ANALYSIS OF THE BILATERAL MULTIVALUED CONTRACTIONS AND ASSOCIATED RESULTS

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The notions of bilateral multivalued contractions, introduced by Yahaya et al. [Fixed points of bilateral multivalued contractions, Filomat, 38(2024), 2835-2846], expanded the concept of bilateral contractions from singlevalued to multivalued mappings. This article presents an enhanced version of these notions and derives new results on fixed points for such mappings. We have suggested an improvement to make the notions simple to understand and also reduced the assertions imposed there on the fixed point results of bilateral contractions.

Keywords: generalized contraction, bilateral multivalued contraction, metric space, Hausdorff distance.

MSC2020: 47H10, 46N20.

1. Introduction and Preliminaries

In the study of functional analysis and applied mathematics, the Picard-Banach-Caccioppoli fixed-point theorem [1, 2] serves as a foundational result, asserting the existence and uniqueness of solutions to the equation $x = Tx$ under specific conditions on the mapping T. This celebrated contraction principle, central to metric fixed-point theory, guarantees both existence and uniqueness of fixed points, making it a powerful tool with applications across differential equations, optimization, and mathematical modeling. Over the decades, researchers have extended this result by either relaxing the contractive conditions or by broadening the ambient space, primarily through modifications to the triangle inequality.

At the heart of many such advancements lie generalized contraction-type inequalities, which characterize fixed points across diverse mathematical structures. These inequalities surpass traditional contraction conditions by incorporating combinations of distance measures between elements. Notable among these are the contraction conditions introduced by Kannan [7, 8], Chatterjea [3], and Ćirić-Reich-Rus [11]. Approaches by Jaggi [6] and Dass-Gupta [5] have also provided unique perspectives, enriching fixed-point theory with new techniques.

In recent years, the concept of bilateral contractions has deepened this domain. Chen *et al.* [4] introduced bilateral contraction inequalities, including Jaggi-type and Dass-Gupta-type bilateral contractions, establishing foundational fixed-point

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results. Building upon these ideas, Taş [13, 14] extended the concept, introducing Jaggi-type bilateral x_0-contractive mappings and Dass-Gupta-type bilateral x_0-contractive mappings, broadening the scope of bilateral contraction mappings to encompass both metric spaces and S-metric spaces, while Karapınar *et al.* [9] developed Ćirić-Caristi-type bilateral contractions by combining Ćirić's generalizations with Caristi's approach. Roy and Saha [12] further expanded these concepts into extended b-metric spaces with interpolative-Caristi-type contractive mappings.

More recently, Yahaya *et al.* [15] extended bilateral contractions to multivalued mappings, enabling fixed-point analysis for set-valued operators under bilateral contraction constraints. This extension is particularly significant, as multivalued mappings naturally arise in applications where solutions are not single-valued. Their work opens new avenues in fixed-point theory, offering fertile ground for both theoretical exploration and applied methodologies in disciplines reliant on fixed-point results.

Before mentioning the findings of Yahaya *et al.* [15], we recall some required notions and notations.

The collection of all nonempty, closed and bounded subsets of Z is denoted by $CB(Z)$, and the Hausdorff distance on $CB(Z)$, induced by the metric ρ, by H_ρ,

$$H_\rho(C, D) = \max\{\sup_{c\in C} \rho(c, D), \sup_{d\in D} \rho(d, C)\}$$

where $\rho(d, C) = \inf\{\rho(d, c) : c \in C\}$.

Throughout this paper, $\mathbb{N}$ denotes the set of all strictly positive integers, i.e. $\mathbb{N} = \{1, 2, 3, ...\}$.

Note that, if (Z, ρ) is a metric space, for single-valued mappings $\zeta\colon Z \to Z$, a point $z^* \in Z$ is called a fixed point if $z^* = \zeta z^*$. Extending this idea to the context of multivalued mappings, where $\zeta\colon Z \to CB(Z)$ assigns to each point $z \in Z$ a set $\zeta z \subset Z$, a fixed point is defined as follows.

Definition 1.1. ([10]) *Let $\zeta\colon Z \to CB(Z)$ be a multivalued mapping. A point $z^* \in Z$ is called a fixed point of ζ if $z^* \in \zeta z^*$.*

The following lemma is required for the main results. It follows from basic principles in metric space theory, and its proof is therefore omitted for brevity.

Lemma 1.1. *Let (Z, ρ) be a metric space. Let C, $D \in CB(Z)$ and $q > 1$. Then for each $c \in C$ with $\rho(c, D) > 0$, there exists $l \in D$ such that $0 < \rho(c, l) \le q\rho(c, D)$.*

Yahaya *et al.* [15] presented the following notions of bilateral multivalued contractions:

Definition 1.2. *Let (Z, ρ) be a metric space. A mapping $\zeta\colon Z \to CB(Z)$ is said to be a Jaggi-type bilateral multivalued contraction provided that there exists a function $\theta\colon Z \to [0, \infty)$ such that $\rho(z, \zeta z) > 0$ implies*

$$H_\rho(\zeta z, \zeta y) \le (\theta(z) - \theta(\zeta z)) \max\Big\{\rho(z, y), \frac{\rho(z, \zeta z)\rho(y, \zeta y)}{1 + \rho(z, y)}\Big\},$$

for each z, $y \in Z$ with $z \ne y$.

