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DIVISIBLE GROUPS DERIVED FROM DIVISIBLE HYPERGROUPS

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The purpose of this paper is to define a new equivalence relation τ^* on divisible hypergroups and to show that this relation is the smallest strongly regular relation (the fundamental relation) on commutative divisible hypergroups. We show that $\tau^* \neq \beta^*$, $\tau^* \neq \gamma^*$ and, we define a divisible hypergroup on every nonempty set. We show that the quotient of a finite divisible hypergroup by τ^* is the trivial divisible group. Moreover, the concept of (self) fundamental divisible group is defined and it is shown that any divisible group is a self fundamental divisible group. Finally, we study complete parts in divisible hypergroups.

Keywords: Divisible (hyper)group, fundamental relation.

MSC2010: 20N20.

1. Introduction

The hyperalgebraic structure theory was firstly introduced in 1934 at the 8th congress of Scandinavian Mathematicians by F. Marty [11]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then, many researchers have worked in this new field of modern algebra and have developed it. Fundamental relations are one of the main tools in algebraic hyperstructures theory, which brings us into the classical algebra. The relation β (resp. the fundamental relation β^*) was introduced on hypergroups by Koskas [9] and was studied mainly by Corsini [4] and Vougiouklis [14]. Freni proved that in hypergroups the relation β is transitive [5]. Recently, Freni introduced the relation γ as a generalization of the relation β and proved that, in hypergroups, the relation β is transitive [6]. Davvaz et al. introduced the smallest equivalence relation ν^* on a hypergroup H such that the quotient $\frac{H}{\nu^*}$, the set of all equivalence classes, is a nilpotent group and they characterized nilpotent groups via strongly regular relations [1]. R. Ameri et al. introduced the smallest equivalence relation ξ^* on a given hypergroup G in a way that the quotient G/ξ^*, the set of all equivalence classes, is an Engel group [3]. Further materials regarding solvable polygroups, solvable groups, subpolygroups and hypergroups are available in the literature too [2, 8]. S. Pianskool et al. defined the concept of divisible hypergroups and investigated some properties of them [12, 13].

In this paper, first we construct divisible hypergroups on every nonempty set. A new strongly regular equivalence relation τ^* on divisible hypergroups is defined and it is shown that this relation is a fundamental relation on commutative divisible hypergroups. Using the concept of a fundamental group, we investigate some of its properties and show that the τ^* is different by β^* and γ^*. Finally, we prove that direct product of divisible hypergroups is a

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divisible hypergroup, too and show that the τ^* is transitive if and only if for all $x \in G, \tau^*(x)$ is a τ-part.

2. **Preliminaries**

In this section, we review some definitions and results from [14], which we need in what follows.

Let G be a nonempty set and $P^*(G)$ be the family of all nonempty subsets of G. Every function $\circ_i : G \times G \longrightarrow P^*(G)$ where $i \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}^*$ is called a *hyperoperation*. For all x, y of G, $\circ_i(x, y)$ is called the *hyperproduct* of x and y. An algebraic system $(G, \circ_1, \circ_2, \ldots, \circ_n)$ is called a *hyperstructure* and a binary structure $(G, \circ)$ endowed with only one hyperoperation is called a *hypergroupoid*. For any two nonempty subsets A and B of G, $A \circ B$ means $\bigcup\limits_{a\in A, b\in B} a \circ b$. Recall that a *hypergroupoid* $(G, \circ)$ is called a *semihypergroup* if for all $x, y, z \in G, (x \circ y) \circ z = x \circ (y \circ z)$ and a semihypergroup $(G, \circ)$ is a *hypergroup* if it satisfies in the *reproduction axiom*, i.e. for all $x \in G, x \circ G = G \circ x = G$. The map $f : G_1 \to G_2$ is called an *inclusion* homomorphism if for all $x, y \in G, f(x \circ y) \subseteq f(x) \circ f(y)$ and is called a *strong* (*good*) homomorphism or briefly a good homomorphism if for all $x, y \in G$, we have $f(x \circ y) = f(x) \circ f(y)$. Let $(G, \circ)$ be a hypergroup and ρ be an equivalence relation on G. For nonempty subsets A and B of G, we denote $A\overline{\overline{\rho}}B \Longleftrightarrow \forall a \in A, b \in B, a\rho b$. The relation ρ is called strongly regular on the left (on the right) if $x\rho y \Longrightarrow (a \circ x)\overline{\overline{\rho}}(a \circ y)$ ($x\rho y \Longrightarrow (x \circ a)\overline{\overline{\rho}}(y \circ a)$, respectively), where $x, y, a \in G$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left. Let $G/\rho = \{\rho(g) \mid g \in G\}$ be the set of all equivalence classes of G with respect to ρ. Define a hyperoperation $*$ on G/ρ as follows:

$$\rho(a) * \rho(b) = \{\rho(c) \mid c \in \rho(a) \circ \rho(b)\}.$$

Theorem 2.1. [4] *Let $(G, \circ)$ be a semihypergroup (hypergroup) and ρ be an equivalence relation on G. Then $(G/\rho, *)$ is a semigroup (group) if and if only ρ is strongly regular.*

The smallest equivalence relation ρ on G, such that $(G/\rho, *)$ is a group is called the *fundamental relation*. Let $\mathcal{U}(G)$ denote the set of all finite products of elements of G. Define relation β on G by

$$a\beta b \Longleftrightarrow \exists\ u \in \mathcal{U}(G) \text{ such that } \{a, b\} \in u.$$

Denote the *transitive closure* of β by β^*. The quotient structure $(\frac{G}{\beta^*}, *)$ is a group, called the *fundamental group* of $(G, \circ)$ [4]. We have:

$$a\beta^* b \Longleftrightarrow \exists z_1 = a, z_2, ..., z_n = b \in G, u_1, u_2.., u_n \in \mathcal{U}, : \ \{z_i, z_{i+1}\} \in u_i, \forall\ 1 \leq i \leq n.$$

Theorem 2.2. [4, 5] *Let $(G, \circ)$ be a hypergroup. Then*
(i) the relation β^* is strongly regular on G and so $(G/\beta^*, *)$ is a group;
(ii) $\beta^* = \beta$.

