ON SOME OPTIMAL INEQUALITIES FOR STATISTICAL SUBMANIFOLDS OF STATISTICAL SPACE FORMS

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In [J. Nonlin. Sc. Appl. 10(2019)], Lee et al. proved some optimal inequalities involving the normalized scalar curvature and normalized δ-Casorati curvatures of submanifolds in a statistical manifold of constant curvature, later generalized by Bansal et al in [Balk. J. Geom. Appl. 24(2019)] to the case of generalized normalized δ-Casorati curvatures. In this paper, we will improve all these inequalities, by considering a more natural curvature tensor field in statistical setting, called the statistical curvature tensor, originally introduced by Opozda in [Ann. Glob. Anal. Geom. 48(2015)]. We also investigate the equality cases of the derived inequalities and give an example.

Keywords: statistical manifold; dual connections; statistical curvature tensor field; generalized normalized δ-Casorati curvature; normalized scalar curvature.

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1. Introduction

In 1890, in his seminal work [5], Casorati introduced the concept of Casorati curvature for surfaces in 3-dimensional Euclidean space as an alternative to the classical Gaussian curvature. Casorati's main motivation was that the new concept corresponds better to human intuition on the notion of curvature, because the Gaussian curvature vanishes for surfaces that intuitively did not look flat. The notion was later extended for hypersurfaces and then for submanifolds of the Riemann manifolds [13]. The Casorati curvature - an extrinsic invariant, extends the concept of principal direction of a hypersurface of a Riemann manifold in case of submanifolds of Riemannian manifolds. In [8], S. Decu, S. Haesen and L. Verstraelen introduced the δ–Casorati normalized curvatures and established some optimal inequalities for these new curvature invariants. These kind of inequalities use both extrinsic notions (such as δ-Casorati curvatures) and intrinsic notions (like scalar curvature). Recent results on this topic were obtained in [2, 14, 15, 16, 21, 25, 27].

On the other hand, in 1985, the notion of statistical manifold arises for the first time in Amari's paper [1] from the need of generalizing the concept of statistical model to statistical manifold. From now on, numerous and valuable studies that adapt the general theory of geometric structures on manifolds to statistical manifolds have been written (see, e.g., [3, 7, 9, 11, 18, 22, 24, 26]). In the spirit of Casorati inequalities established in [8], Lee et. al prove in [17] some similar results in case of statistical manifolds, showing that the normalized scalar curvature is bounded by the Casorati curvatures of submanifolds in a statistical manifold of constant curvature.

In this paper we improve the inequalities established in [17], by considering a curvature tensor which is more natural in a statistical context, namely the statistical curvature tensor introduced by Opozda in [19]. This new tensor was introduced because it has all the

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symmetries that are necessary to a curvature tensor of type $(0,4)$, unlike the classical Riemannian curvature tensor, which in a statistical setting no longer has all these symmetries [19].

The article is divided in four sections, as follows. After the present Introduction, in Section 2 we recall notations, basic definitions and results used throughout the paper. In Section 3 we establish the main results, respectively two optimal inequalities for δ-Casorati normalized curvature of statistical submanifolds in statistical manifolds of constant curvature, known under the name of statistical space forms. In the last section, we will present an example to demonstrate that the equality case can be achieved and we will obtain a characterization of those submanifolds for which the equality case is reached.

2. **Preliminary Facts**

Let us briefly recall some basic facts about terminology in the setting of statistical manifolds (cf. [9, 10, 20]).

Let $(\overline{M},\bar{g})$ be an m-dimensional Riemann manifold with an affine connection $\overline{\nabla}$. Let $\bar{T}$ be the torsion tensor field of type $(1,2)$ of $\bar{\nabla}$.

Definition 2.1. *A pair* $(\overline{\nabla},\bar{g})$ *is called* a statistical structure *on* $\bar{M}$ *if*
(1) $(\overline{\nabla}_X\bar{g})(Y,Z)-(\overline{\nabla}_Y\bar{g})(X,Z)=\bar{g}(\bar{T}(X,Y),Z)$, *for all vector fields* X,Y *and* Z *on* $\bar{M}$*;*
(2) $\bar{T}=0$.