Definition 1.3. *Let (Z, ρ) be a metric space. A mapping $\zeta\colon Z \to CB(Z)$ is said to be a Dass-Gupta-type bilateral multivalued contraction if there exists a function*

$\theta\colon Z \to [0,\infty)$ such that $\rho(z,\zeta z) > 0$ implies

$$H_\rho(\zeta z,\zeta y) \le (\theta(z) - \theta(\zeta z)) \max\left\{\rho(z,y), \frac{(1+\rho(z,\zeta z))\rho(y,\zeta y)}{1+\rho(z,y)}\right\},$$

for each $z,\, y \in Z$.

Definition 1.4. *Let (Z,ρ) be a metric space. A mapping $\zeta\colon Z \to CB(Z)$ is said to be a Ćirić-Caristi-type bilateral multivalued contraction if there exists a function $\theta\colon Z \to [0,\infty)$ such that $\rho(z,\zeta z) > 0$ implies*

$$H_\rho(\zeta z,\zeta y) \le (\theta(z) - \theta(\zeta z)) \max\left\{\rho(z,y), \rho(z,\zeta z), \rho(y,\zeta y), \rho(z,\zeta y), \rho(y,\zeta z)\right\},$$

for each $z,\, y \in Z$.

Yahaya *et al.* [15] proved the following results to ensure the existence of fixed points for the above defined notions:

Theorem 1.1. *Let (Z,ρ) be a complete metric space, and let $\zeta\colon Z \to CB(Z)$ be a Jaggi-type bilateral multivalued contraction so that there exists $k = \sup\{\theta(z) - \theta(\zeta z)\colon \rho(z,y) > 0\}$. Then, there exists a point $z^* \in Z$ such that $z^* \in \zeta z^*$.*

Theorem 1.2. *Let (Z,ρ) be a complete metric space, and let $\zeta\colon Z \to CB(Z)$ be a Dass-Gupta-type bilateral multivalued contraction satisfying the condition that there exists $k = \sup\{\theta(z) - \theta(\zeta z)\colon \rho(z,y) > 0\}$. Then, there exists a point $z^* \in Z$ such that $z^* \in \zeta z^*$.*

Theorem 1.3. *Let (Z,ρ) be a complete metric space, and let $\zeta\colon Z \to CB(Z)$ be a Ćirić-Caristi-type bilateral multivalued contraction satisfying the condition that there exists $k = \sup\{\theta(z) - \theta(\zeta z)\colon \rho(z,y) > 0\}$. Then, there exists a point $z^* \in Z$ such that $z^* \in \zeta z^*$.*

2. Main results

2.1. Discussion on bilateral multivalued contractions and associated results

In the definitions of bilateral multivalued contractions introduced by Yahaya *et al.* [15], the mapping $\zeta\colon Z \to CB(Z)$ is defined as a multivalued function, with $\theta\colon Z \to [0,\infty)$ as a real-valued auxiliary function. Consequently, the expression $\theta(\zeta z)$ represents a set of real numbers, specifically defined as $\theta(\zeta z) = \{\theta(a) : a \in \zeta z\}$. Within this framework, $\theta(z) - \theta(\zeta z)$ may yield multiple values, creating a scenario where inequalities are formulated with a real number on one side and a set of real values on the other. Therefore, it can be represented as $\{\theta(z) - \theta(a) : a \in \zeta z\}$. In this situation, Yahaya et al. [15], studied the inequalities involving a real number on one side and a set of real numbers on the other side. In light of this, we propose a modification to the framework of bilateral multivalued contraction mappings to enhance clarity. Specifically, we suggest redefining these mappings as follows:

- A mapping $\zeta\colon Z \to CB(Z)$ is called a Dass-Gupta-type bilateral multivalued contraction if there exists a function $\theta\colon Z \to [0,\infty)$ such that $\rho(z,\zeta z) > 0$ implies

$$H_\rho(\zeta z,\zeta y) \le (\theta(z) - \theta(a)) \max\left\{\rho(z,y), \frac{(1+\rho(z,\zeta z))\rho(y,\zeta y)}{1+\rho(z,y)}\right\}, \text{ for all } a \in \zeta z, \quad (1)$$

for each $z,\, y \in Z$.

- A mapping $\zeta\colon Z \to CB(Z)$ is called a Jaggi-type bilateral multivalued contraction if there exists a function $\theta\colon Z \to [0,\infty)$ such that $\rho(z,\zeta z) > 0$ implies

$$H_\rho(\zeta z,\zeta y) \le (\theta(z)-\theta(a)) \max\left\{\rho(z,y), \frac{\rho(z,\zeta z)\rho(y,\zeta y)}{1+\rho(z,y)}\right\}, \text{ for all } a \in \zeta z, \tag{2}$$

for each $z,\, y \in Z$.

- A mapping $\zeta\colon Z \to CB(Z)$ is called a Ćirić-Caristi-type bilateral multivalued contraction if there exists a function $\theta\colon Z \to [0,\infty)$ such that $\rho(z,\zeta z) > 0$ implies

$$\begin{aligned} H_\rho(\zeta z,\zeta y) \;\; &\le \;\; (\theta(z)-\theta(a)) \max\Big\{\rho(z,y), \rho(z,\zeta z), \\ &\qquad \rho(y,\zeta y), \rho(z,\zeta y), \rho(y,\zeta z)\Big\} \text{ for all } a \in \zeta z \end{aligned} \tag{3}$$

for each $z,\, y \in Z$.