In [6], Freni introduced the relation $\gamma = \bigcup\limits_{n\geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \Longleftrightarrow \exists\ z_1, \ldots, z_n \in G,\ \exists\ \sigma \in S_n : x \in \prod_{i=}^{n} z_i \text{ and } y \in \prod_{i=}^{n} z_{\sigma(i)},$$

where S_n is the symmetric group of order n. Denote the transitive closure of γ by γ^*. The relation γ^* is a strongly regular relation [6] and γ^* is the least equivalence relation on a hypergroup G, such that the quotient $(G/\gamma^*, *)$ is an abelian group.

3. Divisible hypergroups

The concept of a divisible semihypergroup and a divisible hypergroup was firstly introduced by S. Pianskool, S. Chaopraknoi and Yupaporn Kemprasit in 2005, 2006 [12, 13].

Definition 3.1. [12, 13] *A semihypergroup* $(G, \circ)$ *is said to be divisible if for all* $x \in G$ *and every* $n \in \mathbb{N}^*, x \in \underbrace{y \circ y \circ \ldots \circ y}_{(n-times)}$, *for some* $y \in G$.

For every group G and subgroup H of G, they considered the hypergroups G/H and $G|H$ and showed that:

Theorem 3.1. [12] *If* G *is a divisible group, then both* G/H *and* $G|H$ *are divisible hypergroups.*

Moreover, for the next equivalence relation ρ on every abelian group G:

$$x\rho y \Longleftrightarrow x = y \text{ or } x = y^{-1},$$

they proved that

Theorem 3.2. [13] *If* G *is a divisible group, then* $(G/\rho, \circ)$ *is a divisible hypergroup.*

The concept of divisible elements in semihypergroups (hypergroups) were not introduced in [12, 13] and so, we define it as follows:

Definition 3.2. *Let* $(G, \circ)$ *be a semihypergroup (hypergroup) and* $x \in G$. *We say that* x *is divisible, if for any* $n \in \mathbb{N}^*$ *there exists* $y \in G$ *such that* $x \in y^n = \underbrace{y \circ y \circ \ldots \circ y}_{(n-times)}$.

In this section, for every non-empty set, we construct a divisible (semi) hypergroup and show that, for all $n \in \mathbb{N}$ there exists at least a divisible (semi)hypergroup such that its order is n. To construct infinite divisible (semi)hypergroups, we use divisible groups and applied homomorphisms.

Theorem 3.3. *Let* G *be a nonempty set. Then there exists a binary hyperoperation* $\circ$ *on* G, *such that* $(G, \circ)$ *is a divisible semihypergroup.*

Proof. If $|G| = 1$, the proof is clear. Let $|G| \geq 2$ and $e, f \in G$ be distinct. Now, for all $x, y \in G$ define a hyperoperation $\circ$ on G as follows:

$$x \circ y = \begin{cases} \{e, f\}, & \text{if } y = e, \\ \{y\}, & \text{otherwise.} \end{cases}$$

Associativity: Let $x, y, z \in G$. We consider the following cases:
Case 1: $x = y \neq z$. Then, $(x \circ y) \circ z = (x \circ x) \circ z = \{z\} = x \circ (x \circ z) = x \circ (y \circ z)$.
Case 2: $x = z \neq y$. Then, $(x \circ y) \circ z = (x \circ y) \circ x = \{e, f\} = x \circ (y \circ x)$.
Case 3: $y = z \neq x$. Then, for $y \neq e$, $(x \circ y) \circ z = (x \circ y) \circ x = \{y\} = x \circ (y \circ x)$ and for $y = e$, $(x \circ y) \circ z = (x \circ y) \circ x = \{e, f\} = x \circ (y \circ x)$.
Case 4: $x \neq y \neq z$. If $e \notin \{x, y, z\}$, then $(x \circ y) \circ z = y \circ z = \{z\} = x \circ z = x \circ (y \circ z)$ and if $e \in \{x, y, z\}$, then $(x \circ y) \circ z = y \circ z = \{e, f\} = x \circ z = x \circ (y \circ z)$.
Case 5: $x = y = z$. Then, $(x \circ y) \circ z = y \circ z = \{z\} = x \circ z = x \circ (y \circ z)$.

Now, let $x \in G$. Notice that $x \circ G = \bigcup_{y \in G} = (x \circ x) \cup \bigcup_{x \neq y \in G} = G$, but $G \circ x \neq G$. Hence $(G, \circ)$ is only a semihypergroup. Let $x \in G$. Then for all $n \in \mathbb{N}^*$ we have $x \in \underbrace{x \circ x \circ \ldots \circ x}_{(n-times)}$, whence it follows that $(G, \circ)$ is a divisible semihypergroup. $\square$

Corollary 3.1. *Let $n \in \mathbb{N}^*$. Then there exists at least a divisible semihypergroup $(G, \circ)$ such that $|G| = n$.*

Theorem 3.4. *Let G be a nonempty set. Then there exists a binary hyperoperation $\circ$ on G such that $(G, \circ)$ is a divisible hypergroup.*

Proof. Let $|G| \geq 1$ and $e \in G$. Now, for all $x, y \in G$ define a hyperoperation $\circ$ on G, as follows:

$$x \circ y = \begin{cases} \{e, x\}, & \text{if } x = y, \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

