U.P.B. Sci. Bull., Series A, Vol. 79, Iss. 1, 2017 ISSN 1223-7027

NEW FAST CONVERGENT SEQUENCES OF EULER-MASCHERONI TYPE

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We introduce two new sequences of Euler-Mascheroni type which have fast convergence to the constant γ. Our results extend, improve and unify some existing results in this direction.

Keywords: Euler-Mascheroni constant, speed of convergence, asymptotic expansion.
MSC2010: 33B14, 26A48, 41A60, 41A25

1. Introduction

One of the most known constant in mathematics is the Euler-Mascheroni constant $\gamma = 0.57721566490153286\ldots$, which is defined as the limit of sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n. \tag{1}$$

This sequence has diverse applications in many areas of mathematics, ranging from classical or numerical analysis to number theory, special functions or theory of probability.

There is a huge literature about the sequence $(\gamma_n)_{n\geq 1}$ and the constant γ. Please refer to [3, 4, 5, 6, 7, 8] and all the references therein.

The speed of convergence to γ of sequence $(\gamma_n)_{n\geq 1}$ is very slowly, if we take into account that it converges like n^{-1}. That is why many authors develop studies to improve the speed of convergence of sequence $(\gamma_n)_{n\geq 1}$.

For example, Cesaro [1] proved that for every positive integer $n \geq 1$, there exists a number $c_n \in (0, 1)$ such that the following relation is true:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{2}\log(n^2 + n) - \gamma = \frac{c_n}{6n(n+1)}.$$

Recently, by changing the logarithmic term in (1), Chen and Li [2] introduced the sequences

$$P_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right)$$

and

$$Q_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{4}\log\left[\left(n^2 + n + \frac{1}{3}\right)^2 - \frac{1}{45}\right]$$

and proved that the following inequalities hold:

$$\frac{1}{180(n+1)^4} < \gamma - P_n < \frac{1}{180n^4},$$

and

$$\frac{8}{2835(n+1)^6} < Q_n - \gamma < \frac{8}{2835n^6},$$

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for all integers n, $n \geq 1$.

In section 2, we introduce the sequence

$$\omega_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{r}\log(n^r + bn^{r-1}), \tag{2}$$

where r and b are positive real constants.

Our aim is to find values for r and b which provide a faster convergence of the sequence $(\omega_n)_{n\geq 1}$ to the Euler-Mascheroni constant γ.

In section section 3, we discuss on the faster convergence towards the constant γ of a sequence with logarithmic term involving the constant e. In this part, we make a link between our study and the research work of Mortici [5].

2. A new fast convergent sequence to the constant γ

An important tool for computing the speed of the convergence is Stolz lemma, the case 0/0. Our study is based on a variant of this lemma.

Lemma 2.1. *If $(\omega_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_{n+1} - \omega_n) = l \in \mathbb{R}, \quad k > 1,$$

then there exists the limit

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{1-k}.$$

At first, we calculate the difference $\omega_{n+1} - \omega_n$. After some calculation, we find

$$\omega_{n+1} - \omega_n = \frac{1}{n+1} - \frac{r-1}{r}\log\left(1 + \frac{1}{n}\right) - \frac{1}{r}\left[\log\left(1 + \frac{b+1}{n}\right) - \log\left(1 + \frac{b}{n}\right)\right].$$

Then, we use a computer software to write the expression $\omega_{n+1} - \omega_n$ as power series of n^{-1}. Thus

$$\omega_{n+1} - \omega_n = \frac{2b-r}{2r}\cdot\frac{1}{n^2} + \frac{2r-3b^2-3b}{3r}\cdot\frac{1}{n^3} + \frac{4b^3+6b^2+4b-3r}{4r}\cdot\frac{1}{n^4} + 0\left(\frac{1}{n^5}\right). \tag{3}$$

The best speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is obtained in case when first coefficients of (3) vanish:

$$\frac{2b-r}{2r} = 0, \quad \frac{2r-3b^2-3b}{3r} = 0.$$

We find $b = \frac{1}{3}$ and $r = \frac{2}{3}$. In this case, the coefficient of n^{-4} becomes $\frac{1}{18}$.

