UNIQUE OPTIMAL FUNCTION

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Această lucrare este o abordare originală a teoriei credibilității semi-liniare, din perspectiva functiilor de variabile aleatoare.

Pentru a obține rezultate superioare de credibilitate semi-liniară este utilizată o funcție optimă unică de aproximare f, în loc să considerăm funcții prescrise de aproximare: $f_1, f_2,, \ldots, f_n$.

Aceste performanțe includ cazul $f_p = f$ *pentru toți p în clasa celor mai buni estimatori de credibilitate semi-liniară, cu utilitate în practică.*

This paper is an original approach of the semi-linear credibility theory, from the perspective of the functions of the observable random variables.

In order to obtain better semi-linear credibility results, a unique optimal approximating function f is used, instead of considering prescribed approximating functions: $f_1,, f_2,, \ldots, f_n$.

These performances include the case $f_p = f$ *for all p in the class of the best semi-linear credibility estimators with usefulness in practice.*

Key words: semi-linear credibility model, approximating functions, the structure parameters.

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Introduction

This article is devoted to semi-linear credibility, where on examines functions of the random variables, representing claim amounts, rather than the claim amounts themselves.

So far, credibility estimators were linear functions of the observable random variables.

Semi-linear credibility estimators are linear functions of transformed observations.

The semi-linear credibility model involves the class of linear combinations of given functions of the observable variables, for solving the minimization problems of the type:

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$$MinE\left\{\left[f_0(X_{t+1})-\sum_{p=1}^{n}\sum_{r=1}^{t}c_{pr}f_{pr}(X_r)\right]^2\right\}$$

So the approximation to $f_0(X_{t+1})$ or to $\mu_0(\theta) = E[f_0(X_{t+1})|\theta]$ furnished in the first section, is based on prescribed approximating functions: $f_1, f_2,\ldots,f_n$.

One may either assume the function f_0, f_{pr} are given in advance-as in Section **1**, or one may try to determine the best choice-as Section **2**.

So in the second section we take $f_p = f$ for all p, and try to find the optimal function f.

1. Semi-linear credibility model

Consider a finite sequence $\theta, X_1,...,X_{t+1}$ of random variables. Assume that for fixed θ, the variables $X_1,..., X_{t+1}$ are conditionally independent and identically distributed (conditionally i.i.d.). The variables $X_1,..., X_t$ are observable and θ is the structure variable. The variable X_{t+1} is considered as being not (yet) observable.

We assume that $f_p(X_r)$, p = $\overline{0,n}$, r = $\overline{1,t+1}$ have finite variance. For f_0, we take the function of X_{t+1} we want to forecast.

We use the notation:

$$\mu_p(\theta) = E[f_p(X_r)|\theta] \tag{1.1}$$

(p = $\overline{0,n}$, r = $\overline{1,t+1}$)

This expression does not depend on r).

For this model we define the following structure parameters:

$$m_p = E[\mu_p(\theta)] = E\{E[f_p(X_r)|\theta]\} = E[f_p(X_r)] \tag{1.2}$$

$$a_{pq} = E\{Cov[f_p(X_r), f_q(X_r)|\theta]\} \tag{1.3}$$

$$b_{pq} = Cov[\mu_p(\theta), \mu_q(\theta)] \tag{1.4}$$

$$c_{pq} = Cov[f_p(X_r), f_q(X_r)] \tag{1.5}$$

$$d_{pq} = Cov[f_p(X_r), \mu_q(\theta)] \tag{1.6}$$

for p,q = $\overline{0,n}$. These expressions do not depend on r = $\overline{1,t+1}$.

The structure parameters are connected by the following relations:

$$c_{pq} = a_{pq}+b_{pq} \tag{1.7}$$

$$d_{pq} = b_{pq} \quad (1.8)$$

for p,q = $\overline{0,n}$.

This follows from the covariance relations obtained in the probability theory ,where they are very well-known.