Definition 2.2. *A statistical manifold* $(\overline{M},\bar{g},\overline{\nabla})$ *is a Riemannian manifold, endowed with a pair of torsion free affine connections* $\bar{\nabla}$ *and* $\bar{\nabla}^*$ *such that*

$$Z\bar{g}(X,Y)=\bar{g}(\overline{\nabla}_Z X,Y)+\bar{g}(X,\overline{\nabla}^*_Z Y),\forall X,Y,Z\in\Gamma(T\overline{M}).$$

The connections $\overline{\nabla}$ *and* $\overline{\nabla}^*$ *are called* dual (conjugate) connections*.*

Remark 2.1. It is known that:
(1) $\left(\overline{\nabla}^*\right)^*=\overline{\nabla}$.
(2) If $(\overline{\nabla},\bar{g})$ is a statistical structure, then$(\overline{\nabla}^*,\bar{g})$ is also a statistical structure.
(3) In terms of the Levi-Civita connection $\overline{\nabla}^\circ$, the affine connection $\overline{\nabla}$ has always a dual connection $\overline{\nabla}^*$ satisfying
$$\overline{\nabla}+\overline{\nabla}^*=2\overline{\nabla}^\circ. \tag{1}$$

Let $\overline{R}$ and $\overline{R}^*$ be the curvature tensor fields of $\overline{\nabla}$, respectively $\overline{\nabla}^*$.

Definition 2.3. *A statistical structure* $(\overline{\nabla},g)$ *is said to be of* constant curvature $c\in\mathbb{R}$ *if*

$$\overline{R}(X,Y)Z=c\{\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\},\forall X,Y,Z\in\Gamma(T\overline{M}).$$

Remark 2.2. By a simple computation we deduce that

$$\overline{g}\left(\overline{R}^*(X,Y)Z,W\right)=-\overline{g}\left(Z,\overline{R}(X,Y)Z,W\right).$$

Using the above equation, we deduce that if $(\overline{\nabla},g)$ is of constant curvature c, then so is $(\overline{\nabla}^*,\overline{g})$.

Definition 2.4. *With the notations above, we define* $S\in\Gamma(TM)$ *be* the statistical curvature tensor *as:*

$$S(X,Y)Z=\frac{1}{2}\{R(X,Y)Z+R^*(X,Y)Z\}.$$

Moreover, $(\overline{M},\bar{g},\overline{\nabla})$ is said to be a statistical space form if S has the following expression:

$$S(X,Y)Z=c\{\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\},$$

for all vector fields X, Y, Z on $\overline{M}$, where c is a real constant. Such a space is denoted by $\overline{M}(c)$.

Definition 2.5. *Suppose M^n is a statistical submanifold of dimension n of a statistical manifold N^m of dimension m. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of tangent space T_pM, where $p \in M$, and let $\{e_{n+1}, e_{n+2}, \dots, e_m\}$ be an orthonormal basis of the normal space $T_p^{\perp}M$. Then* the scalar curvature τ at p *is given by:*

$$\tau(p) = \sum_{1 \le i < j \le n} g(S(e_i, e_j)e_j, e_i).$$

It is known that

$$g(S(X,Y)W,Z) = g(S(Z,W)Y,X)$$

and the Gauss equation is

$$\begin{aligned} 2\overline{g}(\overline{S}(X,Y)Z,W) &= 2g(S(X,Y)Z,W) \\ &+\overline{g}(h(X,Z),h^*(Y,W)) + \overline{g}(h^*(X,Z) - h(Y,W)) \\ &-\overline{g}(h^*(X,W),h(Y,Z)) - \overline{g}(h(X,W),h^*(Y,Z)) \end{aligned}$$

with $X, Y, Z, W \in \Gamma(TM)$, where h and h^* stand for the imbedding curvature tensors with respect to the dual connections (see, e.g., [6]). On the other hand, we have the *normalized scalar curvature* ρ defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Let H and H^* be the mean curvature vector fields:

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i).$$

From equation (1) we deduce that:

$$2H^\circ = H + H^*,$$

where H° denotes the mean curvature field of M defined through the second fundamental form h° with respect to the Levi-Civita connection ∇° on M. If $h^\circ = 0$, then the submanifold is called totally geodesic.

On the other hand, it is well-known that the squared mean curvature of the submanifold M in $\overline{M}$ is computed by:

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2,$$

where:

$$h_{ij}^{\alpha} = \bar{g}(h(e_i, e_j), e_\alpha),$$

for $i, j \in \{1, \dots, n+1\}, \alpha \in \{n+1, \dots, m\}$.

