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PROPER BIHARMONIC SUBMANIFOLDS IN A UNIT SPHERE

Tianmin Zhu[1], Shichang Shu[2]

In this article, we study the proper biharmonic submanifolds in a unit sphere S^n. If the submanifolds satisfy certain geometric and rigidity properties, we obtain some characterizations of the two canonical examples of proper biharmonic submanifolds: hyperspheres $S^{n-1}(1/\sqrt{2})$ and the generalized Clifford tori $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = n-1$, $m_1 \neq m_2$.

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1. New theorems of proper biharmonic submanifolds

A biharmonic map is a map $\varphi : M^m \to N^n$ between Riemannian manifolds that is a critical point of the bienergy functional $E_2(\varphi) = \frac{1}{2}\int_{M^m} |\tau(\varphi)|^2 v_g$, where $\tau(\varphi) = \mathrm{tr}\nabla d\varphi$ denotes the tension field of φ. By calculating the first variation of φ, G.Y. Jiang [10] showed that the map φ is biharmonic if and only if its bitension field $\tau_2(\varphi)$ vanishes identically, that is, $\tau_2(\varphi) = \mathrm{tr}(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^{M^m}})\tau(\varphi) - \mathrm{tr}R^{N^n}(d\varphi, \tau(\varphi))d\varphi = 0$, where R^{N^n} is the curvature operator of N^n defined by $R^{N^n}(X,Y)Z = [\nabla^{N^n}_X, \nabla^{N^n}_Y]Z - \nabla^{N^n}_{[X,Y]}Z$. We note that any harmonic map is biharmonic. The non-harmonic biharmonic maps are called proper biharmonic. The submanifolds with biharmonic inclusion map are called biharmonic submanifolds. It is well known that an isometric immersion is minimal if and only if it is harmonic. So the minimal submanifolds are trivially biharmonic. The submanifolds with non-harmonic (non-minimal) biharmonic inclusion map are called proper biharmonic submanifolds.

From the results of G.Y. Jiang [11], we know that if $\varphi : M^m \to S^n$ be an isometric immersion submanifold in a unit sphere S^n with codimension $n-m$, then M^m is biharmonic if and only if for any α, i

$$\sum_{j,k} h^{\alpha}_{jjkk} - \sum_{i,j,k,\beta} h^{\beta}_{jj} h^{\beta}_{ik} h^{\alpha}_{ik} + m \sum_{j} h^{\alpha}_{jj} = 0, \tag{1.1}$$

$$\sum_{j,k,\beta} (2h^{\beta}_{jjk} h^{\beta}_{ik} + h^{\beta}_{jj} h^{\beta}_{kki}) = 0, \tag{1.2}$$

[1]Professor, School of Mathematics and Information Science, Weinan Normal University, Weinan, 714000, Shaanxi, P.R. China, e-mail: wnzhutm@163.com

[2]Professor, School of Mathematics and Information Science, Xianyang Normal University, Xianyang, 712000, Shaanxi, P.R. China, e-mail: shusc163@sina.com

where $1 \leq i, j, k \leq m$, $m+1 \leq \alpha, \beta \leq n$, h^α_{ij} are the components of the second fundamental form of M^m, h^α_{ijk} and h^α_{ijkl} are the first and second covariant derivatives of h^α_{ij} defined by (2.6) and (2.7).

We should note that B.Y. Chen [7], Caddeo– Montaldo–Oniciuc [5, 6], Balmus–Montaldo–Oniciuc [2, 3] and Ou [16] also studied biharmonic maps and biharmonic submanifolds, they obtained (1.1) and (1.2) in several steps and by different sign conventions. From [11], [2] and [4], the canonical examples of proper biharmonic submanifolds in a unit sphere S^n are the small hypersphere $S^{n-1}(1/\sqrt{2}) = \{(x, 1/\sqrt{2}) \in \mathbf{R}^{n+1} | x \in \mathbf{R}^n, |x|^2 = 1/2\} \subset S^n$ and the products $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) = \{(x, y) \in \mathbf{R}^{n+1} | x \in \mathbf{R}^{m_1+1}, y \in \mathbf{R}^{m_2+1}, |x|^2 = |y|^2 = 1/2\} \subset S^n$, where $m_1 + m_2 = n-1$ and $m_1 \neq m_2$. Recently, Balmus–Montaldo–Oniciuc [2] proposed the following:

Conjecture. *The only m-dimensional $(m \geq 3)$ proper biharmonic hypersurfaces in S^{m+1} are the open parts of hyperspheres $S^m(1/\sqrt{2})$ and of the generalized Clifford tori $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

We should notice that the above Conjecture is still open. In this article, we study the proper biharmonic submanifolds in a unit sphere S^n and prove that the above Conjecture is true if the submanifolds satisfy certain geometric and rigidity properties. More precisely, we obtain the following:

