tx_n)) \leq [\theta(\psi(d(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))}.$$
$$\leq [\theta(d(x_0,x_1)]^P,$$

where $P = \prod_{i=1}^n k(d(x_{i-1},x_i))$.
The sequence $(d(x_n,x_{n+1}))_n$ is a decreasing sequence and bounded at below, then it is convergent. Since $\lim_{t\to s^+}\sup k(t) < 1$, then there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n,x_{n+1})) < \delta$, for all $n \geq n_0$. Thus $P \leq \delta^{n-n_0}$, and so we have

$$1 < \theta(d(x_n,x_{n+1})) \leq [\theta(d(x_0,x_1))]^{\delta^{n-n_0}}, \tag{2}$$

for all $n \geq n_0$.
On taking the limit as $n \to \infty$, we get $\lim_{n\to\infty}\theta(d(x_n,x_{n+1})) = 1$, from (θ_2) we obtain $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$.
Now, we prove $\{x_n\}$ is a Cauchy sequence, from (θ_3) there exist $r \in [0,1)$ and $l \in (0,\infty]$

such that $\lim_{n\to\infty} \dfrac{\theta(d(x_n, x_{n+1}) - 1}{(d(x_n, x_{n+1})^r} = l$. If $l < \infty$, let $2\varepsilon = l$, so from the definition of limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have

$$\varepsilon = l - \varepsilon \leq \frac{\theta(d(x_n, x_{n+1}) - 1}{(d(x_n, x_{n+1})^r} = l$$

$$n(d(x_n, x_{n+1}))^r \leq \frac{n\big[\big(\theta(d(x_0, x_1))\big]^{\delta^{n-n_0}} - 1\big)}{\varepsilon}. \tag{3}$$

If $l = \infty$, let A be an arbitrary positive real number, so from the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{(d(x_n, x_{n+1}))^r} > A,$$

which implies that

$$n(d(x_n, x_{n+1}))^r < \frac{n(\theta(d(x_0, x_1))^{\delta^{n-n_0}} - 1)}{A}. \tag{4}$$

Letting $n \to \infty$ in (3)(resp in (4)), we obtain

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^r = 0.$$

From the definition of the limit, there exists $n_2 \geq \max\{n_0, n_1\}$ such that for all $n \geq n_2$, we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}.$$

Then the series $\sum_{n_1}^{\infty} d(x_n, x_{n+1})$ is convergent, hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, so $\{x_n\}$ converges to some $x \in X$.
Now, we prove that x is a fixed point for T, in fact since X is regular so for the sequence $\{x_n\}$ which satisfying $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, by using (1) we get

$$\theta(d(x_{n+1}, Tx)) = \theta(d(Tx_n, Tx)) \leq [\theta(d(x_n, x)]^{k(d(x_n,x))} < \theta(d(x_n, x)),$$

since θ is non decreasing function, we get

$$0 \leq d(x_{n+1}, Tx) < d(x_n, x).$$

Passing to the limit we obtain $d(x, Tx) = 0$ which implies that $x = Tx$. □

Remark 2.1. (1) *If for all $x, y \in X$, we have $\alpha(x, y) \geq 1$, then the fixed point is unique.*
(2) *If for each $x, y \in Fix(T)$(set of fixed point of T), we have $\alpha(x, y) \geq 1$. Then the fixed point is unique.*
(3) *If for all $x, y \in X$, there exists $z \in X$ such $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then the fixed point is unique.*

If $\alpha(x, y) = 1$, for all $x, y \in X$, we get the following corollary.

Corollary 2.1. *Let (X, d) be a complete metric space, $T : X \to X$ be a self mapping if there exist $\theta \in \Theta$,$\psi \in \Psi$, $L \geq 0$ and $k : (0,\infty) \to [0, 1)$ with $\lim_{t\to s^+} \sup k(t) < 1$ for all $s \in (0,\infty)$ such that $d(Tx, Ty) > 0$ implies*

$$\theta(d(Tx, Ty)) \leq [\theta(\psi(M(x, y)) + LN(x, y)]^{k(M(x,y))}, \tag{5}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. Then T has a fixed point.