Upon reflecting on the proofs of Theorem 1.1 [15, Theorem 1], Theorem 1.2 [15, Theorem 2], and Theorem 1.3 [15, Theorem 3], we observed that their approach relied on an additional condition to establish the fixed point results. Specifically, they assumed the existence of a constant $q > 1$ such that $qk < 1$. Consequently, the original statements of these theorems are based on the following assumptions for each of the three classes of mappings:

- C-1: there exists $k = \sup\{\theta(z) - \theta(\zeta z) : \rho(z,y) > 0\}$;
- C-2: there exists $q > 1$ such that $qk < 1$.

It appears that, in Condition C-1, the requirement $\rho(z,y) > 0$ may be unnecessary, as y does not factor into the expression $\theta(z) - \theta(\zeta z)$.

2.2. Improved fixed point results for bilateral multivalued contractions

In this subsection, we will prove the existence of a fixed point for bilateral multivalued contractions without the assumptions C-1 and C-2.

Theorem 2.1. *Let (Z,ρ) be a complete metric space, and let $\zeta\colon Z \to CB(Z)$ be a Dass-Gupta-type bilateral multivalued contraction. Then, there exists a point $z^* \in Z$ such that $z^* \in \zeta z^*$.*

Proof. We start the proof with an arbitrary $z_0 \in Z$. As $\zeta z_0 \in CB(Z)$, we know that $\zeta z_0 \neq \emptyset$, thus we can take an element $z_1 \in \zeta z_0$. If $z_1 = z_0$, then we reach a fixed point of ζ. Suppose that $\rho(z_0,\zeta z_0) > 0$. Then, by (1), for all $a \in \zeta z_0$, we get

$$H_\rho(\zeta z_0,\zeta z_1) \le (\theta(z_0)-\theta(a)) \max\left\{\rho(z_0,z_1), \frac{(1+\rho(z_0,\zeta z_0))\rho(z_1,\zeta z_1)}{1+\rho(z_0,z_1)}\right\}.$$

As $z_1 \in \zeta z_0$, then the above inequality implies that

$$\begin{aligned} \rho(z_1,\zeta z_1) \;\; &\le \;\; \sup_{x\in\zeta z_0} \rho(x,\zeta z_1) \le H_\rho(\zeta z_0,\zeta z_1) \\ &\le \;\; (\theta(z_0)-\theta(z_1)) \max\left\{\rho(z_0,z_1), \frac{(1+\rho(z_0,\zeta z_0))\rho(z_1,\zeta z_1)}{1+\rho(z_0,z_1)}\right\}. \end{aligned} \tag{4}$$

If $z_1 \in \zeta z_1$, then z_1 is a fixed point of ζ, and we get a conclusion of the theorem. Thus, assume that $\rho(z_1, \zeta z_1) > 0$. By Lemma 1.1, for any fixed $q > 1$, there exists $z_2 \in \zeta z_1$ such that $0 < \rho(z_1, z_2) \leq q\rho(z_1, \zeta z_1)$. Using (4) and the above facts, we get

$$\begin{aligned} 0 < \rho(z_1, z_2) &\leq q\rho(z_1, \zeta z_1) \\ &\leq q(\theta(z_0) - \theta(z_1)) \max\left\{\rho(z_0, z_1), \frac{(1 + \rho(z_0, \zeta z_0))\rho(z_1, \zeta z_1)}{1 + \rho(z_0, z_1)}\right\} \\ &\leq q(\theta(z_0) - \theta(z_1)) \max\left\{\rho(z_0, z_1), \frac{(1 + \rho(z_0, z_1))\rho(z_1, z_2)}{1 + \rho(z_0, z_1)}\right\} \\ &= q(\theta(z_0) - \theta(z_1)) \max\left\{\rho(z_0, z_1), \rho(z_1, z_2)\right\}. \end{aligned}$$

This implies

$$0 < \frac{\rho(z_1, z_2)}{q \max\{\rho(z_0, z_1), \rho(z_1, z_2)\}} \leq \theta(z_0) - \theta(z_1).$$

Hence, we get $\theta(z_1) < \theta(z_0)$.

As it has already been assumed that $\rho(z_1, \zeta z_1) > 0$, then, by relation (1), for all $a \in \zeta z_1$, we get that

$$H_\rho(\zeta z_1, \zeta z_2) \leq (\theta(z_1) - \theta(a)) \max\left\{\rho(z_1, z_2), \frac{(1 + \rho(z_1, \zeta z_1))\rho(z_2, \zeta z_2)}{1 + \rho(z_1, z_2)}\right\}.$$

Since $z_2 \in \zeta z_1$, the previous inequality implies that

$$\rho(z_2, \zeta z_2) \leq (\theta(z_1) - \theta(z_2)) \max\left\{\rho(z_1, z_2), \frac{(1 + \rho(z_1, \zeta z_1))\rho(z_2, \zeta z_2)}{1 + \rho(z_1, z_2)}\right\}. \quad (5)$$

