OPTIMIZATION OF INTERNAL FORCED CONVECTION THROUGH A DUCT BASED ON THE SECOND LAW

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In funcție de soluția aleasă pentru schimbătorul de căldură, procesul de convecție interioară poate fi realizat cu diferite distribuții de temperatură ale suprafețelor de separație dintre fluide. Atunci când fluxul total de căldură este impus, fiecare din aceste distribuții influențează ireversibilitățile viscoase şi termice asociate procesului. În majoritatea cazurilor însă, calculul vitezei de generare a entropiei este realizat cu ajutorul temperaturii medii a peretelui conductei. In acest fel, influența distribuției de temperatură a acestuia este in mare parte neglijată.

Utilizând drept criteriu viteza totală de generare a entropiei, obiectivul lucrării este acela de a identifica soluția optimă a procesului convectiv de transfer de căldură printr-o conductă în raport cu distribuția de temperatură a peretelui şi lungimea acestuia. Expresia vitezei totale de generare a entropiei este obținută prin integrarea distribuției sale liniare în lungul curgerii, iar procedura de minimizare are la bază teoria controlului optimal a lui Pontreaghin.

Depending on the heat exchanger solution, the heat transfer in internal forced convection can be fulfilled with different wall temperature distributions. If the overall rate of the heat flux is imposed, each of these distributions influences both, flow friction and heat transfer irreversibilities. In most cases, the calculus of entropy generation rate is performed with the aid of the mean temperature of the walls. In this way, the influence of the wall temperature distribution on irreversibility is neglected.

Using as criterion the overall rate of entropy generation, the objective of this paper is to identify the optimal solution of internal convection heat transfer through a duct with respect to the wall temperature distribution and the duct length. The expression of the overall entropy generation rate is obtained by integrating its linear distribution along the duct and the procedure of minimization relies on the optimal control theory of Pontryagin.

Keywords: second law, temperature distribution, entropy generation

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1. Introduction

Since the fundamental work of Bejan [1], many studies have been performed in the field of thermodynamic design of external or internal forced convection. More often, in the power-generation field, the heat rate $\dot{Q}$ is imposed, because the thermodynamic agent must be heated or cooled in order to reach an imposed temperature T_1 at the end of the process. The thermal and viscous irreversibilities, that accompany the heat transfer process, destroy the flow exergy at a rate that is proportional to the system rate of entropy generation, $\dot{S}_{gen}$ and the rate of lost available work (or lost exergy) results from well known Gouy-Stodola theorem:

$$\dot{W}_{lost} = T_{env}\dot{S}_{gen} \tag{1}$$

where T_{env} represents the environment temperature. The competition between thermal and viscous irreversibilities often allow to identify an optimum size or operating regime for which the rate of lost available work (or entropy generation rate) has a minimum value.

The rate of entropy generation can be computed at bulk or at continuum level. In the first case [1]-[3], the information of flow and heat transfer is obtained with the aid of dimensionless correlations of friction factor and Nusselt or Stanton numbers. Once they are available, this information allows the computation of viscous and thermal component of irreversibility. For the simplest laminar or turbulent convection heat transfer processes, occurring in boundary layers or ducts, the bulk level model could rely on the differential equations and the linear rate of entropy generation may be determined. But for the complex ones, like heat exchangers, the model computes directly the overall rate of entropy generation.

The continuum level [4]-[6] is more sophisticated because it uses the analytical or numerical solutions of velocity and temperature fields for determining the volumetric rate of entropy generation over the entire flow domain. At this level, the model provides a great precision of calculus and a true understanding of the irreversibility structure. For a turbulent convection process, the gap between the bulk and the continuum level of second law analysis can be found in [6].

Relying on the full solution of Navier-Stokes equations, the continuum level cannot be used in an optimization procedure which is based only on differential equations. Therefore, for this work the bulk level method of entropy generation calculus was retained.

There are two motivations to deal with in this paper. First, the entropy generation rate $\dot{S}_{gen}$ depends not only on the system size or mean temperature difference between fluid and wall, but also on the heat exchange solution, that determines the temperature distribution of the wall. Second, the process performances are always compared with the ideal case performances, for which the heat exchange proceeds without irreversibilities. Maybe the ideal case is not the most reasonable for comparison, because practically it can be never reached. One believes that it is useful to find the conditions for minimum exergy destruction first in the simplest case of internal forced convection.