We can state the following

Theorem 2.1. *The following statements hold true:*

i) *If $2b - r \neq 0$, then the speed of convergence of sequence $(\omega_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n^2(\omega_{n+1} - \omega_n) = \frac{2b-r}{2r}, \quad \textit{and} \quad \lim_{n\to\infty} n(\omega_n - \gamma) = \frac{r-2b}{2r} \neq 0.$$

ii) *If $2b - r = 0$, and $2r - 3b^2 - 3b \neq 0$, that is $b \neq \frac{1}{3}$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^3(\omega_{n+1} - \omega_n) = \frac{2r-3b^2-3b}{3r}, \quad \textit{and} \quad \lim_{n\to\infty} n^2(\omega_n - \gamma) = \frac{-2r+3b^2+3b}{6r} \neq 0.$$

iii) *If $2b - r = 0$, and $2r - 3b^2 - 3b = 0$, that is $b = \frac{1}{3}$ and $r = \frac{2}{3}$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-3}, since*

$$\lim_{n\to\infty} n^4(\omega_{n+1} - \omega_n) = \frac{1}{18}, \quad \textit{and} \quad \lim_{n\to\infty} n^3(\omega_n - \gamma) = -\frac{1}{54}.$$

Remark 2.1. For $b = \frac{1}{3}$ and $r = \frac{2}{3}$, the sequence introduced by us has the form

$$\omega_n^0 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{3}{2}\log\left(\sqrt[3]{n^2} + \frac{1}{3\sqrt[3]{n}}\right).$$

If denote by ψ the digamma function, it is well known that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{4}$$

By the elegant article by Mortici and Chen [8], we obtain

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x+1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3}, \quad \forall x > 0. \tag{5}$$

We will prove the following result.

Theorem 2.2. *If (ω_n^0) is defined as above, then the following relations hold:*

$$\frac{1}{54(n+1)^3} < \gamma - \omega_n^0 < \frac{1}{54n^3},$$

for any integer n, $n \geq 1$.

Proof. First of all, we observe that

$$\gamma - \omega_n^0 - \frac{1}{54n^3} = \gamma - \sum_{k=1}^{n} \frac{1}{k} + \frac{3}{2}\log\left(\sqrt[3]{n^2} + \frac{1}{3\sqrt[3]{n}}\right) - \frac{1}{54n^3}.$$

and, by (4),

$$\gamma - \omega_n^0 - \frac{1}{54n^3} = \frac{3}{2}\log\left(\sqrt[3]{n^2} + \frac{1}{3\sqrt[3]{n}}\right) - \psi(n+1) - \frac{1}{54n^3}. \tag{6}$$

This enable us to consider the function

$$F\colon (0,\infty) \to \mathbb{R}, \quad F(x) = \frac{3}{2}\log\left(\sqrt[3]{x^2} + \frac{1}{3\sqrt[3]{x}}\right) - \psi(x+1) - \frac{1}{54x^3},$$

whose formula can be written in a more convenient form as

$$F(x) = \frac{3}{2}\Big(\log(3x+1) - \log 3 - \frac{1}{3}\log x\Big) - \psi(x+1) - \frac{1}{54x^3}.$$

Now, after some calculation and using (5), we get

$$F'(x) > \frac{1}{18x^4(3x+1)} > 0, \quad \forall x > 0.$$

Therefore, the sequence $(F(n))$ is increasing, hence

$$F(n) < \lim F(n) = 0.$$

From (6), it follows that

$$\gamma - \omega_n^0 - \frac{1}{54n^3} < 0.$$

To prove that

$$\gamma - \omega_n^0 - \frac{1}{54(n+1)^3} > 0,$$

we consider the function

$$G\colon [1,\infty) \to \mathbb{R}, \quad G(x) = \frac{3}{2}\log\left(\sqrt[3]{x^2} + \frac{1}{3\sqrt[3]{x}}\right) - \psi(x+1) - \frac{1}{54(x+1)^3}.$$