Just as in the case of considering linear combinations of the observable variables themselves, we can also obtain non-homogeneous credibility estimates, taking as estimators the class of linear combinations of given functions of the observable variables, as shown in the following theorem:

Theorem 1.1 (Optimal non-homogeneous linearized estimator)

The linear combination of 1 and the random variables $f_p(X_r)$ (p = $\overline{1,n}$, r = $\overline{1,t}$) closest to $\mu_0(\theta) = E[f_0(X_{t-1})|\theta]$ and to $f_0(X_{t+1})$ in the least squares sense equals:

$$M = \sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p \quad (1.9),$$

where $z_1,...,z_n$ is a solution to the linear system of equations:

$$\sum_{p=1}^{n} \left[c_{pq} + (t-1)d_{pq}\right] z_p = t d_{0q} \quad (q = \overline{1,n}) \quad (1.10)$$

or to the equivalent linear system of equations:

$$\sum_{p=1}^{n} \left(a_{pq} + t b_{pq}\right) z_p = t b_{0q} \quad (q = \overline{1,n}) \quad (1.11)$$

2. Unique optimal approximating function

The estimator M for $f_0(X_{t+1})$ (or for $\mu_0(\theta)$) of **Theorem 1.1,** can be displayed as: $M = g(X_1)+g(X_2)+...+g(X_t)$ (2.1),

where: $g(x) = \frac{1}{t}\sum_{p=1}^{n} z_p f_p(x) + \frac{1}{t} m_0 - \frac{1}{t}\sum_{p=1}^{n} z_p m_p$ (2.2)

Indeed, we have:

$$M = \sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p = \frac{1}{t}\sum_{p=1}^{n} z_p \sum_{r=1}^{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p =$$

$$= \frac{1}{t}\sum_{p=1}^{n} z_p \left[f_p(X_1) + f_p(X_2) + ... + f_p(X_t)\right] + \frac{1}{t} m_0 t - \frac{1}{t}\left(\sum_{p=1}^{n} z_p m_p\right) t = \left(\frac{1}{t}\sum_{p=1}^{n} z_p \cdot\right.$$

$$\cdot f_p(X_1)+\frac{1}{t}m_0-\frac{1}{t}\sum_{p=1}^{n}z_p m_p\Bigg)+\Bigg(\frac{1}{t}\sum_{p=1}^{n}z_p f_p(X_2)+\frac{1}{t}m_0-\frac{1}{t}\sum_{p=1}^{n}z_p m_p\Bigg)+...+$$

$$+\Bigg(\frac{1}{t}\sum_{p=1}^{n}z_p f_p(X_t)+\frac{1}{t}m_0-\frac{1}{t}\sum_{p=1}^{n}z_p m_p\Bigg)=$$ $g(X_1)+g(X_2)+...+g(X_t)$, as was to be proven.

Let us forget now about this structure of g and look for any function g such that (2.1) is closest to $f_0(X_{t+1})$ (or to $\mu_0(\theta)$).

If only functions g such that $g(X_1)$ has finite variance are considered, then the optimal approximating function g results from the following theorem:

Theorem 2.1. (Optimal approximating function)

$g(X_1)+g(X_2)+...+g(X_t)$ is closest to $f_0(X_{t+1})$ (and to $\mu_0(\theta)$) in the least squares sense, if and only if g is a solution to the equation:

$$g(X_1)+(t-1)E[g(X_2)|X_1]-E[f_0(X_2)|X_1]\equiv 0 \tag{2.3}$$

Proof:

We have to solve the following minimization problem:

$$\underset{f}{Min}\, E\left\{[f_0(X_{t+1})-f(X_1)-f(X_2)-...-f(X_t)]^2\right\} \tag{2.4}$$

Suppose that g denotes the solution to this problem, then we consider:

$$f(X) = g(X)+\alpha h(X) \tag{2.5}$$

with $h(\cdot)$ arbitrary, like in variational calculus.