Theorem 1.1. *Let M^m be a m-dimensional $(m \geq 3)$ proper biharmonic hypersurface in S^{m+1} with constant mean curvature. If the sectional curvature of M^m is nonnegative, then M^m is an open part of $S^m(1/\sqrt{2})$ or of the standard products $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, where $m_1 + m_2 = m$ and $m_1 \neq m_2$.*

For a fixed $\alpha, m+1 \leq \alpha \leq n$, we may choose orthonormal frame field $\{e_1, \ldots, e_m\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$. Putting $\phi^\alpha_{ij} = h^\alpha_{ij} - \frac{1}{m}\mathrm{tr}H^\alpha \delta_{ij}$ and consider the symmetric tensor $\phi = \sum_{i,j,\alpha} \phi^\alpha_{ij} \omega_i \omega_j e_\alpha$, we can easily see that ϕ is traceless and that $|\phi|^2 = |A|^2 - mH^2$, where $h^\alpha_{ij}, |A|^2, \vec{H}$ and H are the components of the second fundamental form, the square of the norm of the shape operator, the mean curvature vector field and the mean curvature of M^m. We know that $|\phi|^2 \equiv 0$ if and only if M^m is totally umbilical. We define a polynomial $P_{H,n-m}(x)$ by

$$P_{H,n-m}(x) = (2 - \frac{1}{n-m})x^2 + \frac{m-2}{\sqrt{m(m-1)}} mHx - m(1 + H^2), \tag{1.3}$$

where $n-m$ is the codimension of M^m. It may be easily checked that $P_{H,n-m}(x) = 0$ has a positive real root. Let $B_{H,n-m}$ be the square of the positive real root. We obtain the following results:

Theorem 1.2. *Let M^m be a m-dimensional $(m \geq 3)$ proper biharmonic hypersurface in S^{m+1} with constant mean curvature. If*

$$|\phi|^2 \leq B_{H,1}, \tag{1.4}$$

then M^m is an open part of hyperspheres $S^m(1/\sqrt{2})$ or of the generalized Clifford tori $S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

Theorem 1.3. *Let M^m be a m-dimensional $(m \geq 3)$ proper biharmonic submanifold in S^n with the codimension $n-m$ $(n-m \geq 2)$ and parallel mean curvature vector field. If*

$$|\phi|^2 \leq \min\{m(1-H^2), B_{H,n-m}\}, \tag{1.5}$$

then M^m is a minimal submanifold of a small hypersphere $S^{n-1}(1/\sqrt{2}) \subset S^n$.

Theorem 1.4. *Let M^m be a m-dimensional $(m \geq 3)$ complete proper biharmonic hypersurface in S^{m+1} with constant scalar curvature $m(m-1)R$ and $\bar{R} = R-1 \geq 0$. If*

$$m\bar{R} \leq \sup|A|^2 \leq (m-1)\frac{m\bar{R}+2}{m-2} + \frac{m-2}{m\bar{R}+2}, \tag{1.6}$$

then

(i) $\sup|A|^2 = m\bar{R}$ and M^m is an open part of hyperspheres $S^m(1/\sqrt{2})$ or

(ii) $\sup|A|^2 = (m-1)\frac{m\bar{R}+2}{m-2} + \frac{m-2}{m\bar{R}+2}$. If the supremum $\sup|A|^2$ is attained at some point of M^m, then M^m is an open part of the generalized Clifford tori $S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

We should notice that Theorem 1.1–Theorem 1.3 hold for all compact and complete non-compact proper biharmonic hypersurfaces (submanifolds) in S^n and Theorem 1.4 holds for all complete non-compact proper biharmonic hypersurfaces in S^n. For the compact case, integrating both sides of (4.12) and by the assertion in the last part of the proof of Theorem 1.4, we conclude that if $m\bar{R} \leq |A|^2 \leq (m-1)\frac{m\bar{R}+2}{m-2} + \frac{m-2}{m\bar{R}+2}$, then M^m is an open part of hyperspheres $S^m(1/\sqrt{2})$ or of the generalized Clifford tori $S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

2. **Basic formulas of submanifolds in S^n**

Let $x : M^m \to S^n$ be a m-dimensional submanifold in an n-dimensional unit sphere S^n. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal basis of M^m with respect to the induced metric, $\{\theta_1, \ldots, \theta_m\}$ are their dual form. Let $e_{m+1}, \ldots, e_n$ be the local orthonormal normal vector field. We make the following convention on the range of indices: $1 \leq i,j,k,l,s \leq m$, $m+1 \leq \alpha, \beta \leq n$. Then the structure equations are

$$dx = \sum_i \theta_i e_i, \tag{2.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{j,\alpha} h^\alpha_{ij} \theta_j e_\alpha - \theta_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij} \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.4}$$

$$m(m-1)(R-1) = m^2H^2 - |A|^2, \tag{2.5}$$

where $|A|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2$, $\vec{H} = \sum_\alpha H^\alpha e_\alpha$, $H^\alpha = \frac{1}{m}\sum_k h^\alpha_{kk}$, $H = |\vec{H}|$, R is the normalized scalar curvature of M^m.