Associativity: Let $x, y, z \in G$. We consider the following cases:
Case 1: x, y, z are distinct. Then, $(x \circ y) \circ z = \{x, y\} \circ z = \{x, y, z\} = (x \circ z) \circ y$.
Case 2: $x = y \neq z$. Then, $(x \circ y) \circ z = \{x, e\} \circ z = \{x, e, z\} = (x \circ z) \circ y$.
Case 3: $x = z \neq y$. Then, $(x \circ y) \circ z = \{x, y\} \circ z = \{x, y, e\} = (x \circ z) \circ y$.
Case 4: $y = z \neq x$. Then, $(x \circ y) \circ z = \{x, y\} \circ z = \{x, y, e\} = (x \circ z) \circ y$.
Case 5: $x = y = z$. Then, $(x \circ y) \circ z = \{x, y\} \circ z = \{x, e\} = (x \circ z) \circ y$. Now, if $x \in G$, then $\bigcup\limits_{x \in G} (x \circ G) = \bigcup\limits_{x,y \in G} (x \circ y) = \bigcup\limits_{x,y \in G} \{x, y\} = G$.
Therefore $(G, \circ)$ is a hypergroup. Let $x \in G$. Then for all $n \in \mathbb{N}, x \in \underbrace{x \circ x \circ \ldots \circ x}_{(n-times)}$, whence it follows that $(G, \circ)$ is a divisible hypergroup. □

Corollary 3.2. *Let $n \in \mathbb{N}^*$. Then there exists at least a divisible hypergroup $(G, \circ)$ such that $|G| = n$.*

Example 3.1. *Let $G = \{e, a, b, c, d\}$. Define a hyperoperation $\circ$ on G as follows:*

$\circ$	e	a	b	c	d
e	$\{e\}$	$\{e, a\}$	$\{e, b\}$	$\{e, c\}$	$\{e, d\}$
a	$\{e, a\}$	$\{a, e\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	$\{e, b\}$	$\{a, b\}$	$\{e, b\}$	$\{b, c\}$	$\{b, d\}$
c	$\{e, c\}$	$\{c, a\}$	$\{b, c\}$	$\{e, c\}$	$\{c, d\}$
d	$\{e, d\}$	$\{a, d\}$	$\{b, d\}$	$\{c, d\}$	$\{e, d\}$

Then $(G, \circ)$ is a divisible hypergroup.

Lemma 3.1. *Let $(G, .)$ be a divisible group. Then for every group $(H, .)$, there exists a binary hyperoperation " $\circ$ " on $G \times H$, such that $(G \times H, \circ)$ is a divisible hypergroup.*

Proof. Let $(H, .)$ be a nonzero group. Define a hyperoperation " $\circ$ " on $G \times H$, as follows:

$$(g, h) \circ (g', h') = \{(g.g', h), (g.g', h')\}$$

Clearly $\circ$ is associative. We verify the reproduction axiom. Let $(g, h) \in (G \times H)$. Since $(g, h) \in (g, h) \circ (1, 1) = \{(g.1, h), (g.1, 1)\} = \{(g, h), (g, 1)\}$, then

$$\begin{aligned} (g, h) \circ (G \times H) = \bigcup_{(g', h') \in G \times H} (g, h) \circ (g', h') &= \bigcup_{(g', h') \in G \times H} \{(g.g', h), (g.g', h')\} \\ &= G \times H \end{aligned}$$

and similarly, it follows that $(G \times H) \circ (g, h) = G \times H$. Thus, $(G \times H, \circ)$ is a hypergroup. Let $(g, h) \in G \times H$ and $n \in \mathbb{N}^*$. Since G is a divisible group, there exists $y \in G$ such that $g = y^n$. Now $(g, h) \in \{(g, h)\} = \underbrace{(y, h) \circ (y, h) \circ \ldots \circ (y, h)}_{(n-times)}$. Therefore $(G \times H, \circ)$ is a divisible hypergroup. □

Remark 3.1. (i) *The divisible hypergroup* $(G \times H, \circ)$ *is called the associated divisible hypergroup to* G *via* H *(or shortly, the associated divisible hypergroup) and it is denoted by* G_H.
(ii) *The mapping* $\varphi : G \longrightarrow G_H$ *defined by* $\varphi(g) = (g, 1)$ *is an embedding.*
(iii) G_H *is a hypergroup with identity.*
(iv) *If* $H = \mathbb{Z}$, *we denote* G_H *by* $\overline{G}$.
(vi) *For* $H = \mathbb{Z}_2, G_H$ *is the smallest associated divisible hypergroup.*

Theorem 3.5. *Let* $(G_1, .)$ *and* $(G_2, .)$ *be isomorphic divisible groups. Then, for every group* $(H, .)$, G_{1_H} *and* G_{2_H} *are isomorphic divisible hypergroups.*

Proof. Let $f : (G_1, .) \longrightarrow (G_2, .)$ be an isomorphism. Define a map $\theta : (G_1 \times H, \circ) \longrightarrow (G_2 \times H, \circ)$ by $\theta(g, h) = (f(g), h)$ where $(g, h) \in G_1 \times H$. Clearly θ is a bijection, now we show that is a good homomorphism. Let $(g_1, h), (g_2, h') \in G_1 \times H$. Then

$$\begin{aligned}
\theta((g_1, h) \circ (g_2, h')) &= \theta(\{(g_1.g_2, h), (g_1.g_2, h')\}) \\
&= \{\theta(g_1.g_2, h), \theta(g_1.g_2, h')\} = \{(f(g_1.g_2), h), (f(g_1.g_2), h')\} \\
&= \{(f(g_1).f(g_2), h), (f(g_1).f(g_2), h')\} \\
&= (f(g_1), h) \circ (f(g_2), h')) \\
&= \theta((g_1, h)) \circ \theta((g_2, h'))
\end{aligned}$$

Therefore θ is an isomorphism and $(G_1 \times H, \circ) \cong (G_2 \times H, \circ)$. □

Example 3.2. $(\mathbb{Q} \times \mathbb{Z}_2, \circ)$ *is the smallest associated divisible hypergroup, by the following hyperoperation:*

$$(g, h) \circ (g', h') = \{(g + g', \overline{0}), (g + g', \overline{1})\}$$

Theorem 3.6. *Let* $(G, \circ)$, $(H, \circ')$ *be hypergroups,* $f : (G, \circ) \to (H, \circ')$ *be a good homomorphism,* $x, y \in G$ *and* $n \in \mathbb{N}^*$. *Then the following statements are satisfied:*
(i) *if* x *is a divisible element in* G, *then* $f(x)$ *is a divisible element in* H,
(ii) *if* f *is an onto and* G *is divisible, then* H *is a divisible hypergroup, too.*