Let:

$$\varphi(\alpha) = E\{[f_0(X_{t+1})-f(X_1)-f(X_2)-...-f(X_t)]^2\} = E\{[f_0(X_{t+1})-g(X_1)-g(X_2)-...-g(X_t)-\alpha\cdot h(X_1)-\alpha h(X_2)-...-\alpha h(X_t)]^2\} \tag{2.6}$$

Clearly for g to be optimal, $\varphi'(0)=0$, so for every choice of h:

$$E\{[f_0(X_{t+1})-g(X_1)-\ldots-g(X_t)]\cdot[h(X_1)+h(X_2)+\ldots+h(X_t)]\}=0 \tag{2.7}$$

must hold. This can be rewritten as:

$E[tf_0(X_2)h(X_1)-tg(X_1)h(X_1)-t(t-1)g(X_2)h(X_1)] = 0$, or:

$$E[h(X_1)\{-g(X_1)-(t-1)E[g(X_2)|X_1]+E[f_0(X_2)|X_1]\}] = 0 \tag{2.8}$$

Because this equation has to be satisfied for every choice of the function h one obtains, the expression in brackets in (2.8) must be identical to zero, which proves (2.3).

An application of Theorem 2.1:

If $X_1,...,X_{t+1}$ can only take the values 0, 1, 2,..., n and $p_{qr} = P(X_1=q, X_2=r)$ for $q,n=\overline{0,n}$, then $g(X_1)+...+g(X_t)$ is closest to $f_0(X_{t+1})$ (and to $\mu_0(\theta)$ in the least squares sense, if and only if for $q = \overline{0,n}$, g(q) is a solution of the linear system:

$$g(q)\sum_{r=0}^{n} p_{qr} +(t-1)\sum_{r=0}^{n} g(r)p_{qr} = \sum_{r=0}^{n} f_0(r)p_{qr} \tag{2.9}$$

Proof:

Theorem 2.1. affirms that:

"$g(X_1)+g(X_2)+\ldots+g(X_t)$ is closest to $f_0(X_{t+1})$ (and to $\mu_0(\theta)$ in the least squares sense, if and only if g is a solution of the equation:

$$g(X_1)+(t-1)E[g(X_2)|X_1]-E[f_0(X_2)|X_1]\equiv 0\text{"} \tag{2.10}$$

Here we have:

$$g(X_1):\begin{pmatrix} g(q) \\ P[g(X_1)=g(q)] \end{pmatrix} = \begin{pmatrix} g(q) \\ P(X_1=q) \end{pmatrix}, q=\overline{0,n} \tag{2.11}$$

and:

$$[g(X_2)|X_1]:\begin{pmatrix} g(r) \\ P[g(X_2)=g(r)\,|\,X_1=q] \end{pmatrix} = \begin{pmatrix} g(r) \\ P(X_2=r\,|\,X_1=q) \end{pmatrix} =$$

$$= \begin{pmatrix} g(r) \\ \dfrac{P(X_1=q, X_2=r)}{P(X_1=q)} \end{pmatrix}, r=\overline{0,n} \tag{2.12}$$

But:

$$P(X_1 = q, X_2 = r)=p_{qr};q,r = \overline{0,n} \tag{2.13}$$

and:

$$P(X_1=q) = P(X_1 = q, \Omega) = P[X_1 = q, \bigcup_{r=0}^{n}(X_2 = r)] = P\left[\bigcup_{r=0}^{n}(X_1 = q, X_2 = r)\right] =$$

$$= \sum_{r=0}^{n} P(X_1 = q, X_2 = r) = \sum_{r=0}^{n} p_{qr}; q = \overline{0,n} \quad (2.14)$$

(see the hypothesis of the application).