2. Optimal problem formulation

Consider a mass flow rate, $\dot{m}$ which passes through a circular duct having the diameter D and the length l. The wall temperature is $T_w(x)$, while the bulk temperature of the stream $T(x)$, varies from T_0 in section x_0= 0, to T_1, in section x_1= l (see also *Fig. 1*). For this internal forced convection, the following hypotheses are considered: a) the thermodynamic agent is considered as ideal gas; b) in the initial section x_0= 0 the flow is already turbulent and fully developed. c) the heat transfer process can be neglected in the flow direction; d) the flow is assumed with nearly constant density ρ, viscosity μ and thermal conductivity λ.

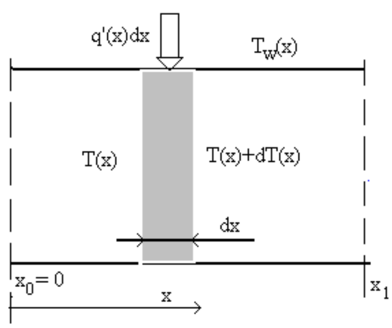

Fig. 1 Internal forced convection through a duct

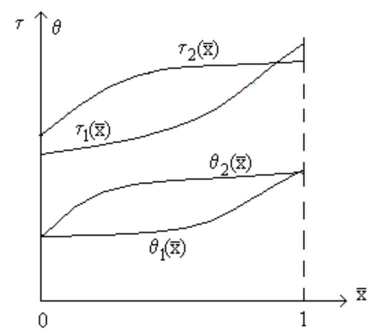

Fig. 2 Different wall and bulk temperature distributions

Using both, the first and the second law of thermodynamics for open systems, the following expressions can be easily obtained:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\bar{x}} = 4St\left[\tau(\bar{x}) - \theta(\bar{x})\right] \qquad (2)$$

$$\frac{1}{\pi\lambda}\dot{S}_{gen} = Nu\frac{[\tau(\bar{x})-\theta(\bar{x})]^2}{\tau(\bar{x})\theta(\bar{x})} + \frac{\xi(Re)\mu^3}{8D^2\rho_0^2\lambda T_0}Re^3\frac{1}{\theta(\bar{x})} \tag{3}$$

in which *St*, *Nu*, *Re* represents Stanton, Nusselt and Reynolds numbers, $\xi(Re)$ is the friction factor and:

$$\bar{x} = x/D \quad ; \quad \tau(\bar{x}) = T_w(D\bar{x})/T_0 \quad ; \quad \theta(\bar{x}) = T(D\bar{x})/T_0$$

Boundaries conditions must be added to these equations. Passing through the duct, the fluid must be heated (or cooled) from T_0 until T_1, so that:

$$\theta(0) = 1 \quad ; \quad \theta(1) = \theta_1 \tag{4}$$

Taking into account the hypotheses made above, both Nusselt and Stanton numbers are constant in the *x* direction due to the similarity of the flow and heat transfer.

If $\dot{m}$ and *D* are fixed, the boundary conditions (4) show that the heat transfer rate:

$$\dot{Q} = D\int_0^{\bar{x}_1} q'(\bar{x})\mathrm{d}\bar{x} = \dot{m}c_pT_0(\theta_1 - 1)$$

is imposed, while the value of the heat transfer rate per unit length q' depends on both the wall temperature distribution $\tau(x)$ and the duct length L=x_1. As is can be seen in *Figure 2*, two different wall temperature distributions $\tau_1(x)$ and $\tau_2(x)$ establish two different bulk temperature distributions of the stream $\theta_1(x)$ and $\theta_2(x)$ through eq. (2) and two distinct values of the overall rate of entropy generation:

$$\dot{S}_{gen} = \pi\lambda D\int_0^{\bar{x}_1}\dot{S}'_{gen}(\bar{x})d\bar{x} \tag{5}$$

through eq. (3). The objective is to find the wall optimal temperature distribution $\tilde{\tau}(\bar{x})$, or/and the length L=x_1, which minimizes the entropy generation rate (5) by verifying eq. (2) with boundary conditions (4).