Moreover, the following properties hold:

$$\begin{gathered} g(S(X,Y)W,Z) = g(S(Z,W)Y,X) \\ g(S(W,Z)Y,X) = -g(S(Z,W)Y,X) \\ g(S(Z,W)X,Y) = -g(S(Z,W)Y,X) \\ S(Z,W)Y + S(W,Y)Z + S(Y,Z)W = 0. \end{gathered}$$

The Casorati curvatures of the submanifold M in $\overline{M}$ are denoted by $\mathcal{C}$ and $\mathcal{C}^*$ and given as:

$$\mathcal{C} = \frac{1}{n}\|h\|^2 = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2,$$

$$\mathcal{C}^* = \frac{1}{n}\|h^*\|^2 = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{*\alpha}\right)^2.$$

Let L be a s-dimensional subspace for $T_pM, s \geq 2$ and $\{e_i, \dots, e_s\}$ be an orthonormal basis of L. Then the Casorati curvature of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{s}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{s}\left(h_{ij}^{\alpha}\right)^2$$

and the normalized δ–Casorati curvatures are given by [9]

$$\delta_{\mathcal{C}}(m-1)|_p = \frac{1}{2}\mathcal{C}|_p + \frac{m+1}{2m}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}_{\mathcal{C}}(n-1)|_p = 2\mathcal{C}|_p - \frac{2n-1}{2n}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

We define analogically the dual normalized δ^*-Casorati curvatures:

$$\delta^*_{\mathcal{C}}(n-1)|_p = \frac{1}{2}\mathcal{C}|_p + \frac{n+1}{2n}\inf\{\mathcal{C}^*(L)|\ L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}^*_{\mathcal{C}}(n-1)|_p = 2\mathcal{C}^*|_p - \frac{2n-1}{2n}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}$$

We can also consider the generalized normalized $\delta-$ Casorati curvatures $\delta_c(r, n-1)$ and $\delta^*_c(r, n-1)$ of statistical submanifold M^m defined by

$$\delta_{\mathcal{C}}(r, n-1)|_p = r\mathcal{C}|_p - \frac{(n-1)(n-r)(n^2-n-r)}{rn}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}^*_{\mathcal{C}}(r, n-1)|_p = r\mathcal{C}^*|_p - \frac{(n-1)(n-r)(n^2-n-r)}{rn}\inf\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < r < n(n-1)$.

Moreover, if $r > n(n-1)$, then the generalized normalized $\delta-$Casorati curvatures $\hat{\delta}_{\mathcal{C}}(r, n-1)$ and $\hat{\delta}^*_{\mathcal{C}}(r, n-1)$ are defined by

$$\hat{\delta}_{\mathcal{C}}(r, n-1)|_p = r\mathcal{C}|_p - \frac{(n-1)(n-r)(n^2-n-r)}{rn}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\hat{\delta}^*_{\mathcal{C}}(r, n-1)|_p = r\mathcal{C}^*|_p - \frac{(n-1)(n-r)(n^2-n-r)}{rn}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}.$$

3. **Main Inequalities**

This section is dedicated to establish the main results of this paper, namely, for obtaining an improvement of the inequalities stated in [17] and [4].

In order to achieve the results, we will use the next result.

Lemma 3.1. [23] *Let* $\Gamma = \left\{\left(x^1, \dots, x^n\right) \in \mathbb{R}^n \middle| x^1 + \dots + x^n = k\right\}$ *be a hyperplane in* $\mathbb{R}^n$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *be a quadratic form given by*

$$f\left(x^1, \dots, x^n\right) = a\sum_{i=1}^{n-1}\left(x^i\right)^2 + b\left(x^n\right)^2 - 2\sum_{1\leq i<j\leq n} x^i j^j,$$

where $a > 0, b > 0$.