If we take $\theta(t) = e^t$ and the logarithm of two sides in corollary 2.2 we obtain the following corollary:

Corollary 2.2. *Let (X, d) be a complete metric space, $T : X \to X$ be a self mapping if there exist $\theta \in \Theta$,$\psi \in \Psi$, $L \geq 0$ and $k : (0, \infty) \to [0, 1)$ with $\lim_{t \to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that $d(Tx, Ty) > 0$ implies*

$$d(Tx, Ty)) \leq k(M(x, y))[\psi(M(x, y)) + LN(x, y)], \tag{6}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. Then T has a fixed point.

Example 2.2. *Let $X = \mathbb{R}$ and $d(x, y) = |x-y|$. Define $T \colon X \to X$ and $\alpha : X \times X \to [0, \infty)$ by*

$$Tx = \begin{cases} \frac{x+1}{2}, & x, y \in [0, 1] \\ \frac{x}{3} + 1, & x < 0 \\ \frac{x}{5}, & x > 1 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, +\infty) \\ \frac{1}{5}, & otherwise \end{cases}$$

Taking $\theta(t) = e^{te^t}$, $\psi(t) = \frac{4}{5}t$ and $k(t) = e^{-\frac{1}{4}t}$.
For $x = 0$ we get $T(0) = \frac{1}{2}$ and $\alpha(0, \frac{1}{2}) = 1$.
For all $x, y \in [0, +\infty)$, we have $\alpha(x, y) = 1$ and $T([0, +\infty)) = (0, +\infty)$, which implies T is α-admissible. Now, we show that the contractive condition is verified. For $x, y \in X$, we have $T([0, 1]) = [\frac{1}{2}, 1] \subset [0, 1]$. Then T is α-admissible.
We discuss the following cases:

(1) *For for all $x, y \in (-\infty, 0)$ with $x \neq y$, we have $d(Tx, Ty) > 0$ and*

$$\ln(\alpha(x, y)) + \frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - \frac{4}{5}d(x,y)} = -\ln(5) + \frac{5}{12} e^{-\frac{7}{15}|x-y|}$$
$$\leq e^{-\frac{1}{4}|x-y|} = k(d(x, y)).$$

(2) *For for all $x, y \in [0, 1]$ with $x \neq y$, we have $d(Tx, Ty) > 0$ and*

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - \frac{4}{5}d(x,y)} = \frac{1}{4} e^{-\frac{3}{10}|x-y|}$$
$$\leq e^{-\frac{1}{4}|x-y|} = k(d(x, y)).$$

(3) *For for all $x, y \in (1, +\infty)$ with $x \neq y$, we have $d(Tx, Ty) > 0$ and*

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - \frac{4}{5}d(x,y)} = \frac{5}{8} e^{-\frac{3}{5}|x-y|}$$
$$\leq e^{-\frac{1}{4}|x-y|} = k(d(x, y)).$$

If $\{x_n\}$ a sequence in $[0, \infty)$ converges to x, it is clear $x \in [0, \infty)$. Then all hypotheses of Theorem 2.1 hold, so T has a fixed point. Here T has two fixed point 1 and 2, since $\alpha(1, 2) = 0 < 1$ and for $x > 1$, or $y < 0$ we have $\alpha(x, y) = 0$.

3. Some sequences

In this section, as consequences we give an existence theorem of a fixed point in complete metric space endowed with a partially order relationship and in complete metric spaces endowed with graph.

Theorem 3.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to X$ be a self mapping. Assume that the following assertions hold:*

(*i*) *For each $x, y \in X$ such that $x \preceq y$ we have $Tx \preceq Ty$.*
(*ii*) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(*iii*) *There exist* $\theta \in \Theta$,$\psi \in \Psi$, $L \geq 0$ *and* $k : (0,\infty) \to [0,1)$ *satisfies* $\lim_{t\to s^+} \sup k(t) < 1$ *for all* $s \in (0,\infty)$ *such that*

$$\theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^{k(M(x,y))},$$

for all $x, y \in X$ *with* $x \preceq y$ *and* $d(Tx,Ty) > 0$. *Then* T *has a fixed point.*