The first covariant derivatives h^α_{ijk} and the second covariant derivatives h^α_{ijkl} of h^α_{ij} are defined by

$$\sum_k h^\alpha_{ijk}\theta_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\theta_{ki} + \sum_k h^\alpha_{ik}\theta_{kj} + \sum_\beta h^\beta_{ij}\theta_{\beta\alpha}, \tag{2.6}$$

$$\sum_l h^\alpha_{ijkl}\theta_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\theta_{li} + \sum_l h^\alpha_{ilk}\theta_{lj} + \sum_l h^\alpha_{ijl}\theta_{lk} + \sum_\beta h^\beta_{ijk}\theta_{\beta\alpha}. \tag{2.7}$$

Then, we have the Codazzi equations and the Ricci identities

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.8}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_s h^\alpha_{sj} R_{sikl} + \sum_s h^\alpha_{is} R_{sjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{2.9}$$

The Ricci equations are

$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}). \tag{2.10}$$

Define the first, second covariant derivatives of the mean curvature vector field $\vec{H} = \sum_\alpha H^\alpha e_\alpha$ in the normal bundle $N(M^m)$ as follows

$$\sum_i H^\alpha_{,i}\theta_i = dH^\alpha + \sum_\beta H^\beta \theta_{\beta\alpha}, \tag{2.11}$$

$$\sum_j H^\alpha_{,ij}\theta_j = dH^\alpha_{,i} + \sum_j H^\alpha_{,j}\theta_{ji} + \sum_\beta H^\beta_{,i}\theta_{\beta\alpha}. \tag{2.12}$$

Let f be a smooth function on M^m. The first, second covariant derivatives $f_i, f_{i,j}$ and the Beltrami-Laplace of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{i,j}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.13}$$

In general, for a matrix $\mathcal{A} = (a_{ij})$ we denote by $N(\mathcal{A})$ the square of the norm of $\mathcal{A}$, that is, $N(\mathcal{A}) = \mathrm{tr}(\mathcal{A} \cdot \mathcal{A}^t) = \sum_{i,j}(a_{ij})^2$. Clearly, $N(\mathcal{A}) = N(\mathcal{T}^t\mathcal{A}\mathcal{T})$ for any orthogonal matrix $\mathcal{T}$.

3. **Formulas of proper biharmonic submanifolds**

From (1.1) and (1.2), we may easily obtain that if $\varphi : M^m \to S^n$ be an isometric immersion submanifold in a unit sphere S^n with codimension $n-m$, then M^m is biharmonic if and only if for any α, i

$$\sum_k H^\alpha_{,kk} - \sum_{i,k,\beta} H^\beta h^\beta_{ik} h^\alpha_{ik} + mH^\alpha = 0, \tag{3.1}$$

$$2\sum_{k,\beta} h^\beta_{ik} H^\beta_{,k} + m\sum_\beta H^\beta H^\beta_{,i} = 0. \tag{3.2}$$

If $n-m=1$, M^m is a biharmonic hypersurface if and only if for any i

$$\Delta H - (|A|^2 - m)H = 0, \tag{3.3}$$

$$2\sum_k h_{ik} H_{,k} + mHH_{,i} = 0, \tag{3.4}$$

where A is the Weingarten shape operator, H the mean curvature, Δ the Beltrami-Laplace operator on M^m defined by (2.13) and $H_{,i}$ is defined by (2.11).

Define tensors

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}, \tag{3.5}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \phi_{ij}^{\alpha}\phi_{ij}^{\beta}, \quad \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}. \tag{3.6}$$

Then the $((n-m)\times(n-m))$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{m+1},\ldots,e_n$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha}\delta_{\alpha\beta}. \tag{3.7}$$

By a direct calculation, we have

$$\sum_k \phi_{kk}^{\alpha} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - mH^{\alpha}H^{\beta}, \quad |\phi|^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha} = |A|^2 - mH^2, \tag{3.8}$$

$$\sum_{i,j,k,\alpha} h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} = \sum_{i,j,k,\alpha} \phi_{kj}^{\beta}\phi_{ij}^{\alpha}\phi_{ik}^{\alpha} + 2\sum_{i,j,\alpha} H^{\alpha}\phi_{ij}^{\alpha}\phi_{ij}^{\beta} + H^{\beta}|\phi|^2 + mH^2H^{\beta}. \tag{3.9}$$