Then the conditioned extreme problem:

$$\min_{(x^1,\dots,x^n)\in\Gamma} f$$

has a global solution:

$$\begin{cases} x^1 = x^2 = \ldots = x^{n-1} = \frac{k}{a+1} \\ x^n = \frac{k}{b+1} = \frac{n-1}{b}\left(\frac{k}{a+1}\right) = (a-n+2)\frac{k}{a+1} \end{cases}$$

provided that

$$b = \frac{n-1}{a-n+2}.$$

Theorem 3.1. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then:*

i. The normalized δ-Casorati curvatures $\delta_C(r,n-1)$ and $\delta_C^*(r,n-1)$ satisfy

$$\rho \leq \frac{2\delta_C^{\circ}(r,n-1)}{n(n-1)} + \frac{\mathcal{C}}{2(n-1)} - \frac{4n}{n-1}\left\|H^{\circ}\right\|^2 + \frac{2n}{n-1}g(H,H^*) + c, \tag{2}$$

where

$$2\delta_C^0(r,n-1) = \delta_C(r,n-1) + \delta_C^*(r,n-1)$$

and

$$2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*.$$

ii. The normalized δ-Casorati curvatures $\hat{\delta}_C(r,n-1)$ and $\hat{\delta}_C^*(r,n-1)$ satisfy

$$\rho \leq \frac{2\hat{\delta}_C^{\circ}(r,n-1)}{n(n-1)} + \frac{\mathcal{C}}{2(n-1)} - \frac{4n}{n-1}\left\|H^{\circ}\right\|^2 + \frac{2n}{n-1}g(H,H^*) + c, \tag{3}$$

where

$$2\hat{\delta}_C^0(r,n-1) = \hat{\delta}_C(r,n-1) + \hat{\delta}_C^*(r,n-1).$$

Proof. Using the Gauss equation we obtain:

$$\begin{aligned} 2\bar{g}(\bar{S}(X,Y)Z,W) &= 2g(S(X,Y)Z,W) + \bar{g}\left(h(X,Z),h^*(Y,W)\right) \\ &+ \bar{g}\left(h^*(X,Z),h(Y,W)\right) - \bar{g}\left(h(X,W),h(Y,Z)\right) - \\ &\bar{g}\left(h(X,W),h^*(Y,Z)\right), X,Y,Z,W \in \Gamma(TM). \end{aligned} \tag{4}$$

Replacing

$$X = e_i, \quad Y = e_j, \quad Z = e_j, \quad W = e_i$$

in (4) and using the symmetry of the curvature tensor, we obtain by summation over $1 \leq i < j \leq n$:

$$2\tau = n(n-1)c + 2n^2 g(H,H^*) - 2\sum_{1\leq i<j\leq n}\bar{g}(h(e_i,e_j),h^*(e_i,e_j)). \tag{5}$$

But

$$g\left(H+H^*, H+H^0\right) = \|H\|^2 + \left\|H\right\|^* + 2g\left(H,H^*\right) \tag{6}$$

and inserting (6) in (5) we get

$$\begin{aligned} n(n-1)c &= 2\tau - 4n^2\|H\|^2 \\ &+ n^2\left(\|H\|^2 + \left\|H^*\right\|^2 + 2\sum_{1\leq i<j\leq n}\bar{g}(h(e_i,e_j),h^*(e_i,e_j))\right) \end{aligned} \tag{7}$$

In the following, we are going to compute the last sum in (7) in terms of Casorati curvatures $\mathcal{C}$ and $\mathcal{C}^*$. In order to obtain the desired equation, we will use the relation between h and h^*.

We know that:

$$2\bar{\nabla}^{\circ} = \bar{\nabla} + \bar{\nabla}^*. \tag{8}$$

But

$$\begin{cases} \bar{\nabla}^{\circ}_X Y = \nabla^{\circ}_X Y + h^{\circ}(X,Y) \\ \bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \\ \bar{\nabla}_X Y^* = \nabla^*_X Y + h^*(X,Y) \end{cases} \tag{9}$$

We replace (9) in (8) and we obtain that:

$$2\nabla_X Y + 2h^{\circ}(X,Y) = \nabla_X Y + h(X,Y) + \nabla^*_X Y + h^*(X,Y),\ \forall X,Y \in X(M)$$