Proof. Define $\alpha : X \times X \to \mathbb{R}_+$ by

$$\alpha\colon X \times X \to [0,+\infty), \quad \alpha\,(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (*iii*), we have $x \preceq y$ so $\alpha(x,y) = 1 \geq 1$, which implies T is a generalized almost (α,ψ,θ)-contraction.
Also from (*i*) for $x \in X$ and $y \in Tx$ such that $x \preceq y$, i.e., $\alpha(x,y) = 1 \geq 1$ we have $tx \preceq Ty$, then T is α-admissible.
From (*ii*) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0 = x_1$, which implies $\alpha(x_0,x_1) \geq 1$.
From [21] every ordering space is regular. Hence all hypotheses of Theorem 2.1 are satisfied, then T has a fixed point. □

Now, we present the existence of a fixed point for self mapping from a complete metric space X, endowed with a graph. Consider a graph G such that the set $V\,(G)$ of its vertices coincides with X and the set $E\,(G)$ of its edges contains all loops; that is, $E\,(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V\,(G)\,, E\,(G))$.

Theorem 3.2. *Let* (X,d) *be a complete metric space endowed with a graph* G, *that is* $G = (V(G),E(G))$, *where* $V(G)$ *is its vertices and* $E(G)$ *its edges, moreover suppose the* G *ha no parallels edges and* $T : X \to X$ *be a self mapping. Assume that the following assertions hold:*

(*i*) *For each* $x, y \in X$ *such that* $(x,y) \in E(G)$ *we have* $(Tx,Ty) \in E(G)$.
(*ii*) *There exist* $x_0 \in X$ *such that* $(x_0,Tx_0) \in E(G)$;
(*iii*) *There exist* $\theta \in \Theta$,$\psi \in \Psi$, $L \geq 0$ *and* $k : (0,\infty) \to [0,1)$ *such that* $\lim_{t\to s^+} \sup k(t) < 1$ *for all* $s \in (0,\infty)$ *satisfies*

$$\theta(d(Tx,Ty)) \leq [\theta(\psi(M(x,y)) + LN(x,y)]^{k(M(x,y))},$$

for all $x, y \in X$ *with* $(x,y) \in E(G)$ *and* $d(Tx,Ty) > 0$. *Then* T *has a fixed point.*

Proof. Define $\alpha : X \times X \to \mathbb{R}_+$ by

$$\alpha\colon X \times X \to [0,+\infty), \quad \alpha\,(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (*iii*), we have $(x,y) \in E(G)$ so $\alpha(x,y) = 1 \geq 1$, which implies T is a generalized almost (α,ψ,θ)-contraction.
Also from (*i*) for $x \in X$ and $y \in Tx$ such that $(x,y) \in E(G)$, i.e., $\alpha(x,y) = 1 \geq 1$ we have $(Tx,Ty) \in E(G)$, then T is α-admissible.
From (*ii*) there exist $x_0 \in X$ such that $(x_0,Tx_0) \in E(G)$, which implies $\alpha(x_0,Tx_0) \geq 1$.
From [16] every metric space endowed with graph is regular. Then all hypotheses of Theorem 2.1 are satisfied, then T has a fixed point. □

4. **Application to fractional differential equations**

Consider the following boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)),\ t \in J = [0,1], \\ x(0) = 0, \\ x(1) = \lambda \int_0^1 g(s, x(s))ds, \end{cases} \tag{7}$$

where ${}^cD^q$ with $1 < q \leq 2$ is the Caputo fractional derivative, $\lambda > 0$ and $f : J \times \mathbb{R} \to \mathbb{R}$.
Let $X = C(J, \mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the uniform convergence norm $\|x\|_\infty = \sup\{|x(t)|, t \in J\}$.