Proof. (i) Since $x \in G$ is divisible, then for all $n \in \mathbb{N}^*$ there exists $y \in G$ such that $x \in y^n$ and so $f(x) \in f(y^n) = f(y)^n$. Thus $f(x)$ is a divisible element in H.
(ii) Let $y \in H$ and $n \in \mathbb{N}^*$. Then there exists $x \in G$ such that $y = f(x)$. Since G is divisible then x is divisible and by $(i), y$ is divisible in H and so H is a divisible hypergroup. □

4. New strongly regular equivalence relation τ^* on (divisible) hypergroups

In this section, we introduce a new equivalence relation on a (divisible) hypergroup, which we denote by τ^* . We prove that τ^* is the smallest strongly regular relation on a divisible hypergroup and the elements of the quotient group are divisible. Moreover, we show that τ^* is a fundamental relation on commutative divisible hypergroups. We give some examples for which $\tau^* \neq \beta^*$ and $\tau^* \neq \gamma^*$.

Definition 4.1. *Let* $(G; \circ)$ *be a hypergroup. Set* $\tau_1 = \{(x, x) \mid x \in G\}$ *and for every integer* $n \geq 2$, τ_n *is defined as follows:*

$$x \tau_n y \Longleftrightarrow \exists\ a_1, \ldots, a_n \in G \text{ and } \sigma \in S_n \text{ such that } \{x, y\} \subseteq \bigodot_{i=1}^{n} a_i,$$

and $a_i \in a^3_{\sigma(j)}$, *for some* $1 \leq i, j \leq n$, $i \neq \sigma(j)$.

Obviously for every $n \geq 1$ the relation τ_n is symmetric, so $\tau = \bigcup_{n\geq 1} \tau_n$ is a reflexive and symmetric relation. Let τ^* be the *transitive closure* of τ. In the following theorem we show that τ^* is a strongly regular relation.

Theorem 4.1. *Let $(G, \circ)$ be a hypergroup. Then τ^* is a strongly regular relation on G.*

Proof. Let $x, y \in G$ and $x\ \tau^* y$. Then there exist $a_0, a_1, \ldots, a_k \in G$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}^*$, such that $a_0 = x$, $a_k = y$ and

$$x = a_0\ \tau_{n_1}\ a_1\ \tau_{n_2}\ a_2 \tau_{n_3} \ldots \tau_{n_{k-2}} a_{k-2}\ \tau_{n_{k-1}}\ a_{k-1}\ \tau_{n_k}\ a_k = y,$$

where $k \in \mathbb{N}^*$. For all $1 \leq i \leq k$, $a_{i-1}\ \tau_{n_i}\ a_i$, there exist $z_{qr} \in G$ and $\sigma \in S_n$, such that $\{a_{i-1}, a_i\} \subseteq \bigodot_{l=1}^{n_i} z_{i-1l}$, where $a_i \in a^3_{\sigma(j)}$ and $z_{i-1m} \in z^3_{i-1\sigma(j)}$, for some $1 \leq i, j \leq n$, where $i \neq \sigma(j)$. Moreover for all $0 \leq q \leq k-1$ we have $1 \leq r \leq k$. Now, let $s \in G$. Then for all $1 \leq i \leq k$, $a_{i-1} \circ s \subseteq \bigodot_{l=1}^{n_i} z_{i-1l} \circ s$ and simillary $a_i \circ s \subseteq \bigodot_{l=1}^{n_i} z_{i-1l} \circ s$. Now for all $1 \leq i \leq k$ and for all $u \in a_{i-1} \circ s$, $v \in a_i \circ s$, we have $u\ \tau_{n_{i+1}}\ v$, and so for all $z \in a_0 \circ s = x \circ s$, $w \in a_n \circ s = y \circ s$, we have $z\ \tau^*\ w$. Then τ^* is a strongly right regular and similarly is a strongly left regular relation. Therefore, τ^* is a strongly regular relation. □

Corollary 4.1. *Let $(G, \circ)$ be a divisible hypergroup. Then τ^* is a strongly regular relation on G.*

Example 4.1. *Let $G = \{1, 2, 3\}$. Then $(G, \circ)$ is a hypergroup, which is not divisible.*

$\circ$	1	2	3
1	$\{1\}$	$\{2\}$	$\{3\}$
2	$\{2\}$	$\{1,3\}$	$\{2\}$
3	$\{3\}$	$\{2\}$	$\{1\}$

Clearly $\{1,3\} \subseteq 2\circ 2 \cup 1\circ 2\circ 2 \cup 3\circ 2\circ 2 \cup 1\circ 3\circ 2\circ 2$ *so* $1\beta 3$. *But* $1 \notin 2\circ 2\circ 2 \cup 3\circ 3\circ 3, 2 \notin 1\circ 1\circ 1 \cup 3\circ 3\circ 3$ *and* $3 \notin 1\circ 1\circ 1 \cup 2\circ 2\circ 2$, *whence* $(1,3) \notin \tau$ *and so* $\tau \neq \beta$.

Remark 4.1. *Let $(G, \circ)$ be a hypergroup. Then by Example 4.1, it follows that $\beta \neq \tau \neq \gamma$ and so $\beta^* \neq \tau^* \neq \gamma^*$.*

Lemma 4.1. *Let $(G, \circ)$ be a divisible hypergroup and $a, b \in G$. Then there exist $g, g' \in G$ such that $b \in g^3$ and so $a \circ b \subseteq g \circ g' \circ b$.*

Proof. Since $b \in G$ and G is a divisible hypergroup, there exists $g \in G$ such that $b \in g^3$. Now by the reproduction axiom, there exists $g' \in G$ such that $a \in g \circ g'$. Thus $a \circ b \subseteq g \circ g' \circ b$. □

Theorem 4.2. *Let $(G, \circ)$ be a hypergroup. Then*
(i) $\tau^* \subseteq \beta^*$;
(ii) if G is divisible, then $\tau^* = \beta^*$;
(iii) if G is commutative and divisible, then $\tau^* = \gamma^*$.