If $z_2 \in \zeta z_2$, then z_2 is a fixed point of ζ, and we obtain the conclusion of the theorem. Thus, assume that $\rho(z_2, \zeta z_2) > 0$. For the assumed $q > 1$, there exists $z_3 \in \zeta z_2$ such that $0 < \rho(z_2, z_3) \leq q\rho(z_2, \zeta z_2)$. Using (5) and the above facts, we get

$$\begin{aligned} 0 < \rho(z_2, z_3) &\leq q\rho(z_2, \zeta z_2) \\ &\leq q(\theta(z_1) - \theta(z_2)) \max\left\{\rho(z_1, z_2), \frac{(1 + \rho(z_1, \zeta z_1))\rho(z_2, \zeta z_2)}{1 + \rho(z_1, z_2)}\right\} \\ &\leq q(\theta(z_1) - \theta(z_2)) \max\left\{\rho(z_1, z_2), \frac{(1 + \rho(z_1, z_2))\rho(z_2, z_3)}{1 + \rho(z_1, z_2)}\right\} \\ &= q(\theta(z_1) - \theta(z_2)) \max\left\{\rho(z_1, z_2), \rho(z_2, z_3)\right\}. \end{aligned}$$

This implies

$$0 < \frac{\rho(z_2, z_3)}{q \max\{\rho(z_1, z_2), \rho(z_2, z_3)\}} \leq \theta(z_1) - \theta(z_2).$$

Hence, we get $\theta(z_2) < \theta(z_1)$.

Working with the above methodology, we construct a sequence $\{z_n\}$ in Z having the following characteristics:
(i) $z_n \in \zeta z_{n-1}$, for all $n \in \mathbb{N}$;
(ii) $z_n \neq z_{n-1}$, for all $n \in \mathbb{N}$;
(iii)

$$0 < \frac{\rho(z_n, z_{n+1})}{q \max\{\rho(z_{n-1}, z_n), \rho(z_n, z_{n+1})\}} \leq \theta(z_{n-1}) - \theta(z_n), \text{ for all } n \in \mathbb{N};$$

(iv) $\theta(z_n) < \theta(z_{n-1})$, for all $n \in \mathbb{N}$.

As $\{\theta(z_n)\}$ is a decreasing sequence of positive real numbers, it will converge to a real number $l \geq 0$. By (iii), we have

$$0 < \frac{\rho(z_n, z_{n+1})}{q \max\{\rho(z_{n-1}, z_n), \rho(z_n, z_{n+1})\}} \quad \leq \quad \theta(z_{n-1}) - \theta(z_n), \text{ for all } n \in \mathbb{N}.$$

By letting $n \to \infty$, we obtain

$$\lim_{i \to \infty} \frac{\rho(z_i, z_{i+1})}{\max\{\rho(z_{i-1}, z_i), \rho(z_i, z_{i+1})\}} = 0.$$

The existence of the above limit implies that, for each $h \in (0, 1)$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{\rho(z_i, z_{i+1})}{\max\{\rho(z_{i-1}, z_i), \rho(z_i, z_{i+1})\}} \leq h, \text{ for all } i \geq n_0.$$

Thus, we get

$$\rho(z_i, z_{i+1}) \leq h \max\{\rho(z_{i-1}, z_i), \rho(z_i, z_{i+1})\} \text{ for all } i \geq n_0. \tag{6}$$

Here, we can claim that $\max\{\rho(z_{i-1}, z_i), \rho(z_i, z_{i+1})\} = \rho(z_{i-1}, z_i)$ for all $i \geq n_0$. On the contrary, assume that there is $k_0 \geq n_0$ such that

$$\max\{\rho(z_{k_0-1}, z_{k_0}), \rho(z_{k_0}, z_{k_0+1})\} = \rho(z_{k_0}, z_{k_0+1}).$$

Then, by (6), we get

$$\rho(z_{k_0}, z_{k_0+1}) \leq h\rho(z_{k_0}, z_{k_0+1}).$$

This implies $\rho(z_{k_0}, z_{k_0+1}) = 0$, that is $z_{k_0} = z_{k_0+1}$, which is a contradiction to (ii). Hence, our assumption is false.

Thus, we obtain

$$\rho(z_i, z_{i+1}) \leq h\rho(z_{i-1}, z_i), \text{ for all } i \geq n_0.$$

This inequality yields the following relation, by reverse substitution:

$$\rho(z_i, z_{i+1}) \leq h^{i-n_0+1} \rho(z_{n_0-1}, z_{n_0}), \text{ for all } i \geq n_0. \tag{7}$$

From (7), and by the triangle inequality, for each $p > r \geq n_0$, we get

$$\rho(z_r, z_p) \leq \sum_{j=r}^{p-1} \rho(z_j, z_{j+1}) \leq \sum_{j=r}^{\infty} h^{j-n_0+1} \rho(z_{n_0-1}, z_{n_0}).$$

Thus, $\lim_{r,p\to\infty} \rho(z_r, z_p) = 0$. Hence, $\{z_n\}$ is a Cauchy sequence in Z, so it also converges to an element z^* in Z, because (Z, ρ) is a complete metric space.