From (3.1), we have $\sum_k H^{\alpha}H_{,kk}^{\alpha} - \sum_{i,k,\beta} H^{\alpha}H^{\beta}h_{ik}^{\beta}h_{ik}^{\alpha} + m(H^{\alpha})^2 = 0$. We take a suitable choice of $e_{m+1},\ldots,e_n$, such that $\sigma_{\alpha\beta} = \sigma_{\alpha}\delta_{\alpha\beta}$, then

$$\begin{aligned}\frac{1}{2}\Delta H^2 =& \frac{1}{2}\sum_{\alpha}\Delta(H^{\alpha})^2 = \sum_{\alpha,k}(H_{,k}^{\alpha})^2 + \sum_{\alpha,k} H^{\alpha}H_{k,k}^{\alpha} \\ =& |\nabla^{\perp}\vec{H}|^2 + \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\sigma_{\alpha\beta} - mH^2 = |\nabla^{\perp}\vec{H}|^2 + \sum_{\alpha}(H^{\alpha})^2\sigma_{\alpha} - mH^2 \\ \leq& |\nabla^{\perp}\vec{H}|^2 + \sum_{\alpha}(H^{\alpha})^2\sum_{\beta}\sigma_{\beta} - mH^2 = |\nabla^{\perp}\vec{H}|^2 + (|A|^2 - m)H^2.\end{aligned} \tag{3.10}$$

If the codimension $n-m=1$, from (3.3), we note that for proper biharmonic hypersurfaces

$$\frac{1}{2}\Delta H^2 = |\nabla^{\perp}\vec{H}|^2 + (|A|^2 - m)H^2. \tag{3.11}$$

Taking covariant derivative for k on (3.2), we have

$$\sum_{j,\beta} h_{ij,k}^{\beta}H_{,j}^{\beta} + \sum_{j,\beta} h_{ij}^{\beta}H_{,jk}^{\beta} = -\frac{m}{2}\sum_{\beta} H_{,k}^{\beta}H_{,i}^{\beta} - \frac{m}{2}\sum_{\beta} H^{\beta}H_{,ik}^{\beta}. \tag{3.12}$$

Setting $k=i$ in (3.12) and taking sum for i, we have

$$\begin{aligned}\sum_{i,j,\beta} h_{ij,i}^{\beta}H_{,j}^{\beta} + \sum_{i,j,\beta} h_{ij}^{\beta}H_{,ji}^{\beta} =& -\frac{m}{2}\sum_{i,\beta}(H_{,i}^{\beta})^2 - \frac{m}{2}\sum_{i,\beta} H^{\beta}H_{,ii}^{\beta} \\ =& -\frac{m}{2}(|\nabla^{\perp}\vec{H}|^2 + \sum_{i,\beta} H^{\beta}H_{,ii}^{\beta}) = -\frac{m}{4}\Delta H^2,\end{aligned}$$

$$\sum_{i,j,\beta} h_{ij}^{\beta}H_{,ji}^{\beta} = -\sum_{j,\beta} m(H_{,j}^{\beta})^2 - \frac{m}{4}\Delta H^2 = -m|\nabla^{\perp}\vec{H}|^2 - \frac{m}{4}\Delta H^2. \tag{3.13}$$

On the other hand, we have

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} = |\nabla h|^2 + \sum_{i,j,\alpha} h^\alpha_{ij}(mH^\alpha)_{i,j}$$
$$+ \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) + \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}.$$

From (3.13), we see that

$$\frac{1}{2}\Delta(|A|^2 + \frac{1}{2}m^2H^2) = |\nabla h|^2 - m^2|\nabla^\perp \vec{H}|^2 \tag{3.14}$$
$$+ \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) + \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}.$$

From [15] and [18], we have

Proposition 3.1. *Let M^m be a complete Riemannian manifold whose Ricci curvature is bounded from below. If f is a C^2-function bounded from above on M^m, then for any $\varepsilon > 0$, there is a point $x \in M^m$ such that*

$$\sup f - \varepsilon < f(x), \quad |\nabla f(x)| < \varepsilon, \quad \Delta f(x) < \varepsilon. \tag{3.15}$$

4. Proof of theorems

Proof of Theorem 1.1. Since for hypersurfaces, H is constant if and only if $\nabla^\perp \vec{H} = 0$, from (3.14), we have

$$\frac{1}{2}\Delta(|A|^2 + \frac{1}{2}m^2H^2) = |\nabla h|^2 - m^2|\nabla^\perp \vec{H}|^2 + \sum_{i,j,k,l} h_{ij}(h_{kl}R_{lijk} + h_{li}R_{lkjk}) \tag{4.1}$$
$$= |\nabla h|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2,$$

where we choose a local orthonormal basis $\{e_1, \ldots, e_m\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (3.11), we see that for constant mean curvature proper biharmonic hypersurfaces $|A|^2 = m$. Thus, if the sectional curvature is nonnegative, from (4.1), we infer that $R_{ijij}(\lambda_i - \lambda_j)^2 = 0$, that is, $(1 + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0$. This implies that M^m has at most two distinct principal curvatures. From Theorem 4.3 of [2], we know that M^m is an open part of $S^m(1/\sqrt{2})$ or of the standard products $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, where $m_1 + m_2 = m$ and $m_1 \neq m_2$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. From (4.1), Lemmas in [14] or [1] and by a standard calculation (see [1]), we have

$$\frac{1}{2}\Delta(|A|^2 + \frac{1}{2}m^2H^2) = |\nabla\phi|^2 + |\phi|^2(m(H^2+1) - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| - |\phi|^2). \tag{4.2}$$