For calculation, we use a more convenient form of G as

$$G(x) = \frac{3}{2}\log(3x+1) - \frac{1}{2}\log x - \frac{3}{2}\log 3 - \psi(x+1) - \frac{1}{54(x+1)^3}.$$

Using the same technique as above, we find that $G'(x) < 0, \quad \forall x > 1$. This means that the sequence $G(n)$ is decreasing, hence

$$G(n) > \lim G(n) = 0,$$

and this completes the proof. □

In the following we introduce the sequence $(R_n)_{n\geq 1}$,

$$R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{r}\log(n^r + bn^{r-1} + cn^{r-2}), \tag{7}$$

r, b and c being positive real constants.

Note that for $c = 0$ we find the sequence (ω_n), introduced in (2).

We also calculate the difference

$$R_{n+1} - R_n = \frac{1}{n+1} - \frac{r-2}{r}\log\left(1 + \frac{1}{n}\right) - \frac{1}{r}\left[\log\left(1 + \frac{b+2}{n} + \frac{b+c+1}{n^2}\right) - \log\left(1 + \frac{b}{n} + \frac{c}{n^2}\right)\right].$$

Using a computer software to write the expression $R_{n+1} - R_n$ as power series of n^{-1}, we obtain

$$\begin{aligned} R_{n+1} - R_n &= \frac{2b-r}{2r}\cdot\frac{1}{n^2} + \frac{2r - 3b^2 - 3b + 6c}{3r}\cdot\frac{1}{n^3} \\ &+ \frac{-3r - 12bc + 4b^3 + 6b^2 + 4b - 12c}{4r}\cdot\frac{1}{n^4} \\ &+ \frac{4r + 2 - 5(b+2)(b+c+1)^2 + 5bc^2 + 5(b+2)^3(b+c+1) - 5b^3c + b^5 - (b+2)^5}{5r}\cdot\frac{1}{n^5} \\ &+ 0\left(\frac{1}{n^6}\right). \end{aligned}$$

We vanish the first three coefficients, and find

$$\begin{cases} \dfrac{2b-r}{2r} = 0 \\ \dfrac{2r - 3b^2 - 3b + 6c}{3r} = 0 \\ \dfrac{-3r - 12bc + 4b - 12c + 4b^3 + 6b^2}{4r} = 0. \end{cases}$$

The solution of this system is $b = 1$, $c = \dfrac{1}{3}$ and $r = 2$.

We also have $\lim\limits_{n\to\infty} n^5(R_{n+1} - R_n) = \dfrac{1}{45}$ and using Lemma 2.1, we obtain $\lim\limits_{n\to\infty} n^4(R_n - \gamma) = -\dfrac{1}{180}$. Therefore, by our method, we obtain that the sequence

$$R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right)$$

has the speed of convergence n^{-4}.

Remark 2.2. Note that the sequence $(R_n)_{n\geq 1}$ is the sequence $(P_n)_{n\geq 1}$ introduced by Chen and Li in [2]. Therefore, we proved that this one is the unique sequence of the form (7), which has the speed of convergence n^{-4}.

3. **A new fast convergent sequence with logarithmic term involving the constant** e

Our aim in this section is to discuss on the faster convergence towards the constant γ of a sequence with logarithmic term involving the constant e. In this respect, we shall refer as starting point the work of Mortici [5]. In this research article, he introduced and studied the speed of convergence to γ for sequences of the form

$$\mu_n^* = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \log\big(\exp(a/(n+b)) - 1\big) - \log a.$$

Adapting our initial sequence $(\omega_n)_{n\geq 1}$ for $r = 2$, we define a new sequence

$$\mu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{2}\log\left(\frac{\exp(a/(n^2+bn)) - 1}{a}\right).$$