Next, we will show that $z^* \in \zeta z^*$. Suppose that $\rho(z^*, \zeta z^*) > 0$. Then, by (1), for each $n \geq 0$, we get, for all $a \in \zeta z^*$, that

$$\begin{aligned} H_\rho(\zeta z^*, \zeta z_n) &\leq (\theta(z^*) - \theta(a)) \max\left\{\rho(z^*, z_n), \frac{(1 + \rho(z^*, \zeta z^*))\rho(z_n, \zeta z_n)}{1 + \rho(z^*, z_n)}\right\} \\ &\leq (\theta(z^*) - \theta(a)) \max\left\{\rho(z^*, z_n), \frac{(1 + \rho(z^*, \zeta z^*))\rho(z_n, z_{n+1})}{1 + \rho(z^*, z_n)}\right\}. \end{aligned}$$

By letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} H_\rho(\zeta z^*, \zeta z_n) = 0$. Hence, we conclude that $z^* \in \zeta z^*$. □

In the next result, we will discuss the existence of a fixed point for Jaggi-type bilateral multivalued contraction.

Theorem 2.2. *Let* (Z,ρ) *be a complete metric space, and let* $\zeta\colon Z\to CB(Z)$ *be a Jaggi-type bilateral multivalued contraction. Then, there exists a point* $z^*\in Z$ *such that* $z^*\in\zeta z^*$.

Proof. If $\zeta\colon Z\to CB(Z)$ is a Jaggi-type bilateral multivalued contraction mapping, then, by (2), we obtain, for $\rho(z,\zeta z)>0$, that

$$\begin{aligned} H_\rho(\zeta z,\zeta y) &\le (\theta(z)-\theta(a))\max\Big\{\rho(z,y),\frac{\rho(z,\zeta z)\rho(y,\zeta y)}{1+\rho(z,y)}\Big\}\\ &\le (\theta(z)-\theta(a))\max\Big\{\rho(z,y),\frac{(1+\rho(z,\zeta z))\rho(y,\zeta y)}{1+\rho(z,y)}\Big\},\ \text{for all}\, a\in\zeta z,\end{aligned}$$

for each z, $y\in Z$. Hence, $\zeta\colon Z\to CB(Z)$ is a Dass-Gupta-type bilateral multivalued contraction mapping. Therefore, by Theorem 2.1, there exists a point $a\in Z$ such that $a\in\zeta a$. □

The result given below studies the existence of a fixed point for Ćirić-Caristi-type bilateral multivalued contraction.

Theorem 2.3. *Let* (Z,ρ) *be a complete metric space, and let* $\zeta\colon Z\to CB(Z)$ *be a Ćirić-Caristi-type bilateral multivalued contraction. Then, there exists a point* $z^*\in Z$ *such that* $z^*\in\zeta z^*$.

Proof. For any $z_0\in Z$, the set ζz_0 is not void. Thus, we have at least one point $z_1\in\zeta z_0$. If $z_1=z_0$, then z_0 is a fixed point of ζ. To proceed further with the proof, we assume that $\rho(z_0,\zeta z_0)>0$. Then, by relation (3), we are led, for any $a\in\zeta z_0$, to

$$\begin{aligned} H_\rho(\zeta z_0,\zeta z_1) \le\ & (\theta(z_0)-\theta(a))\max\Big\{\rho(z_0,z_1),\rho(z_0,\zeta z_0),\\ & \rho(z_1,\zeta z_1),\rho(z_0,\zeta z_1),\rho(z_1,\zeta z_0)\Big\}.\end{aligned}$$

As $z_1\in\zeta z_0$, then the previous inequality implies that

$$\begin{aligned} \rho(z_1,\zeta z_1) &\le \sup_{x\in\zeta z_0}\rho(x,\zeta z_1)\le H_\rho(\zeta z_0,\zeta z_1)\\ &\le (\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,\zeta z_1),\rho(z_0,\zeta z_1)\Big\}. \quad (8)\end{aligned}$$

If $z_1\in\zeta z_1$, then z_1 is a fixed point of ζ and the proof is completed. Assume now that $\rho(z_1,\zeta z_1)>0$. For any fixed $q>1$, there exists $z_2\in\zeta z_1$ such that $0<\rho(z_1,z_2)\le q\rho(z_1,\zeta z_1)$.

Using (8) and the above facts, we get

$$\begin{aligned} 0<\rho(z_1,z_2) &\le q\rho(z_1,\zeta z_1)\\ &\le q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,\zeta z_1),\rho(z_0,\zeta z_1)\Big\}\\ &\le q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,z_2),\rho(z_0,z_2)\Big\}\\ &\le q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,z_2),\rho(z_0,z_1)+\rho(z_1,z_2)\Big\}\\ &\le 2q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,z_2)\Big\}.\end{aligned}$$

This implies

$$0 < \frac{\rho(z_1, z_2)}{2q \max\{\rho(z_0, z_1), \rho(z_1, z_2)\}} \leq \theta(z_0) - \theta(z_1).$$

Hence, we get $\theta(z_1) < \theta(z_0)$. By repeating the above steps, we can obtain a sequence $\{z_n\}$ in Z with the following properties:
(i) $z_n \in \zeta z_{n-1}$, for all $n \in \mathbb{N}$;
(ii) $z_n \neq z_{n-1}$, for all $n \in \mathbb{N}$;
(iii)

$$0 < \frac{\rho(z_n, z_{n+1})}{2q \max\{\rho(z_{n-1}, z_n), \rho(z_n, z_{n+1})\}} \leq \theta(z_{n-1}) - \theta(z_n), \text{ for all } n \in \mathbb{N};$$

(iv) $\theta(z_n) < \theta(z_{n-1})$, for all $n \in \mathbb{N}$.