Since M^m is proper biharmonic hypersurface with constant mean curvature, we have $|A|^2 = m$. From (4.2) and the assumption of Theorem 1.2, we see that the right hand side of (4.2) is nonnegative. Thus $|\phi|^2(m(H^2+1) - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| - |\phi|^2) = 0$. This implies that the equalities in Lemma of [1] hold. Thus we see that M^m has at most two distinct principal curvatures and the multiplicities of the two distinct principal curvatures are 1 and $m-1$. From Theorem 4.3 of [2], we know that M^m is an open part of $S^m(1/\sqrt{2})$ or of the standard products $S^1(1/\sqrt{2})\times S^{m-1}(1/\sqrt{2})$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. From (2.10), we have

$$\begin{aligned}\sum_{\alpha,\beta,k}(R_{\beta\alpha jk})^2 &= \sum_{\alpha,\beta,i,j,k}(h^\beta_{ji}h^\alpha_{ik}-h^\beta_{ki}h^\alpha_{ij})R_{\beta\alpha jk} = \sum_{\alpha,\beta,i,j,k}h^\beta_{ji}h^\alpha_{ik}R_{\beta\alpha jk}\\ &\quad-\sum_{\alpha,\beta,i,j,k}h^\beta_{ki}h^\alpha_{ij}R_{\beta\alpha jk} = -2\sum_{\alpha,\beta}\sum_{i,j,k}h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k}h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk} &= -\frac{1}{2}\sum_{\alpha,\beta,k}(R_{\beta\alpha jk})^2 = -\frac{1}{2}\sum_{\alpha,\beta,j,k}(\sum_l h^\beta_{jl}h^\alpha_{lk}-\sum_l h^\alpha_{jl}h^\beta_{lk})^2 \qquad (4.3)\\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}(\sum_l \phi^\beta_{jl}\phi^\alpha_{lk}-\sum_l \phi^\alpha_{jl}\phi^\beta_{lk})^2 = -\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{\mathcal{A}}_\alpha\tilde{\mathcal{A}}_\beta-\tilde{\mathcal{A}}_\beta\tilde{\mathcal{A}}_\alpha),\end{aligned}$$

where $\tilde{\mathcal{A}}_\alpha := (\phi^\alpha_{ij}) = (h^\alpha_{ij}-H^\alpha\delta_{ij})$.

From (2.4), (2.10), (3.6), (3.8), (3.9) and (4.3), we have

$$\begin{aligned}&\sum_\alpha\sum_{i,j,k,l}h^\alpha_{ij}(h^\alpha_{kl}R_{lijk}+h^\alpha_{li}R_{lkjk}) \qquad (4.4)\\ =&m|\phi|^2-\sum_{\alpha,\beta}\sum_{i,j,k,l}h^\alpha_{ij}h^\beta_{ij}h^\alpha_{lk}h^\beta_{lk}+m\sum_{\alpha,\beta}\sum_{i,j,k}H^\beta h^\beta_{kj}h^\alpha_{ij}h^\alpha_{ik}+\sum_{\alpha,\beta,i,j,k}h^\alpha_{ji}h^\beta_{ik}R_{\beta\alpha jk}\\ =&m|\phi|^2-\sum_{\alpha,\beta}\sigma^2_{\alpha\beta}+m\sum_{\alpha,\beta}\sum_{i,j,k}H^\beta\phi^\beta_{kj}\phi^\alpha_{ij}\phi^\alpha_{ik}+2m\sum_{\alpha,\beta}\sum_{i,j}H^\alpha H^\beta\phi^\alpha_{ij}\phi^\beta_{ij}\\ &+m\sum_\beta(H^\beta)^2|\phi|^2+m^2H^2\sum_\beta(H^\beta)^2-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{\mathcal{A}}_\alpha\tilde{\mathcal{A}}_\beta-\tilde{\mathcal{A}}_\beta\tilde{\mathcal{A}}_\alpha)\\ =&m|\phi|^2-\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}+mH^2|\phi|^2+m\sum_{\alpha,\beta}\sum_{i,j,k}H^\beta\phi^\beta_{kj}\phi^\alpha_{ij}\phi^\alpha_{ik}\\ &-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{\mathcal{A}}_\alpha\tilde{\mathcal{A}}_\beta-\tilde{\mathcal{A}}_\beta\tilde{\mathcal{A}}_\alpha).\end{aligned}$$