As above, we write

$$\mu_{n+1} - \mu_n = \frac{1}{n+1} - \frac{1}{2}\left(\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{b+1}{n}\right) - \log\left(1+\frac{b}{n}\right)\right)$$
$$+\frac{1}{2}\log\left(\frac{\exp(a/(n+1)(n+b+1)) - 1}{\frac{a}{(n+1)(n+b+1)}}\right) - \frac{1}{2}\log\left(\frac{\exp(a/(n(n+b))) - 1}{\frac{a}{n(n+b)}}\right).$$

Now, we use a computer software to obtain the following representation in power series:

$$\begin{aligned}\mu_{n+1} - \mu_n = {} & \frac{b-1}{2n^2} + \left(1 - \frac{3b^2+3b+2}{6} - \frac{a}{2}\right)\frac{1}{n^3} \\ & + \left(-\frac{3}{4} + \frac{2b^3+3b^2+2b}{4} + \frac{3a}{4} + \frac{3ab}{4}\right)\frac{1}{n^4} \\ & + \left(\frac{9+b^5-(b+1)^5}{10} - a - \frac{3}{2}ab - ab^2 - \frac{a^2}{12}\right)\frac{1}{n^5} + 0\left(\frac{1}{n^6}\right).\end{aligned} \tag{8}$$

We cancel the first coefficients of (8),

$$b = 1, \quad 1 - \frac{3b^2+3b+2}{6} - \frac{a}{2} = 0.$$

Solving with respect to a and b, we find that the solution is $a = -\frac{2}{3}$ and $b = 1$. In this case, the coefficient of n^{-4} is also 0 and the coefficient of n^{-5} is $\frac{13}{135}$.

Therefore, we have proven this result.

Theorem 3.1. *The following statements hold good:*

i) *If $b \neq 1$, then the speed of convergence of sequence $(\mu_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n^2(\mu_{n+1} - \mu_n) = \frac{b-1}{2}, \quad \textit{and} \quad \lim_{n\to\infty} n(\mu_n - \gamma) = \frac{1-b}{2}.$$

ii) *If $b = 1$ and $a \neq \frac{-2}{3}$, then the speed of convergence of sequence $(\mu_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^3(\mu_{n+1} - \mu_n) = -\frac{1}{3} - \frac{a}{2}, \quad \textit{and} \quad \lim_{n\to\infty} n^2(\mu_n - \gamma) = \frac{1}{2}\left(\frac{a}{2} + \frac{1}{3}\right).$$

iii) *If $b = 1$ and $a = -\frac{2}{3}$, then the speed of convergence of sequence $(\mu_n)_{n\geq 1}$ is n^{-4}, since*

$$\lim_{n\to\infty} n^5(\mu_{n+1} - \mu_n) = \frac{13}{135}, \quad \textit{and} \quad \lim_{n\to\infty} n^4(\mu_n - \gamma) = -\frac{13}{540}.$$

Remark 3.1. For $b = 1$ and $a = -\frac{2}{3}$, the sequence $(\mu_n)_{n\geq 1}$ has the form

$$\mu_n^0 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{2}\log\left(\frac{1-\exp\big(-2/(3n(n+1))\big)}{\frac{2}{3}}\right).$$

We notice that the sequence (μ_n^0) has the logarithm term involving the constant e.

Remark 3.2. The new sequence (μ_n^0) converges to γ like n^{-4}, while the sequence of Mortici introduced in [5] converges to γ like n^{-3}. We deduce that the approximation $\gamma \simeq \mu_n^0$ is more accurate than $\gamma \simeq \mu_n^*$.

Using a similar technique as in the proof of Theorem 2.2, we find

Theorem 3.2. *If* (μ_n^0) *is defined as above, then the following relations hold:*

$$-\frac{13}{540n^4} < \mu_n^0 - \gamma < -\frac{13}{540n^4} + \frac{13}{270n^5},$$

for any integer n, $n \geq 1$.

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