As in the proof of the Theorem 2.1, we say that $\{z_n\}$ is Cauchy and convergent to an element z^* in Z.

We now prove that $z^* \in \zeta z^*$. As $\rho(z_n, \zeta z_n) > 0$ for all $n \geq 0$, by (3), for each $n \geq 0$ and for any $a \in \zeta z_n$, we get

$$\begin{aligned} H_\rho(\zeta z_n, \zeta z^*) \leq (\theta(z_n) - \theta(a)) \max\Big\{&\rho(z_n, z^*), \rho(z_n, \zeta z_n), \\ &\rho(z^*, \zeta z^*), \rho(z_n, \zeta z^*), \rho(z^*, \zeta z_n)\Big\}. \end{aligned} \tag{9}$$

Inequality (9) also yields that

$$\begin{aligned} H_\rho(\zeta z_n, \zeta z^*) &\leq (\theta(z_n) - \theta(z_{n+1})) \max\Big\{\rho(z_n, z^*), \rho(z_n, z_{n+1}), \\ &\qquad \rho(z^*, \zeta z^*), \rho(z_n, \zeta z^*), \rho(z^*, z_{n+1})\Big\} \\ &\leq (\theta(z_n) - \theta(z_{n+1})) \max\Big\{\rho(z_n, z^*), \rho(z_n, z_{n+1}), \\ &\qquad \rho(z^*, \zeta z^*), \rho(z_n, z^*) + \rho(z^*, \zeta z^*), \rho(z^*, z_{n+1})\Big\}, \end{aligned}$$

for each $n \geq 0$.

Letting $n \to \infty$ in the last inequality, we get $\lim_{n\to\infty} H_\rho(\zeta z^*, \zeta z_n) = 0$, as $(\theta(z_n) - \theta(z_{n+1})) \to 0$, when $n \to \infty$. Hence, we conclude that $z^* \in \zeta z^*$. □

2.3. Almost bilateral multivalued contractions and associated fixed point theorems

This subsection deals with an extended form of bilateral multivalued contractions and related fixed point results. The following definition provides the concept of almost bilateral multivalued contraction mappings.

Definition 2.1. *Let (Z, ρ) be a metric space. Then:*

- *A mapping $\zeta\colon Z \to CB(Z)$ is called a Dass-Gupta-type almost bilateral multivalued contraction provided that there exists a function $\theta\colon Z \to [0, \infty)$ such that $\rho(z, \zeta z) > 0$ implies*

$$\begin{aligned} H_\rho(\zeta z, \zeta y) \leq (\theta(z) - \theta(a)) \max\left\{\rho(z, y), \frac{(1 + \rho(z, \zeta z))\rho(y, \zeta y)}{1 + \rho(z, y)}\right\} \\ +(\theta(z) + \theta(a))\rho(z, \zeta y)\rho(y, \zeta z), \text{ for all } a \in \zeta z, \end{aligned} \tag{10}$$

for each $z,\, y \in Z$.

- *A mapping* $\zeta\colon Z \to CB(Z)$ *is called a Jaggi-type almost bilateral multivalued contraction if there exists a function* $\theta\colon Z \to [0,\infty)$ *such that* $\rho(z,\zeta z) > 0$ *implies*
$$\begin{aligned} H_\rho(\zeta z,\zeta y) &\le (\theta(z)-\theta(a))\max\Big\{\rho(z,y), \frac{\rho(z,\zeta z)\rho(y,\zeta y)}{1+\rho(z,y)}\Big\} \\ &\qquad +(\theta(z)+\theta(a))\rho(z,\zeta y)\rho(y,\zeta z),\ \textit{for all}\ a\in\zeta z, \end{aligned}$$
for each $z,\, y \in Z$.
- *A mapping* $\zeta\colon Z \to CB(Z)$ *is called a Ćirić-Caristi-type almost bilateral multivalued contraction if there exists a function* $\theta\colon Z \to [0,\infty)$ *such that* $\rho(z,\zeta z) > 0$ *implies*
$$\begin{aligned} H_\rho(\zeta z,\zeta y) &\le (\theta(z)-\theta(a))\max\Big\{\rho(z,y),\rho(z,\zeta z),\rho(y,\zeta y),\rho(z,\zeta y),\rho(y,\zeta z)\Big\} \\ &\qquad +(\theta(z)+\theta(a))\rho(z,\zeta y)\rho(y,\zeta z),\ \textit{for all}\ a\in\zeta z \qquad (11) \end{aligned}$$
for each $z,\, y \in Z$.

The following result deals with Dass-Gupta-type almost bilateral multivalued contraction mappings.

Theorem 2.4. *Let* (Z,ρ) *be a complete metric space, and let* $\zeta\colon Z \to CB(Z)$ *be a Dass-Gupta-type almost bilateral multivalued contraction. Then, there exists a point* $z^* \in Z$ *such that* $z^* \in \zeta z^*$.