For a fixed $\alpha, m+1\le\alpha\le n$, we can take a local orthonormal frame field $\{e_1,\ldots,e_m\}$ such that $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$. Then, $\phi^\alpha_{ij}=\mu^\alpha_i\delta_{ij}$ with $\mu^\alpha_i=\lambda^\alpha_i-H^\alpha$, $\sum_i\mu^\alpha_i=0$. Let $\sum_i(\phi^\beta_{ii})^2=\tau_\beta$. Then $\tau_\beta\le\sum_{i,j}(\phi^\beta_{ij})^2=\tilde{\sigma}_\beta$. Since $\sum_i\phi^\beta_{ii}=0$, $\sum_i\mu^\alpha_i=0$ and $\sum_i(\mu^\alpha_i)^2=\tilde{\sigma}_\alpha$. From Lemma 3.3 of [8] and Lemma 2.5 of [17] we have that

$$\sum_{\alpha,\beta}\sum_{i,j,k}H^{\alpha}\phi_{ij}^{\alpha}\phi_{kj}^{\beta}\phi_{ik}^{\beta}=\sum_{\beta,\alpha}\sum_{i,j,k}H^{\beta}\phi_{ij}^{\beta}\phi_{kj}^{\alpha}\phi_{ik}^{\alpha}=\sum_{\alpha,\beta}H^{\beta}\sum_{i}\phi_{ii}^{\beta}(\mu_i^{\alpha})^2 \tag{4.5}$$
$$\geq-\frac{m-2}{\sqrt{m(m-1)}}\sum_{\alpha,\beta}|H^{\beta}|\tilde{\sigma}_{\alpha}\sqrt{\tau_{\beta}}\geq-\frac{m-2}{\sqrt{m(m-1)}}\sum_{\alpha}\tilde{\sigma}_{\alpha}\sum_{\beta}|H^{\beta}|\sqrt{\tilde{\sigma}_{\beta}}$$
$$\geq-\frac{m-2}{\sqrt{m(m-1)}}|\phi|^2\sqrt{\sum_{\beta}(H^{\beta})^2\sum_{\beta}\tilde{\sigma}_{\beta}}=-\frac{m-2}{\sqrt{m(m-1)}}|H||\phi|^3.$$

From the well known inequality of Lemma 1 in [9], (3.6), (3.7), we have

$$-\sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2-\sum_{\alpha,\beta}N(\tilde{\mathcal{A}}_{\alpha}\tilde{\mathcal{A}}_{\beta}-\tilde{\mathcal{A}}_{\beta}\tilde{\mathcal{A}}_{\alpha})=-\sum_{\alpha}\tilde{\sigma}_{\alpha}^2-\sum_{\alpha,\beta}N(\tilde{\mathcal{A}}_{\alpha}\tilde{\mathcal{A}}_{\beta}-\tilde{\mathcal{A}}_{\beta}\tilde{\mathcal{A}}_{\alpha}) \tag{4.6}$$
$$\geq-\sum_{\alpha}\tilde{\sigma}_{\alpha}^2-2\sum_{\alpha\neq\beta}\tilde{\sigma}_{\alpha}\tilde{\sigma}_{\beta}=-2(\sum_{\alpha}\tilde{\sigma}_{\alpha})^2+\sum_{\alpha}\tilde{\sigma}_{\alpha}^2$$
$$\geq-2|\phi|^4+\frac{1}{n-m}(\sum_{\alpha}\tilde{\sigma}_{\alpha})^2=-(2-\frac{1}{n-m})|\phi|^4.$$

Therefore, from (3.14), (4.3)–(4.6), we have

$$\frac{1}{2}\Delta(|A|^2+\frac{1}{2}m^2H^2)\geq|\nabla h|^2-m^2|\nabla^{\perp}\vec{H}|^2 \tag{4.7}$$
$$+|\phi|^2\{m+mH^2-\frac{m-2}{\sqrt{m(m-1)}}mH|\phi|-(2-\frac{1}{n-m})|\phi|^2\}.$$

From (1.5), we see that $|A|^2\leq m$. Since the mean curvature vector field is parallel, that is $\nabla^{\perp}\vec{H}=0$, we know that H is nonzero constant, by (3.10), we have $|A|^2=m$. From (1.5) again, we have $|\phi|^2\leq B_{H,n-m}$. Thus, the right hand side of (4.7) is nonnegative. We conclude from (4.7) that $|\phi|^2\{m+mH^2-\frac{m-2}{\sqrt{m(m-1)}}mH|\phi|-(2-\frac{1}{n-m})|\phi|^2\}=0$. Thus, $|\phi|^2=0$ or $m+mH^2-\frac{m-2}{\sqrt{m(m-1)}}mH|\phi|-(2-\frac{1}{n-m})|\phi|^2=0$. In the first case, M^m is totally umbilical and $|A|^2=mH^2$. Since $|A|^2=m$, we have $m=mH^2$ and $H=1$. From Theorem 2.10 of [2], we know that M^m is a minimal submanifold of a small hypersphere $S^{n-1}(1/\sqrt{2})\subset S^n$. In the second case, M^m is not totally umbilical and the equalities in (4.7), (4.6), (4.5) and Lemma 1 of [9] hold. Thus, we have $\nabla h=0$, $(n-m)\sum_{\alpha}\tilde{\sigma}_{\alpha}^2=(\sum_{\alpha}\tilde{\sigma}_{\alpha})^2$, that is