The relations (2.13) and (2.14) imply:

$$[g(X_2)|X_1]: \begin{pmatrix} g(r) \\ \dfrac{p_{qr}}{\sum_{r=0}^{n} p_{qr}} \end{pmatrix}, r = \overline{0,n} \quad (2.15)$$

(see (2.12)).

So:

$$E[g(X_2)|X_1] = \sum_{r=0}^{n} g(r) \frac{p_{qr}}{\sum_{r=0}^{n} p_{qr}} = \frac{\sum_{r=0}^{n} g(r) p_{qr}}{\sum_{r=0}^{n} p_{qr}} \quad (2.16)$$

Also, we have:

$$[f_0(X_2)|X_1]: \begin{pmatrix} f_0(r) \\ P(f_0(X_2) = f_0(r) \mid X_1 = q) \end{pmatrix} = \begin{pmatrix} f_0(r) \\ P(X_2 = r \mid X_1 = q) \end{pmatrix} =$$

$$= \begin{pmatrix} f_0(r) \\ \dfrac{P(X_1 = q, X_2 = r)}{P(X_1 = q)} \end{pmatrix} = \begin{pmatrix} f_0(r) \\ \dfrac{p_{qr}}{\sum_{r=0}^{n} p_{qr}} \end{pmatrix}, r = \overline{0,n} \quad (2.17)$$

(see (2.13) and (2.14)).

So:

$$E[f_0(X_2)|X_1]=\sum_{r=0}^{n} f_0(r)\frac{p_{qr}}{\sum_{r=0}^{n} p_{qr}} = \frac{\sum_{r=0}^{n} f_0(r)p_{qr}}{\sum_{r=0}^{n} p_{qr}} \tag{2.18}$$

Inserting (2.11), (2.16) and (2.18) in (2.10) one obtains:

$$g(q)+(t-1)\frac{\sum_{r=0}^{n} g(r)p_{qr}}{\sum_{r=0}^{n} p_{qr}} - \frac{\sum_{r=0}^{n} f_0(r)p_{qr}}{\sum_{r=0}^{n} p_{qr}} = 0, \forall q = \overline{0,n}, \text{ or:}$$

$g(q)\sum_{r=0}^{n} p_{qr} + (t-1)\sum_{r=0}^{n} g(r)p_{qr} = \sum_{r=0}^{n} f_0(r)p_{qr}, \forall q = \overline{0,n}$, as was to be proven (see (2.9)).

Conclusions

An original paper which suggests a way of thought for semi-linear credibility theory development, founded on analysis of the functions of the observable random variables.

This line of thought fits perfectly within the framework of the greatest accuracy credibility theory.

The point we want to emphasize is that the approximation to $f_0(X_{t+1})$ or to $\mu_0(\theta)$ based on a unique optimal approximating function f is always better than the one furnished in **Section 1.** based on prescribed approximating functions $f_1,\ldots, f_n$.

The usefulness of the latter approximation is that it is easy to apply, since it is sufficient to know estimates for the parameters a_{pq}, b_{pq} appearing in the credibility factors z_p.

In this article we try to demonstrate what kind of data is needed to apply semi-linear credibility theory.

The purpose followed in this paper is to get better semi-linear credibility results, using a unique optimal approximating function f, instead of considering prescribed approximating functions $f_1,\ldots, f_n$.

These performances lead to easily computable premiums and so, with usefulness in practice.

We give a rather explicit description of the input data for the semi-linear credibility model used, only to show that in practical situations, there will always be enough data to apply semi-linear credibility theory to a real insurance portfolio.

This paper shows that the mathematical theory is really a useful tool-perhaps the only existing tool-for the study of semi-linear credibility models.

So the mathematical properties of conditional expectations and conditional covariances become useful in the more complicated credibility models.

The fact that it is based on complicated mathematics involving variational calculus, needs not bother the user more than it does when he applies statistical tools like discriminatory analysis, scoring models, GLIM and SAS.

These techniques can be applied by anybody on his own field of endeavor, be it economics, medicine, or insurance.

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