Proof. For an arbitrary $z_0 \in Z$, it is clear that ζz_0 is not empty. Then, we consider $z_1 \in \zeta z_0$. If $z_1 = z_0$, then z_0 is a fixed point of ζ. Suppose that $\rho(z_0,\zeta z_0) > 0$, then by (10), we get
$$\begin{aligned} H_\rho(\zeta z_0,\zeta z_1) &\le (\theta(z_0)-\theta(a))\max\Big\{\rho(z_0,z_1), \frac{(1+\rho(z_0,\zeta z_0))\rho(z_1,\zeta z_1)}{1+\rho(z_0,z_1)}\Big\} \\ &\qquad +(\theta(z_0)+\theta(a))\rho(z_0,\zeta z_1)\rho(z_1,\zeta z_0),\ \text{for all}\, a\in\zeta z_0. \qquad (12) \end{aligned}$$

As $z_1 \in \zeta z_0$, then (12) implies that
$$\begin{aligned} \rho(z_1,\zeta z_1) &\le (\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1), \frac{(1+\rho(z_0,\zeta z_0))\rho(z_1,\zeta z_1)}{1+\rho(z_0,z_1)}\Big\} \\ &\qquad +(\theta(z_0)+\theta(z_1))\rho(z_0,\zeta z_1)\rho(z_1,\zeta z_0) \\ &\le (\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1), \frac{(1+\rho(z_0,z_1))\rho(z_1,\zeta z_1)}{1+\rho(z_0,z_1)}\Big\} \\ &= (\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,\zeta z_1)\Big\}. \qquad (13) \end{aligned}$$

If $z_1 \in \zeta z_1$, then z_1 is a fixed point of ζ. To continue the proof, we assume that $\rho(z_1,\zeta z_1) > 0$. For any fixed $q > 1$, we say that there exists $z_2 \in \zeta z_1$ such that $0 < \rho(z_1,z_2) \le q\rho(z_1,\zeta z_1)$. Using (13) and the above facts, we get
$$\begin{aligned} 0<\rho(z_1,z_2) &\le q\rho(z_1,\zeta z_1) \\ &\le q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,\zeta z_1)\Big\} \\ &\le q(\theta(z_0)-\theta(z_1))\max\Big\{\rho(z_0,z_1),\rho(z_1,z_2)\Big\}. \end{aligned}$$

This implies

$$0 < \frac{\rho(z_1, z_2)}{q \max\{\rho(z_0, z_1), \rho(z_1, z_2)\}} \leq \theta(z_0) - \theta(z_1).$$

Hence, we obtain $\theta(z_1) < \theta(z_0)$.

The above mentioned proof technique yields a sequence $\{z_n\}$ in Z with the following conditions:
(i) $z_n \in \zeta z_{n-1}$, for all $n \in \mathbb{N}$;
(ii) $z_n \neq z_{n-1}$, for all $n \in \mathbb{N}$;
(iii)

$$0 < \frac{\rho(z_n, z_{n+1})}{q \max\{\rho(z_{n-1}, z_n), \rho(z_n, z_{n+1})\}} \leq \theta(z_{n-1}) - \theta(z_n), \text{ for all } n \in \mathbb{N};$$

(iv) $\theta(z_n) < \theta(z_{n-1})$, for all $n \in \mathbb{N}$.

From this stage, by following the same steps as in the proof of the Theorem 2.1, we can conclude that $\{z_n\}$ is a Cauchy sequence in Z. Now, the completeness of Z guarantees the existence of some z^* in Z such that $z_n \to z^*$ as $n \to \infty$.

Next, we will show that $z^* \in \zeta z^*$. Suppose that $\rho(z^*, \zeta z^*) > 0$. Then, by (10), for each $n \geq 0$ and for all $a \in \zeta z^*$, we get

$$\begin{aligned} H_\rho(\zeta z^*, \zeta z_n) &\leq (\theta(z^*) - \theta(a)) \max\left\{\rho(z^*, z_n), \frac{(1 + \rho(z^*, \zeta z^*))\rho(z_n, \zeta z_n)}{1 + \rho(z^*, z_n)}\right\} \\ &\quad + (\theta(z^*) + \theta(a))\rho(z^*, \zeta z_n)\rho(z_n, \zeta z^*) \\ &\leq (\theta(z^*) - \theta(a)) \max\left\{\rho(z^*, z_n), \frac{(1 + \rho(z^*, \zeta z^*))\rho(z_n, \zeta z_n)}{1 + \rho(z^*, z_n)}\right\} \\ &\quad + (\theta(z^*) + \theta(a))\rho(z^*, z_{n+1})[\rho(z_n, z^*) + \rho(z^*, \zeta z^*)]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} H_\rho(\zeta z^*, \zeta z_n) = 0$. Hence, we conclude that $z^* \in \zeta z^*$. □

The existence of a fixed point for Jaggi-type almost bilateral multivalued contraction mappings is now easily reached. The following result provides the existence of a fixed point for Ćirić-Caristi-type almost bilateral multivalued contraction mappings.