$$\tilde{\sigma}_{m+1}=\cdots=\tilde{\sigma}_n, \tag{4.8}$$

$$N(\tilde{\mathcal{A}}_{\alpha}\tilde{\mathcal{A}}_{\beta}-\tilde{\mathcal{A}}_{\beta}\tilde{\mathcal{A}}_{\alpha})=2N(\tilde{\mathcal{A}}_{\alpha})N(\tilde{\mathcal{A}}_{\beta}),\quad \alpha\neq\beta, \tag{4.9}$$

and

$$\sum_{\beta}|H^{\beta}|\sqrt{\tilde{\sigma}_{\beta}}=|H||\phi|. \tag{4.10}$$

From (4.8), (4.10) and the assumption $n-m\geq 2$, we have $\sqrt{\tilde{\sigma}_{m+1}}\sum_{\beta}|H^{\beta}|=\sqrt{\sum_{\beta}(H^{\beta})^2}\sqrt{\sum_{\beta}\tilde{\sigma}_{\beta}}=\sqrt{(n-m)\tilde{\sigma}_{m+1}}\sqrt{\sum_{\beta}(H^{\beta})^2}$. Since M^m is not totally umbilical, we have $\tilde{\sigma}_{m+1}\neq 0$. Thus, we have $(\sum_{\beta}|H^{\beta}|)^2=(n-m)\sum_{\beta}(H^{\beta})^2$, that is,

$$|H^{m+1}| = \cdots = |H^n|. \tag{4.11}$$

From Lemma 1 of [9], we know that at most two of $\tilde{\mathcal{A}}_\alpha = (\phi^\alpha_{ij}), \alpha = m+1, \cdots, n$ are different from zero. If all of $\tilde{\mathcal{A}}_\alpha = (\phi^\alpha_{ij})$ are zero, which is a contradiction with M^m is not totally umbilical. If only one of them, say $\tilde{\mathcal{A}}_\alpha$, is different from zero, which is contradiction with (4.8). Therefore, we may assume that $\tilde{\mathcal{A}}_{m+1} = \lambda\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_{m+2} = \mu\tilde{B}$, $\lambda, \mu \neq 0$, $\tilde{\mathcal{A}}_\alpha = 0$, $\alpha \geq m+3$, where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are defined in Lemma 1 of [9]. From (4.10), we have $(\sqrt{2}\lambda|H^{m+1}| + \sqrt{2}\mu|H^{m+2}|)^2 = H^2|\phi|^2 = \sum_\alpha (H^\alpha)^2(2\lambda^2 + 2\mu^2)$. Thus, from (4.11), we have $(H^{m+1})^2(\lambda+\mu)^2 = (n-m)(H^{m+1})^2(\lambda^2+\mu^2)$, that is, $(H^{m+1})^2[(n-m-1)\lambda^2 - 2\lambda\mu + (n-m-1)\mu^2] = 0$. Since $\lambda, \mu \neq 0$, we infer that $H^{m+1} = 0$. Thus, from (4.11), we have $H^\alpha = 0, m+1 \leq \alpha \leq n$, that is, $\vec{H} = 0$, M^m is a minimal submanifold in S^n. This is a contradiction with that M^m is a proper biharmonic submanifold in S^n. We complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Since the scalar curvature $m(m-1)R$ is constant and $\bar{R} = R - 1 \geq 0$, from (1.6), (2.5), (3.14), (4.2) and $|\nabla h|^2 \geq m^2|\nabla^\perp \vec{H}|^2$ (see Lemma 3.2 of [13]), by a standard calculation, we have

$$\begin{aligned}\frac{3}{4}\Delta|A|^2 \geq & |\phi|^2(m(H^2+1) - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| - |\phi|^2) \\ \geq & \frac{m-1}{m}(|A|^2 - m\bar{R})\{m + m\bar{R} - \frac{m-2}{m}(|A|^2 - m\bar{R}) \\ & - \frac{m-2}{m}\sqrt{[|A|^2 + m(m-1)\bar{R}](|A|^2 - m\bar{R})}\} \geq 0.\end{aligned} \tag{4.12}$$

From (1.6) and (1.5) of Main Theorem in [12], we know that

$$\begin{aligned}\mathrm{Ric} \geq & \frac{m-1}{m}\{m(H^2+1) - \frac{m(m-2)}{\sqrt{m(m-1)}}H|\phi| - |\phi|^2\} \\ = & \frac{m-1}{m}\{m + m\bar{R} - \frac{m-2}{m}(|A|^2 - m\bar{R}) \\ & - \frac{m-2}{m}\sqrt{[|A|^2 + m(m-1)\bar{R}](|A|^2 - m\bar{R})}\} \geq 0.\end{aligned}$$

Therefore, we know that the Ricci curvature is bounded from below.