Proof. We prove only (ii), (iii) ; (i) follows immediately.
(ii) By (i), $\tau^* \subseteq \beta^*$. Let $x, y \in G$. Since G is divisible, by Lemma 4.1, $x\beta^* y$ implies that there exist $n \in \mathbb{N}^*$ and $a_1, a_2, \ldots, a_n \in G$ such that for some $1 \leq i \neq j \leq n$, $a_i \in a_j^3$ and $\{x, y\} \subseteq \bigodot_{i=1}^{n} a_i$. Thus $\tau^* \supseteq \beta^*$ and so $\tau^* = \beta^*$.
(iii) Since G is commutative, we have $\gamma^* = \beta^*$ and by (ii) we get that $\gamma^* = \tau^*$. □

Example 4.2. *Define a hyperoperation* $\circ$ *on* $\mathbb{Q} \times \mathbb{Z}$ *as follows:*

$$(g,h) \circ (g',h') = \{(g+g',h),(g+g',h')\}.$$

Clearly $(\mathbb{Q} \times \mathbb{Z}, \circ)$ *is an infinite divisible hypergroup. For any* $(m/n,k),(r/s,l) \in \mathbb{Q} \times \mathbb{Z}$ *we have*

$$\begin{aligned}(m/n,k)\beta^*(r/s,l) &\iff \exists\, u \in \mathcal{U}(\mathbb{Q}\times\mathbb{Z}) \text{ such that } \{(m/n,k),(r/s,l)\} \in u \\ &\iff m/n = r/s\end{aligned}$$

and so

$$\beta^*(m/n,k) = \{m/n\} \times \mathbb{Z}. \tag{1}$$

For $g,g' \in \mathbb{Q}$ *and* $h,h' \in \mathbb{Z}$ *we have*

$$(m/n,k) \in (g,h) \circ (g',h') \iff m/n = g+g' \text{ and } k \in \{h,h'\}.$$

Now if $(r/s,l) \in \tau^*(m/n,k)$ *then* $(r/s,l) \in (p/q,p) \circ (3p'/q',p')$ *and* $l \in \{p,p'\}$*, so* $r/s = (pq'+3p'q)/(qq')$*. It follows that*

$$\tau^*((m/n,k)) = \{m/n\} \times \mathbb{Z}. \tag{2}$$

Theorem 4.3. *Let* $(G,\circ)$ *be a divisible hypergroup. Then* $(G/\tau^*;*)$ *is a group, of which elements are all divisible.*

Proof. By Theorem 4.1, τ^* is a strongly regular equivalence relation and by Theorem 2.1, $(G/\tau^*,*)$ is a group. Let $x \in G$ and $n \in \mathbb{N}^*$. Since $(G,\circ)$ is a divisible hypergroup, there exists $y \in G$ such that $x \in y^n$ and so $\tau^*(x) = \tau^*(y^n) = (\tau^*(y))^n$. □

Theorem 4.4. *Let* $(G,\circ)$ *be a divisible hypergroup. Then* τ^* *is the smallest strongly regular equivalence relation on* G*, such that* G/τ^* *is a group.*

Proof. It follows by Theorem 4.2, (*ii*). □

Example 4.3. *Consider a divisible semihypergroup* $(G,\circ)$ *and its quotient on* τ^* *as follows:*

$\circ$	e	a	b	c	d
e	$\{e,a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
a	$\{e,a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{e,a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
c	$\{e,a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
d	$\{e,a\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$

and

$*$	$\tau^*(e)$	$\tau^*(b)$	$\tau^*(c)$	$\tau^*(d)$
$\tau^*(e)$	$\tau^*(e)$	$\tau^*(b)$	$\tau^*(c)$	$\tau^*(d)$
$\tau^*(b)$	$\tau^*(e)$	$\tau^*(b)$	$\tau^*(c)$	$\tau^*(d)$
$\tau^*(c)$	$\tau^*(e)$	$\tau^*(b)$	$\tau^*(c)$	$\tau^*(d)$
$\tau^*(d)$	$\tau^*(e)$	$\tau^*(b)$	$\tau^*(c)$	$\tau^*(d)$

.

It is easy to see that $(G/\tau^*,*)$ *is a semigroup such that all its elements are divisible.*

Clearly $(G,\circ)$ *is not commutative and for all* $x \neq y \in G$*, we have* $\tau^*(x) * \tau^*(y) \neq \tau^*(y) * \tau^*(x)$*.*

Corollary 4.2. *Let* $(G,\circ)$ *be a commutative divisible hypergroup. Then* $(G/\tau^*,*)$ *is a divisible group.*

Example 4.4. *Define a hyperoperation* $\circ$ *on* $\mathbb{Z}(2^\infty) \times \mathbb{Z}_2$ *as follows:*

$$(\overline{a/2^i},h) \circ (\overline{b/2^j},h') = \{(\overline{a(2^j)+b(2^i)}/2^{i+j},\overline{0}),(\overline{a(2^j)+b(2^i)}/2^{i+j},\overline{1})\} \text{ where } i,j \in \mathbb{N}.$$

Clearly $(\mathbb{Z}(2^\infty)\times\mathbb{Z}_2,\circ)$ *is a commutative divisible hypergroup and for all* $(\overline{a/2^i},h) \in \mathbb{Z}(2^\infty)\times\mathbb{Z}_2$ *we get that* $\tau^*((\overline{a/2^i},h)) = \{(\overline{a/2^i},\overline{0}),(\overline{a/2^i},\overline{1})\}$*. Now we define a map* $\theta : ((\mathbb{Z}(2^\infty)\times\mathbb{Z}_2,\circ)/\tau^*,*) \longrightarrow (\mathbb{Z}(2^\infty),+)$ *by* $\theta(\tau^*(\overline{a/2^i},h)) = \overline{a/2^i}$*.*

Clearly θ *is well-defined and an isomorphism, so* $((\mathbb{Z}(2^\infty)\times\mathbb{Z}_2,\circ)/\tau^*,*) \cong (\mathbb{Z}(2^\infty),+)$ *is a divisible group.*

Theorem 4.5. *Let* $(G, \circ)$, $(H, \circ')$ *be hypergroups,* $f : (G, \circ) \to (H, \circ')$ *be a homomorphism,* $x, y \in G$ *and* $n \in \mathbb{N}^*$. *Then*
(i) $x\tau^* y$ *implies that* $f(x)\tau^* f(y)$;
(ii) *if* $(G, \circ) \cong (H, \circ')$, *then* $G/\tau^* \cong H/\tau^*$.