Theorem 2.5. *Let (Z, ρ) be a complete metric space, and let $\zeta\colon Z \to CB(Z)$ be a Ćirić-Caristi-type almost bilateral multivalued contraction. Then, there exists a point $z^* \in Z$ such that $z^* \in \zeta z^*$.*

Proof. For any $z_0 \in Z$, the set ζz_0 is not empty, and let $z_1 \in \zeta z_0$. Assume that $\rho(z_0, \zeta z_0) > 0$; otherwise, z_0 is a fixed point of ζ and the proof is completed. Thus, by (11), we get

$$\begin{aligned} H_\rho(\zeta z_0, \zeta z_1) &\leq (\theta(z_0) - \theta(a)) \max\Big\{\rho(z_0, z_1), \rho(z_0, \zeta z_0), \rho(z_1, \zeta z_1), \\ &\quad \rho(z_0, \zeta z_1), \rho(z_1, \zeta z_0)\Big\} \\ &\quad + (\theta(z_0) + \theta(a))\rho(z_0, \zeta z_1)\rho(z_1, \zeta z_0), \text{ for all } a \in \zeta z_0. \end{aligned} \tag{14}$$

As $z_1 \in \zeta z_0$, then (14) implies that

$$\rho(z_1, \zeta z_1) \leq (\theta(z_0) - \theta(z_1)) \max\Big\{\rho(z_0, z_1), \rho(z_1, \zeta z_1), \rho(z_0, \zeta z_1)\Big\}. \tag{15}$$

Again, assume that $\rho(z_1, \zeta z_1) > 0$; otherwise, $z_1 \in \zeta z_1$, and the proof is done. For any fixed $q > 1$, we get $z_2 \in \zeta z_1$ such that $0 < \rho(z_1, z_2) \leq q\rho(z_1, \zeta z_1)$. Using (15) and the above facts, we get

$$\begin{aligned} 0 < \rho(z_1, z_2) &\leq q\rho(z_1, \zeta z_1) \\ &\leq q(\theta(z_0) - \theta(z_1)) \max\Big\{\rho(z_0, z_1), \rho(z_1, \zeta z_1), \rho(z_0, \zeta z_1)\Big\} \\ &\leq q(\theta(z_0) - \theta(z_1)) \max\Big\{\rho(z_0, z_1), \rho(z_1, z_2), \rho(z_0, z_2)\Big\} \\ &\leq q(\theta(z_0) - \theta(z_1)) \max\Big\{\rho(z_0, z_1), \rho(z_1, z_2), \rho(z_0, z_1) + \rho(z_1, z_2)\Big\} \\ &\leq 2q(\theta(z_0) - \theta(z_1)) \max\Big\{\rho(z_0, z_1), \rho(z_1, z_2)\Big\}. \end{aligned}$$

This implies

$$0 < \frac{\rho(z_1, z_2)}{2q \max\{\rho(z_0, z_1), \rho(z_1, z_2)\}} \leq \theta(z_0) - \theta(z_1).$$

Hence, we get $\theta(z_1) < \theta(z_0)$. By repeating the above steps, we get a sequence $\{z_n\}$ in Z with the following properties:
(i) $z_n \in \zeta z_{n-1}$, for all $n \in \mathbb{N}$;
(ii) $z_n \neq z_{n-1}$, for all $n \in \mathbb{N}$;
(iii)

$$0 < \frac{\rho(z_n, z_{n+1})}{2q \max\{\rho(z_{n-1}, z_n), \rho(z_n, z_{n+1})\}} \leq \theta(z_{n-1}) - \theta(z_n), \text{ for all } n \in \mathbb{N};$$

(iv) $\theta(z_n) < \theta(z_{n-1})$, for all $n \in \mathbb{N}$.

By the above facts, it is easy to conclude that $\{z_n\}$ is Cauchy and convergent to an element z^* in Z.

As $\rho(z_n, \zeta z_n) > 0$ for all $n \geq 0$, by using (11), for each $n \geq 0$, we get

$$\begin{aligned} H_\rho(\zeta z_n, \zeta z^*) &\leq (\theta(z_n) - \theta(a)) \max\Big\{\rho(z_n, z^*), \rho(z_n, \zeta z_n), \rho(z^*, \zeta z^*), \\ &\qquad \rho(z_n, \zeta z^*), \rho(z^*, \zeta z_n)\Big\} \\ &\qquad + (\theta(z_n) + \theta(a))\rho(z_n, \zeta z^*)\rho(z^*, \zeta z_n), \text{ for all } a \in \zeta z_n. \end{aligned}$$

The above inequality also yields that

$$\begin{aligned} H_\rho(\zeta z_n, \zeta z^*) &\leq (\theta(z_n) - \theta(z_{n+1})) \max\Big\{\rho(z_n, z^*), \rho(z_n, z_{n+1}), \rho(z^*, \zeta z^*), \\ &\qquad \rho(z_n, \zeta z^*), \rho(z^*, z_{n+1})\Big\} \\ &\qquad + (\theta(z_n) + \theta(z_{n+1}))\rho(z_n, \zeta z^*)\rho(z^*, z_{n+1}), \text{ for all } n \geq 0. \end{aligned}$$

By taking the limit when $n \to \infty$ in the previous inequality, we get that $\lim_{n\to\infty} H_\rho(\zeta z^*, \zeta z_n) = 0$, since $(\theta(z_n) - \theta(z_{n+1})) \to 0$ as $n \to \infty$. Hence, we conclude that $z^* \in \zeta z^*$. □

3. **Conclusions**

This work presents a study on bilateral multivalued contractions. By refining the definitions and relaxing certain restrictive conditions, we established improved fixed-point results applicable to a wider class of mappings. These findings enhance the clarity and utility of existing frameworks while extending their relevance to broader developments in fixed-point theory.

Acknowledgement

Yahya Almalki extends his appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding his work through Large Research Project under grant number RGP2/430/45.

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