By identifying the normal components in the above relation we find:

$$2h^{\circ}(X,Y) = h(X,Y) + h^*(X,Y).$$

hence we have:

$$2h^{\circ} = h + h^*. \tag{10}$$

On the other hand, it is obvious that

$$\bar{g}\left(\sum_{\alpha} h^{\alpha}_{ij} e_{\alpha}, \sum_{\beta} h^{*\beta}_{ij} e_{\beta}\right) = \sum_{\alpha}\sum_{\beta} h^{\alpha}_{ij} h^{*\beta}_{ij} g(e_{\alpha}, e_{\beta})$$

and therefore we derive

$$\bar{g}\left(h(e_i,e_j), h^*(e_i,e_j)\right) = \sum_{\alpha} h^{\alpha}_{ij} h^{*\alpha}_{ij}.$$

We know:

$$\mathcal{C}^0 = \frac{1}{n}\sum_{\alpha}\sum_{i,j}\left(h^{\circ\alpha}_{ij}\right)^2$$

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha}\sum_{i,j}\left(h^{\alpha}_{ij}\right)^2$$

$$\mathcal{C}^* = \frac{1}{n}\sum_{\alpha}\sum_{i,j}\left(h^{*\alpha}_{ij}\right)^2$$

Due to the fact that

$$2h^0 = h + h^*$$

we deduce

$$2h^{\circ\alpha}_{ij} = h^{\alpha}_{ij} + h^{*\alpha}_{ij},$$

which implies

$$4\left(h^{\circ\alpha}_{ij}\right)^2 = \left(h^{\alpha}_{ij}\right)^2 + \left(h^{*\alpha}_{ij}\right)^2 + 2h^{\alpha}_{ij}h^{*\alpha}_{ij}.$$

So, we have:

$$4\sum_{\alpha}\sum_{i,j}\left(h^{\circ\alpha}_{ij}\right)^2 = \sum_{\alpha}\sum_{i,j}\left(h^{\alpha}_{ij}\right)^2 + \sum_{\alpha}\sum_{i,j}\left(h^{*\alpha}_{ij}\right)^2 + 2\sum_{\alpha}\sum_{i,j}h^{\alpha}_{ij}h^{*\alpha}_{ij}$$

In this way:

$$\begin{aligned} 2\sum_{i,j} g\left(h(e_i,e_j), h^*(e_i,e_j)\right) &= 2\sum_{\alpha}\sum_{i,j} h^{\alpha}_{ij}h^{*\alpha}_{ij} \\ &= 4\sum_{\alpha}\sum_{i,j}\left(h^{*\alpha}_{ij}\right)^2 - \sum_{\alpha}\sum_{i,j}\left(h^{\alpha}_{ij}\right)^2 - \sum_{\alpha}\sum_{i,j}\left(h^{*\alpha}_{ij}\right)^2 \\ &= 4n\mathcal{C}^0 - n\mathcal{C} - n\mathcal{C}^* \end{aligned}$$

Hence we derive:

$$2\sum_{1\le i<j\le n} \bar{g}(h(e_i,e_j), h^*(e_i,e_j)) = 4n\mathcal{C}^{\circ} - n\mathcal{C} - n\mathcal{C}^*$$

and inserting the above equation in (5) we arrive at

$$n(n-1)c = 2\tau - 4n^2\left\|H\|^{\circ}\right\|^2 + n^2\left(\|H\|^2 + \left\|H^*\right\|^2 + 4n\mathcal{C}^0 - n(\mathcal{C}+\mathcal{C}^*)\right). \quad (11)$$

We define at this moment the polynomial

$$\begin{aligned} P &= 2r\mathcal{C}^{\circ} + \frac{2(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}^{\circ}(L) \\ &\quad -2T - n^2\left(\|H\|^2 + \left\|H^*\right\|^2\right) \\ &\quad +n(\mathcal{C}+\mathcal{C}^*) + n(n-1)c. \end{aligned}$$

Using relation (11), we obtain:

$$\begin{aligned} P &= \sum_{\alpha}\Big[\frac{2r}{n}\sum_{i,j=1}^{n}\left(h_{ij}^{0\alpha}\right)^2 + \sum_{i,j=1}^{n-1}\frac{2(n+r)(n^2-n-r)}{rn}\left(h_{ij}^{0\alpha}\right)^2 + \sum_{i,j=1}^{n}\left[\left(h_{ij}^{\alpha}\right)^2 + \left(h_{ij}^{*\alpha}\right)^2\right] \\ &\quad -4n^2\left\|H^*\right\|^2 + 4n\mathcal{C}^0 - n\left(\mathcal{C}+\mathcal{C}^*\right)\Big] \\ &= \sum_{\alpha}\Big[\left(\frac{2(n-1)(n+r)}{r} - 1\right)\sum_{i=1}^{n-1}\left(h_{ii}^{0\alpha}\right)^2 \\ &\quad + \frac{4(n-1)(n+r)}{r}\sum_{i<j}^{n-1}\left(h_{ij}^{0\alpha}\right)^2 + 4\left(\frac{r}{n}+1\right)\sum_{i=1}^{n-1}\left(h_{in}^{0\alpha}\right)^2 \\ &\quad + \frac{4r}{n}\left(h_{nn}^{0\alpha}\right)^2 - 4\sum_{i,j} h_{ii}^{0\alpha}h_{jj}^{0\alpha}\Big]. \end{aligned}$$

Thus, we have the following inequality

$$\begin{aligned} P &\geq \sum_{\alpha}\Big[\left(\frac{2(n-1)(n+r)}{r} - 1\right)\sum_{i=1}^{n-1}\left(h_{ii}^{0\alpha}\right)^2 \\ &\quad + \frac{4r}{n}\left(h_{nn}^{\alpha}\right)^2 - 4\sum_{i,j} h_{ij}^{0\alpha}h_{jj}^{0\alpha}\Big]. \end{aligned} \quad (12)$$

Hence, it follows from (12) that

$$P \geq \sum_{\alpha} P_{\alpha},$$

where

$$P_{\alpha} = \left(\frac{2(n-1)(n+r)}{r} - 1\right)\sum_{i=1}^{n-1}\left(h_{ii}^{0\alpha}\right)^2 + \frac{4r}{n}\left(h_{nn}^{0\alpha}\right)^2 - 4\sum_{i,j} h_{ij}^{0\alpha}h_{jj}^{0\alpha}.$$

Using Lemma 3.1 for the conditioned extreme problem:

$$\min_{(h_{11}^{0\alpha},\ldots,h_{nn}^{0\alpha})\in\Gamma_{\alpha}} \frac{P_{\alpha}}{2},$$

where

$$\Gamma_{\alpha} = \{(h_{11}^{0\alpha},\ldots,h_{nn}^{0\alpha}) | h_{11}^{0\alpha} + \ldots + h_{nn}^{0\alpha} = k^{\alpha}\},$$

for a constant k^{α}, one obtains the global solution:

$$h_{11}^{0\alpha} = \ldots = h_{n-1,n-1}^{0\alpha} = \frac{2k^{\alpha}r}{2(n-1)(n+r)-r},\ h_{nn}^{0\alpha} = \frac{k^{\alpha}n}{2r+n}.$$

But a direct computation shows that

$$P_{\alpha}\left(\frac{2k^{\alpha}r}{2(n-1)(n+r)-r},\ldots,\frac{2k^{\alpha}r}{2(n-1)(n+r)-r},\frac{k^{\alpha}n}{2r+n}\right) = 0.$$

Hence, it turns out that

$$P \geq \sum_{\alpha} P_{\alpha}\left(\frac{2k^{\alpha}r}{2(n-1)(n+r)-r},\dots,\frac{2k^{\alpha}r}{2(n-1)(n+r)-r},\frac{k^{\alpha}n}{2r+n}\right)=0.$$

Now, $P \geq 0$ leads to:

$$\begin{aligned}2\tau \quad &\leq \quad 2r\mathcal{C}^0+\frac{2(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}^{\circ}(L)\\ &\quad -n^2\left(\|H\|^2+\|H^*\|^2\right)+n\left(\mathcal{C}+\mathcal{C}^*\right)+n(n-1)c\end{aligned}$$

so we have:

$$\begin{aligned}\frac{2\delta}{n(n-1)} \quad &\leq \quad \frac{2r}{n(n-1)}\mathcal{C}^0+\frac{2(n+r)(n^2-n-r)}{rn^2}\mathcal{C}^0(L)\\ &\quad +\frac{1}{n-1}(\mathcal{C}+\mathcal{C}^*)-\frac{n}{n-1}\left(\|H\|^2+\|H^*\|^2\right)+c\end{aligned}$$