Proof. It is immediate. □

5. **Fundamental divisible groups**

In this section, we introduce (self) fundamental divisible groups and fundamental divisible hypergroups and we point out on the relations between these concepts. We prove that every divisible group is a fundamental divisible group and the fundamental divisible group obtained from a finite divisible hypergroup is trivial. Moreover, we show that isomorphic divisible hypergroups have isomorphic fundamental divisible groups and we construct a self fundamental divisible group.

Definition 5.1. (i) *A group* $(G, .)$ *is said to be a fundamental group if there exists a nontrivial hypergroup say,* $(H, \circ)$ *such that* $(\frac{(H,\circ)}{\tau^*}, *) \cong (G, .)$. *In other words, it is equal to a fundamental group of a nontrivial hypergroup, up to an isomorphism.*
(ii) *A fundamental group* $(G, .)$ *is said to be a self fundamental group if there exists a nontrivial hyperoperation* $\circ$ *on* G *such that* $(\frac{(G,\circ)}{\tau^*}, *) \cong (G, .)$.

Theorem 5.1. *Let* $(G, \circ)$ *be a finite divisible hypergroup and* ρ *be a strongly regular equivalence relation on* G. *Then* $(G/\rho, *)$ *is the trivial divisible group.*

Proof. Let $(G, \circ)$ be a finite divisible hypergroup (by Corollary 3.2, there exists). Thus $((G, \circ)/\tau^*, *)$ is a finite divisible group. Since there are not finite divisible groups, we obtain that $((G, \circ)/\tau^*, *) \cong 1$, where 1 is the trivial group. □

Corollary 5.1. *The fundamental group of every finite divisible hypergroup is the trivial divisible group.*

Example 5.1. *Consider an equivalence relation* R *on* $\mathbb{Q}$ *and the hyperoperation* $\circ$ *on* $\mathbb{Q}/R$ *defined as follows:*

$$xRy \Longleftrightarrow y + x = 2x \ \text{or} \ \ y + x = 0 \ \text{and} \ R(x) \circ R(y) = \{R(x + y), R(x - y)\}.$$

Clearly $(\mathbb{Q}/R, \circ)$ *is a divisible hypergroup and for all* $x, y \in \mathbb{Q}$, $R(x)\tau^* R(y)$. *It follows that* $((\mathbb{Q}/R, \circ)/\tau^*, *)$ *is a trivial divisible group.*

Remark 5.1. *By Example 5.1, the converse of Corollary 5.1, does not hold.*

Theorem 5.2. *Every divisible group is a fundamental divisible group.*

Proof. Let $(G, .)$ be a divisible group. By Lemma 3.1, for all group $(H, .)$, $(G \times H, \circ)$ is a divisible hypergroup. Let $(x, y), (x', y') \in G \times H$. If $(x, y)\tau^*(x', y')$, then there exist $g_1, g_2 \in G$ such that $\{(x, y), (x', y')\} \subseteq (g_1, y)\circ(g_2, y')$ and $g_2 = g_1^3$ or $g_1 = g_2^3$. Without losing generality, let $g_2 = g_1^3$; thus $x = x' = g_1^4$ and so $\tau^*(x, y) = \{(x, z) \mid z \in H\}$. Now, we define a map $\varphi : (\frac{(G\times H,\circ)}{\tau^*}, *) \longrightarrow (G, .)$ by $\varphi(\tau^*(g, h)) = g$. Since for all $(g, h), (g', h') \in G \times H$, $\tau^*((g, h)) = \tau^*((g', h'))$ if and only if $g = g'$ if and only if $\varphi(\tau^*(g, h)) = \varphi(\tau^*(g', h'))$, it follows that φ is well-defined and one to one. Let $(g, h), (g', h') \in G \times H$. Then

$$\begin{aligned} \varphi(\tau^*(g, h) * \tau^*(g', h')) = \varphi(\tau^*(g.g', h)) &= \varphi(\tau^*(g.g', h')) \\ &= g.g' = \varphi(\tau^*(g, h)).\varphi(\tau^*(g', h')). \end{aligned}$$

Thus, φ is a homomorphism. Clearly φ is onto. Therefore, φ is an isomorphism and then $(\frac{(G\times H,\circ)}{\tau^*}, *) \cong (G, .)$. □

Example 5.2. *Let* $(\mathbb{Q}\times\mathbb{Z},\circ)$ *be the divisible hypergroup which is defined in Example 4.2. By Equations 1 and 2,* $(\mathbb{Q}\times\mathbb{Z})/\beta^* = (\mathbb{Q}\times\mathbb{Z})/\gamma^* = (\mathbb{Q}\times\mathbb{Z})/\tau^* = \mathbb{Q}\times\mathbb{Z}$. *Now define a map* $\theta : ((\mathbb{Q}\times\mathbb{Z})/\tau^*, *) \longrightarrow (\mathbb{Q},+)$ *by* $\theta(\tau^*(m/n,k)) = m/n$. *Clearly* θ *is an isomorphism and so* $(\mathbb{Q}\times\mathbb{Z}/\tau^*, *) \cong (\mathbb{Q},+)$. *Therefore, the divisible group* $(Q,+)$ *is a self fundamental divisible group.*

Theorem 5.3. *Let* G *and* H *be two sets such that* $|G| = |H|$. *If* $(G,\circ)$ *is a divisible hypergroup, then there exists a hyperoperation "* $\circ'$ *" on* H*, such that* $(G,\circ)$ *and* $(H,\circ')$*, are isomorphic divisible hypergroups.*