Lemma 4.1. *A function x is a solution of the problem (7) if and only if, x is a solution of the following integral equation:*

$$x(t) = \lambda t \int_0^1 g(s, x(s))ds, + \int_0^1 G(t,s) f(s, x(s))ds,$$

for all $t \in J$, where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} - t(1-s)^{q-1}, & 0 \leq s \leq t \leq 1 \\ -t(1-s)^{q-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{8}$$

Proof. We have

$$I^q({}^cD^q x(t)) = x(t) - c_0 - c_1 t = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds$$

by the boundary values we get

$$x(0) = c_0 = 0$$

$$x(1) = c_1 + \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, x(s)ds = \lambda \int_0^1 g(s, x(s))ds.$$

Then

$$c_1 = -\frac{1}{\Gamma(q)} \int_0^1 ((1-s)^{q-1} f(s, x(s))ds + \lambda \int_0^1 g(s, x(s))ds,$$

which gives

$$x(t) = -\frac{t}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, x(s))ds + \lambda t \int_0^1 g(s, x(s))ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds$$

which implies that

$$x(t) = \int_0^1 G(t,s) f(s, x(s))ds + \lambda t \int_0^1 g(s, x(s))ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} - t(1-s)^{q-1}, & 0 \leq s \leq t \leq 1 \\ -t(1-s)^{q-1} & 0 \leq t \leq s \leq 1. \end{cases}$$

□

$$\int_0^1 G(t,s)ds = \frac{1}{\Gamma(q)}[\int_0^1 (t-s)^{q-1} - t(1-s)^{q-1}ds - \int_t^1 t(1-s)^{q-1}ds$$
$$= \frac{1}{\Gamma(q)}[t^q + 1] \leq \frac{2}{\Gamma(q)}.$$

Let $G_0 = \frac{2}{\Gamma(q)}$.

Assume that the following assumptions hold:

(A_1) : For $x, y \in X$ with $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$.
(A_2) : There exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.
(A_3) : There exist $\psi, \varphi \in L^1(J)$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$|f(t, x_1(t)) - f(t, x_2(t))| \leq \varphi(t)\|x_1 - x_2\|),$$

$$|g(t, x_1(t)) - g(t, x_2(t)| \leq \psi(t)\|x_1 - x_2\|),$$

where $c_0 = G_0\|\varphi\|_{L^1} + \lambda\|\psi\|_{L^1} < \frac{1}{2}$.

Theorem 4.1. *Under the assumptions* $(A_1) - (A_3)$*, the problem* (7) *has a solution in* X*.*

Proof. For $(x, y) \in E(G)$ and for all $t \in J$ we have

$$|Tx(t) - Ty(t)| = |\int_0^1 G(t, s)(f(s, x(s)) - f(s, y(s)))ds + \lambda t \int_0^1 (g(s, x(s)) - g(s, y(s))ds)|$$

$$\leq (G_0\|\varphi\|_{L^1} + \lambda\|\phi\|_{L^1})|x - y|,$$

which implies that

$$\|Tx(t) - Ty(t)\|_\infty \leq G_0(\|\varphi\|_{L^1} + \|\phi\|_{L^1})\|x - y\|_\infty,$$

so

$$d(Tx, Ty) \leq (G_0\|\varphi\|_{L^1} + \|\phi\|_{L^1})d(x, y) = c_0 M(x, y),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$. Hence we have

$$e^{\sqrt{d(Tx,Ty)}} \leq (e^{\sqrt{2c_0 M(x,y)}})^{\frac{\sqrt{2}}{2}}.$$

Then all the hypotheses of Theorem 2.1 are satisfied with $\psi(t) = \frac{1}{2}t$, $k = \frac{\sqrt{2}}{2}$, $L = 0$ and $\theta(t) = e^{\sqrt{t}}$, so T has a fixed point which is a solution of problem (7). $\square$

Remark 4.1. *In the previous application, we didn't need the fourth hypothesis of Theorem 2.1, since in* [16] *the author mentioned that every metric space endowed with a graph is a regular space.*

Conclusion

In this work, we have presented a fixe point theorem by using a combination of some concepts to obtain a new type of θ-contractions in which our study improves and generalizes some results. An example has been given to illustrate our outcomes and as a consequences, we have gave some fixed point results on a metric space endowed with a partial ordering or with a graph. Finally, we have applied our new theorem to ensure the existence of solutions for a boundary value problem of fractional differential equations with integral boundary conditions under weaker conditions.

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