In view of

$$\mathcal{C}+\mathcal{C}^*=2\mathcal{C}^{\circ},$$

the above equation implies that:

$$\begin{aligned}\rho \quad &\leq \quad \frac{2}{n(n-1)}\left[r\mathcal{C}^0+A(r,n-1)\mathcal{C}^{\circ}(L)\right]\\ &\quad +\frac{1}{2(n-1)}\mathcal{C}^0-\frac{n}{n-1}\left(\|H\|^2+\|H^*\|^2\right)+c,\end{aligned}$$

where

$$A(r,n-1)=\frac{(n-1)(n-r)[n^2-n-r]}{rn}.$$

But we have that

$$\|H\|^2+\|H^*\|=4\left\|H^{\circ}\right\|^2-2\bar{g}(H,H^*)$$

and hence we get the inequality

$$\begin{aligned}\rho \quad &\leq \quad \frac{2}{n(n-1)}\left[r\mathcal{C}^{\circ}+A(r,n-1)+\mathcal{C}^{\circ}(L)\right]\\ &\quad +\frac{1}{2(n-1)}\mathcal{C}^{\circ}-\frac{4n}{n-1}\left\|H^{\circ}\right\|^2+\frac{4n}{n-1}\bar{g}(H,H^*)+c.\end{aligned}$$

Passing to the infimum over all hyperplanes, we derive:

$$\rho \leq \frac{2\delta_c^{\circ}(r,n-1)}{n(n-1)}+\frac{\mathcal{C}^0}{2(n-1)}-\frac{4n}{n-1}\left\|H^{\circ}\right\|^2+\frac{2n}{n-1}g(H,H^*)+c,$$

which is nothing but (2). Similarly, by passing to the supremum, we obtain (3). □

Remark 3.1. As an immediate consequence of the above theorem, we obtain the following result:

Corollary 3.1. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then:*

i. The normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\delta_C^*(n-1)$ satisfy:

$$\rho \leq 2\delta_C^{\circ}(n-1)+\frac{\mathcal{C}}{2(n-1)}-\frac{4n}{n-1}\left\|H^{\circ}\right\|^2+\frac{2n}{n-1}g(H,H^*)+c. \tag{13}$$

where

$$2\delta_C^0(n-1)=\delta_C(n-1)+\delta_C^*(n-1).$$

ii.The normalized δ-Casorati curvatures $\hat{\delta}_C(n-1)$ and $\hat{\delta}^*_C(n-1)$ satisfy:

$$\rho \leq 2\hat{\delta}^{\circ}_C(n-1) + \frac{\mathcal{C}}{2(n-1)} - \frac{4n}{n-1}\left\|H^{\circ}\right\|^2 + \frac{2n}{n-1}g(H,H^*) + c. \tag{14}$$

where

$$2\hat{\delta}^0_C(n-1) = \hat{\delta}_C(n-1) + \hat{\delta}^*_C(n-1).$$

Proof. Replacing $r = \frac{n(n-1)}{2}$ in (2) and $r = n(n-1)$ in (3) we obtain (13) and (14) because

$$\delta_C(\frac{n(n-1)}{2}, n-1) = n(n-1)\delta_C(n-1),$$

$$\delta^*_C(\frac{n(n-1)}{2}, n-1) = n(n-1)\delta^*_C(n-1),$$

$$\hat{\delta}_C(2n(n-1), n-1) = n(n-1)\hat{\delta}_C(n-1),$$

$$\hat{\delta}^*_C(2n(n-1), n-1) = n(n-1)\hat{\delta}^*_C(n-1).$$

□

Remark 3.2. We note that Theorem 3.1 generalizes [4, Theorem 1.1], while the previous Corollary generalizes [17, Theorem 1.1], by considering the statistical curvature tensor instead of the classical Riemannian curvature tensor.

4. **Investigation of the equality cases**

In the following, we are going to elucidate the geometry of the submanifolds that achieve equality in the inequalities established in the previous section. First, we prove the next result.