Proof. Since $|G| = |H|$, then there exists a bijection $\varphi : G \longrightarrow H$. For all $h_1, h_2 \in H$, define the hyperoperation " $\circ'$ " on H as follows:

$$h_1 \circ' h_2 = \varphi(g_1 \circ g_2).$$

First we show that "$\circ'$" is well-defined. Let $(h_1,h_2) = (h'_1,h'_2)$, where $h_i = \varphi(g_i)$, $h'_i = \varphi(g'_i)$ and $g_i, g'_i \in G$ for $1 \le i \le 2$. Then $h_i = h'_i$ implies that $\varphi(g_i) = \varphi(g'_i)$. Since φ is a bijection then clearly $g_i = g'_i$ and so $g_1 \circ' g_2 = \varphi(g_1\circ g_2) = \varphi(g'_1 \circ g'_2) = h'_1 \circ' h'_2$. Moreover,

$$\varphi(g_1 \circ g_2) = \varphi(g_1) \circ' \varphi(g_2). \tag{3}$$

Moreover, $(H,\circ')$ is a hypergroup. Let $g_1, g_2 \in G$. Then, by Equation (3), φ is a homomorphism. Therefore, by Theorem 3.6, $(H,\circ')$ is a divisible hypergroup, φ is a homomorphism and then is an isomorphism. □

Corollary 5.2. *Let* $(G,.)$ *be a non-finite divisible group. Then there exists a hyperoperation "* $\circ$ *" on* G *such that* $((\frac{(G,\circ)}{\tau^*}, *) \cong (G,.))$

Proof. Consider the divisible hypergroup $(G\times\mathbb{Z}_2,\circ)$. By Theorem 5.2,

$$(\frac{(G\times\mathbb{Z}_2,\circ)}{\tau^*}, *) \cong (G,.).$$

Since G is infinite, it follows that $|G| = |G\times\mathbb{Z}_2|$ and by Theorem 5.3, there exists a hyperoperation " $\circ'$ " on G , such that $(G,\circ')$ and $(G\times\mathbb{Z}_2,\circ)$ are isomorphic divisible hypergroups. We have

$$(G,.) \cong (\frac{(G\times\mathbb{Z}_2,\circ)}{\tau^*}, *) \cong (\frac{(G,\circ')}{\tau^*}, *).$$

Therefore, $(G,.)$ is a fundamental group of itself and so is a self fundamental divisible group. □

Definition 5.2. *A hypergroup* $(H,\circ)$ *is said to be fundamental divisible if its fundamental group is a divisible group.*

Example 5.3. $(\mathbb{Q},\circ)$ *is a fundamental divisible hypergroup, where* $\circ$ *is defined in Theorem 5.2.*

Theorem 5.4. *Let* $(H,\circ)$ *be a commutative hypergroup. Then* $(H,\circ)$ *is a fundamental divisible hypergroup if and only if it is a divisible hypergroup.*

Proof. Let $(H,\circ)$ be a divisible hypergroup, $y \in H$ and $n \in \mathbb{N}^*$. By definition of τ^*, $(\frac{(H,\circ)}{\tau^*}, *)$ is an abelian group and there exists $x \in H$ such that $y \in x^n$. Then

$$\tau^*(y) = \tau^*(x^n) = \tau^*(\underbrace{x\circ x\circ \ldots \circ x}_{(n-times)}) = \tau^*(x) * \tau^*(x) * \ldots * \tau^*(x) = \tau^*(x)^n.$$

Hence $(\frac{(H,\circ)}{\tau^*}, *)$ is a divisible group and so $(H,\circ)$ is a fundamental divisible hypergroup.

Conversely, let $(H,\circ)$ be a hypergroup such that it is fundamental divisible, $x \in H$ and $n \in \mathbb{N}^*$. Then $\tau^*(x) \in H/\tau^*$. Since H/τ^* is a divisible group, we obtain $\tau^*(y) \in H/\tau^*$

such that $\tau^*(x) = \tau^*(y)^n = \tau^*(\underbrace{y \circ y \circ \ldots \circ y}_{(n-times)})$. It follows that $x \in y^n$ and so $(H, \circ)$ is a divisible hypergroup.

□

Example 5.4. *Consider the divisible hypergroup $(\mathbb{Q} \times \mathbb{Z}_2, \circ)$, defined in Example 3.2. It is easy to see that $(\mathbb{Q}, +) \cong ((\mathbb{Q} \times \mathbb{Z}_2, \circ)/\tau^*, *)$ and so $(\mathbb{Q} \times \mathbb{Z}_2, \circ)$ is a fundamental divisible hypergroup.*

Corollary 5.3. *Direct product of divisible hypergroups is a divisible hypergroup.*

Proof. Let $(H, \circ)$ and $(K, \circ')$ be divisible hypergroups. Then we define a map $\theta : (\frac{(H,\circ)\times(K,\circ')}{\tau^*}, *) \longrightarrow (\frac{(H,\circ)}{\tau^*} \times \frac{(K,\circ')}{\tau^*}, *')$ by $\theta(\tau^*(h, k)) = (\tau^*(h), \tau^*(k))$. Clearly θ is an isomorphism and so $(\frac{(H,\circ)\times(K,\circ')}{\tau^*}, *) \cong (\frac{(H,\circ)}{\tau^*} \times \frac{(K,\circ')}{\tau^*}, *')$. Since $(\frac{(H,\circ)}{\tau^*}, *)$ and $(\frac{(K,*)}{\tau^*}, *)$ are divisible groups, it follows that $\frac{(H,\circ)}{\tau^*} \times \frac{(K,*)}{\tau^*}$ and $(\frac{(H,\circ)\times(K,\circ')}{\tau^*}, *)$ are divisible groups. Hence by Theorem 5.4, $(H, \circ) \times (K, \circ')$ is a divisible hypergroup. The converse is similar. □

Transitivity condition of τ

In what follows, we determine when the relation τ is transitive. The following results are similar to those relating to the relations β^* and γ^* and so, we do not give anymore the proofs.