Now we consider the following smooth function on M^m defined by $f = -(|A|^2 + a)^{-1/2}$, where $a (> 0)$ is a real number. Obviously, f is bounded, so we can apply Proposition 3.1 to f. For any $\varepsilon > 0$, there is a point $x \in M^m$, such that at which f satisfies (3.15). By a simple and direct calculation, we have

$$f\Delta f = 3|df|^2 - \frac{1}{2}f^4\Delta|A|^2. \tag{4.13}$$

From (3.15) and (4.13), we have

$$\frac{1}{2}\Delta|A|^2(x) = f^{-4}(x)[3|df(x)|^2 - f(x)\Delta f(x)] < f^{-4}(x)[3\varepsilon^2 - \varepsilon f(x)]. \tag{4.14}$$

Thus, for any convergent sequence $\{\varepsilon_m\}$ with $\varepsilon_m > 0$ and $\lim_{m\to\infty}\varepsilon_m = 0$, there exists a point sequence $\{x_m\}$ such that the sequence $\{f(x_m)\}$ converges to f_0 (we can take a subsequence if necessary) and satisfies (3.15), hence, $\lim_{m\to\infty}\varepsilon_m[3\varepsilon_m -$

$f(x_m)] = 0$. From the definition of supremum and (3.15), we have $\lim_{m\to\infty} f(x_m) = f_0 = \sup f$ and hence the definition of f gives rise to $\lim_{m\to\infty} |A|^2(x_m) = \sup |A|^2$.

From (4.12) and (4.14), we have

$$\begin{aligned} f^{-4}(x_m)\frac{3}{2}[3\varepsilon_m^2 - \varepsilon_m f(x_m)] > &\frac{3}{4}\Delta|A|^2(x_m) \\ \geq &\frac{m-1}{m}(|A|^2(x_m) - m\bar{R})\{m + m\bar{R} - \frac{m-2}{m}(|A|^2(x_m) - m\bar{R}) \\ &- \frac{m-2}{m}\sqrt{[|A|^2(x_m) + m(m-1)\bar{R}](|A|^2(x_m) - m\bar{R})}\} \geq 0. \end{aligned} \tag{4.15}$$

Putting $m \to \infty$ in (4.15), we have

$$\begin{aligned} &\frac{m-1}{m}(\sup|A|^2 - m\bar{R})\{m + m\bar{R} - \frac{m-2}{m}(\sup|A|^2 - m\bar{R}) \\ &- \frac{m-2}{m}\sqrt{[\sup|A|^2 + m(m-1)\bar{R}](\sup|A|^2 - m\bar{R})}\} = 0. \end{aligned}$$

Thus, we have (i) $\sup|\phi|^2 = \frac{m-1}{m}(\sup|A|^2 - m\bar{R}) = 0$ and M^m is totally umbilical, that is, M^m has one distinct principal curvature, from Theorem 4.3 of [2], we know that M^m is an open part of $S^m(1/\sqrt{2})$, or (ii) $m + m\bar{R} - \frac{m-2}{m}(\sup|A|^2 - m\bar{R}) = \frac{m-2}{m}\sqrt{[\sup|A|^2 + m(m-1)\bar{R}](\sup|A|^2 - m\bar{R})}$ that is,

$$\sup|A|^2 = (m-1)\frac{m\bar{R}+2}{m-2} + \frac{m-2}{m\bar{R}+2}. \tag{4.16}$$

From (4.16) and (4.12), we know that $|A|^2$ is a subharmonic function on M^m. If the supremum $\sup|\phi|^2$ is attained at some point of M^m, by the maximum principle, we have $|A|^2 = constant$. Thus, (4.12) becomes equality. This implies that the equalities in Lemma of [1] hold. Thus we see that M^m has two distinct principal curvatures with multiplicities 1 and $m-1$. From Theorem 4.3 of [2], we know that M^m is an open part of the standard products $S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$. This completes the proof of Theorem 1.4. □

5. **Conclusions**

This article studies the proper biharmonic submanifolds in a unit sphere S^n. If the submanifolds satisfy certain geometric and rigidity properties, some characterizations of the two canonical examples of proper biharmonic submanifolds $S^{n-1}(1/\sqrt{2})$ and $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = n-1$, $m_1 \neq m_2$ are obtained, which give some partly affirmative answer to the Conjecture proposed by [2]. We notice that Theorem 1.1–Theorem 1.3 hold for all compact and complete non-compact proper biharmonic hypersurfaces (submanifolds) in S^n and Theorem 1.3 generalizes Theorem 1.2 to the case that the codimension $n - m \geq 2$. Therefore, we conclude some new rigidity Theorems of the proper biharmonic hypersurfaces (submanifolds) in S^n.

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