Theorem 4.1. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then the case of equality of any of the inequalities* (2) *and* (3) *occurs at a point $x \in M$ if and only if the imbedding curvature tensors h and h^* are related at x by $h^* = -h$.*

Proof. Suppose first that the equality of (2) is satisfied at $x \in M$. Then a straightforward computation shows that the critical point h^c of the polynomial P defined in the proof of Theorem 3.1 has all components null, that is $h^{c\alpha}_{ij} = 0$, for $\alpha = n+1, ..., m$ and $i, j = 1, ..., n$. But as $P \geq 0$ and $P(h^c) = 0$, we deduce that h^c is in fact a minimum point for P. Taking into account that $2h^0 = h + h^*$, we get that at $x \in M$, the imbedding tensors h^* and h are related by $h^* = -h$. Conversely, if $h^* = -h$ at a point $x \in M$, then $h^0 = 0$ at x and the inequality (2) is trivially satisfied with equality sign. Similarly, we obtain the same necessary and sufficient condition for the equality case of (3). □

Corollary 4.1. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then the case of equality of any of the inequalities* (2) *and* (3) *occurs identically at any point $x \in M$ if and only if M is a totally geodesic submanifold of $\overline{M}^m(c)$ with respect to the Levi-Civita connection of the metric $\bar{g}$.*

Proof. Suppose that the equality case of (2) or (3) holds identically on M. Then the above theorem implies $h^* = -h$ and equation (10) leads to $h^{\circ} = 0$. Hence, M is a totally geodesic submanifold of $\overline{M}^m(c)$.

Conversely, if M is a totally geodesic submanifold of $\overline{M}^m(c)$, then (2) and (3) are trivially satisfied with equality. □

Applying Theorem 4.1 and Corollary 4.1 in the particular case of the normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\delta^*_C(n-1)$, respectively $\hat{\delta}_C(n-1)$ and $\hat{\delta}^*_C(n-1)$, we derive the next consequences.

Corollary 4.2. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then the case of equality of any of the inequalities* (13) *and* (14) *occurs at a point $x \in M$ if and only if the imbedding curvature tensors h and h^* are related at x by $h^* = -h$.*

Corollary 4.3. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then the case of equality of any of the inequalities* (13) *and* (14) *occurs identically at any point $x \in M$ if and only if M is a totally geodesic submanifold of $\overline{M}^m(c)$ with respect to the Levi-Civita connection of the metric $\bar{g}$.*

Example 4.1. Consider the upper half-space:

$$\mathbb{H}^{m+1} = \{(x_1, ..., x_{m+1}) \in \mathbb{R}^{m+1} | x_{m+1} > 0\}$$

equipped with the natural metric

$$\bar{g} = \frac{1}{(x_{m+1})^2} \sum_{i=1}^{m+1} (dx_i)^2.$$

We take the affine connection $\bar{\nabla}$ given by

$$\bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0,\ 1 \leq i \neq j \leq m,$$

$$\bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_{m+1}} = \bar{\nabla}_{\frac{\partial}{\partial x_{m+1}}} \frac{\partial}{\partial x_i} = 0,\ 1 \leq i \leq m,$$

$$\bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \frac{2}{x_{m+1}} \frac{\partial}{\partial x_{m+1}},\ 1 \leq i \leq m,$$

$$\bar{\nabla}_{\frac{\partial}{\partial x_{m+1}}} \frac{\partial}{\partial x_{n+1}} = \frac{1}{x_{m+1}} \frac{\partial}{\partial x_{m+1}}.$$

It is known that $(\bar{\nabla}, \bar{g})$ is a statistical structure of constant curvature 0 on $\mathbb{H}^{m+1}$ (see [12]). Then in view of Remark 2.2 it follows that $(\bar{\nabla}^*, \bar{g})$ is also a statistical structure of constant curvature 0 on $\mathbb{H}^{m+1}$. Therefore we conclude that $\mathbb{H}^{m+1}$ is a statistical space form of constant curvature 0. We consider now an immersion $i : \mathbb{R}^n \to \mathbb{H}^{m+1}$ defined by

$$i(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0),$$

where $n \leq m$. Then it is easy to see that i is a totally geodesic immersion providing a natural example of statistical submanifold satisfying the equality cases of all inequalities stated above, namely (2), (3), (13) and (14).

5. **Conclusions**

In this work, we have generalized some optimal inequalities obtained in [4, 17] concerning the normalized scalar curvature and (generalized) normalized δ-Casorati curvatures of statistical submanifolds in a statistical space form, by considering the statistical curvature tensor instead of the Riemannian curvature tensor, a more natural curvature tensor in statistical setting as it was suggested by Opozda in [19]. Moreover, we have proved that the equality cases of these inequalities hold identically only for totally geodesic submanifolds. We also provided an example to illustrate the main results.

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