Definition 5.3. *Let G be a hypergroup and M be a nonempty subset of G. M is called τ-part if for all $n \in \mathbb{N}^*$, $a_1, \ldots, a_n \in G$ and all $\sigma \in S_n$, where for some $1 \leq i, j \leq n$, $i \neq \sigma(j)$ we have $a_i \in a^3_{\sigma(j)}$, then $\bigodot\limits_{i=1}^{n} a_i \cap M \neq \emptyset \Longrightarrow \bigodot\limits_{i=1}^{n} a_i \subseteq M$.*

Lemma 5.1. *Let M be a nonempty subset of a hypergroup G. Then the following conditions are equivalent:*
(i) M is a τ-part of G;
(ii) $x \in M$ and $x\ \tau\ y$ imply $y \in M$;
(iii) $x \in M$ and $x\ \tau^*\ y$ imply $y \in M$.

Theorem 5.5. *Let G be a hypergroup and $x \in G$. Then the following conditions are equivalent:*
(i) τ is a transitive relation;
(ii) $\tau^*(x)$ is a τ-part.

Definition 5.4. *Let $(H, \circ)$ be a divisible hypergroup and A be a subset of G. We denote by $T(A)$ the complete closure of A, which is the smallest complete part of G, that contains A. Denote $K_1(A) = A$ and for all $n \geq 1$ denote*

$$K_{n+1}(A) = \Big\{ x \in G \mid \exists\ p \in \mathbb{N}^*, \sigma \in S_p \text{ and } g_1, ..., g_p \in G, \text{where } \exists\ 1 \leq i, j \leq p, \; i \neq \sigma(j), \text{ such that } g_i \in a^3_{\sigma(j)},\ x \in \bigodot_{i=1}^{p} g_i \text{ and } K_n(A) \cap \bigodot_{i=1}^{p} g_i \neq \emptyset \Big\}$$

and $K(A) = \bigcup\limits_{n\geq 1} K_n(A)$.

Theorem 5.6. *Let $(G, \circ)$ be a divisible hypergroup and $A \subseteq G$. Then*
(i) $T(A) = K(A)$;
(ii) $K(A) = \bigcup\limits_{a\in A} K(a)$.

Theorem 5.7. *Let* $(G, \circ)$ *be a divisible hypergroup and* $x, y \in G$. *Then*
(i) for all $n \geq 2$ *we have* $K_n(K_2(x)) = K_{n+1}(x)$;
(ii) $x \in K_n(y) \Longleftrightarrow y \in K_n(x)$.

6. Conclusions

In this paper we investigate divisible hypergroups and some of their new useful properties. We define a new equivalence relation τ^* on a divisible hypergroup and we prove that:
(1) τ^* is strongly regular, while τ is not transitive in hypergroups and so is not an equivalence relation.
(2) In general $\tau^* \neq \beta^*$ and $\tau^* \neq \gamma^*$.
(3) τ^* is the smallest strongly equivalence relation such that the corresponding quotient structure is a group and such that all its elements are divisible.
(4) In commutative divisible hypergroups τ^* is a fundamental relation.
(5) We construct a divisible hypergroup on every nonempty set.
(6) We define the concept of a fundamental divisible group and we show that all divisible groups are self fundamental divisible groups.
(7) The quotient of a finite divisible hypergroup with respect to a strongly regular equivalence relation is a trivial group.
(8) We prove that a direct product of divisible hypergroups is a divisible hypergroup.
(9) Considering of the concept of a τ-part in a divisible hypergroup , we analyse when τ is transitive.

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REFERENCES

[1] *H. Aghabozorgi, B. Davvaz and M. Jafarpour*, Nilpotent groups derived from hypergroups, J. Algebra., **382** (2013), 177-184.

[2] *H. Aghabozorgi, B. Davvaz and M. Jafarpour*, Solvable polygroups and derived subpolygroups, Comm. in Algebra., **41(8)** (2013), 3098-3107.

[3] *R. Ameri and E. Mohammadzadeh*, Engel groups derived from hypergroups, European J. Combin., **44** (2015), 191-197.

[4] *P. Corsini*, Prolegomena of Hypergroup theory, Hecond Edition, Aviani Editor, 1993.

[5] *D. Freni*, Une note sur le coeur dun hypergroupe et sur la cloture transitive β^* de β, Riv. diMat. Pura Appl., **8** (1991), 153-156.

[6] *D. Freni*, A new characterization of the derived hypergroup via strongly regular equivalences, Comm. Algebra., **30** (2002), No. 8, 3977-3989.

[7] *M. Jafarpour, H. Aghabozorgi and B. Davvaz*, On Nilpotent and Solvable Polygroups, Bull. Iranian Math. Soc., **39** (2013), No. 3, 487-499.

[8] *M. Jafarpour, H. Aghabozorgi and B. Davvaz*, Solvable groups derived from hypergroups, J. Algebra Appl., **15(4)** (2016), 16500167, (9 pages).

[9] *M. Koskas*, Groupoides, demi-hypergroupes et hypergroupes, J. Math. Pures Appl., **49 (9)** (1970), 155-192.

[10] *L. Leoreanu*, Commutative hypergroups associated with arbitrary hypergroups, J.of Disc. Math. Sc. Crypt., **6** (2003), N. 2-3, 195-198.

[11] *F. Marty*, Sur une generalization de la notion de groupe $8th$ Congres Math. Scandinaves, Stockholm, (1934), 45-49.

[12] *S. Pianskool, A. Wasanawichit and Y. Kemprasit*, Certain Divisible Hypergroups, Gen. Math., **13** (2005), No. 3, 15-22.

[13] *S. Pianskool, S. Chaopraknoi and Y. Kemprasit*, Divisibility of Some Hypergroups Defined from Groups, Gen. Math., **14** (2006), No. 1, 77-86.

[14] *T. Vougiouklis*, Hyperstructures and their representations, Hadronic Press Inc, 1994.