Mathematically speaking, eq. (5) represents a functional. Therefore one defines ***T*** as the set of partially continuous functions $\tau : [0,1] \to \Omega$, which, introduced in eq. (2), ensure the boundary conditions (4) for the functions $\theta(x)$:

$$\boldsymbol{T} = \left\{ \tau \ / \ \frac{d\theta}{d\bar{x}} = 4St(\tau - \theta) \ ; \theta(0) = 1, \ \theta(1) = \theta_1 \right\} \tag{6}$$

On this set the functional $\boldsymbol{J}$: $\boldsymbol{T} \rightarrow \boldsymbol{R}$ is defined as:

$$\boldsymbol{J}(\tau, \bar{x}_1) = \frac{1}{\pi \lambda D} \dot{S}_{gen} = \int_0^{\bar{x}_1} \left\{ Nu \frac{[\tau(\bar{x}) - \theta(\bar{x})]^2}{\tau(\bar{x})\theta(\bar{x})} + \frac{\xi(\mathrm{Re})\mu^3}{8D^2 \rho_0^2 \lambda T_0} \mathrm{Re}^3 \frac{1}{\theta(\bar{x})} \right\} d\bar{x} \tag{7}$$

There are two problems for this functional: a) the problem with fixed length for which one finds only the optimal wall temperature distribution $\tilde{\tau} \in \boldsymbol{T}$ that minimizes the functional $\boldsymbol{J}$ at $\bar{x}_1$ fixed; b) the problem with free length, for which $\tilde{\tau} \in \boldsymbol{T}$ and $\bar{x}_1$ that realizes the absolute minimum value of $\boldsymbol{J}$ are looked for.

3. Optimal control theory applied to convection heat transfer

The solution of these problems can be found in many ways. It is useful to consider $\tau(x)$ as the control and $\theta(x)$ as the answer of thermodynamic system at this applied control. Then, it is possible to obtain the solution to this problem by using the optimal control theory [7].

The Hamilton Pontryagin function for the assumed problems is defined by:

$$H(\tau, \theta, \psi) = -Nu \frac{(\tau - \theta)^2}{\theta \tau} - (a \,\mathrm{Re})^3 \frac{1}{\theta} + 4St\psi(\tau - \theta) \tag{8}$$

where a is a constant with respect to x:

$$a = \sqrt[3]{\xi(\mathrm{Re})\mu^3 / (8D^2 \rho^2 \lambda T_0)}$$

and $\psi = \psi(x)$ is an auxiliary function that satisfies the following differential system:

$$\frac{d\theta}{d\bar{x}} = \frac{\partial H}{\partial \psi} \quad ; \quad \frac{\mathrm{d}\psi}{\mathrm{d}\bar{x}} = -\frac{\partial H}{\partial \theta} \tag{9}$$

which has the boundary conditions (4).

If it exists, the function $\tilde{\tau} \in \boldsymbol{T}$ that minimizes the functional (7) is named optimal control and the solution of eq. (2) corresponding to this function is called optimal trajectory. The specific form of the optimal control $\tilde{\tau}(\bar{x})$ is established using Pontryagin's principle of maximum [7]. For our problems, this principle is equivalent to:

$$\partial H\left(\tau, \tilde{\theta}, \tilde{\psi}\right) / \partial \tau = 0 \tag{10}$$

but is essential to mention that the solution of this equation must belong to the set ***T***. Solving the above equation it results:

$$\tilde{\tau}^2 = \frac{Nu\tilde{\theta}^2}{Nu - 4St\tilde{\psi}\tilde{\theta}} \tag{11}$$

Using eq. (11), the differential system (9) becomes:

$$\frac{d\tilde{\theta}}{d\tilde{x}} = 4St\tilde{\theta}\left[\sqrt{\frac{Nu}{Nu - 4St\tilde{\psi}\tilde{\theta}}} - 1\right]; \qquad \frac{d\tilde{\psi}}{d\tilde{x}} = 4St\tilde{\psi}\left[1 - \sqrt{\frac{Nu}{Nu - 4St\tilde{\psi}\tilde{\theta}}}\right] - \frac{(a\,\mathrm{Re})^3}{\tilde{\theta}^2} \tag{12}$$

and has the boundary conditions (4). There are two differential equations with two boundary conditions so that the system (12) can be solved.

4. The problem with fixed length

Being rather complicated, the form of this system does not allow finding an analytic solution. Of course it is possible to solve it numerically very easy, but the numerical solution is not always able to show some important features of the problem. This is the reason for adding a supplementary assumption that refers to the length of the duct. If the length of the duct has a “relatively short” value, the temperature difference $\Delta\theta = \tau - \theta$, that must be applied to the fluid to reach at outlet the temperature T_1, has a high value. In this case, thermal dissipation is very high comparatively with viscous dissipation that can be neglected. In the opposite case, when the length of the duct has a “relatively long” value, the temperature differences become small and the approximation $\tau\theta = (\theta + \Delta\theta)\theta \cong \theta^2$ works. These are two specific cases for which an analytic solution can be found. It remains to establish when a duct has “relatively short” or “relatively long” length, because its length has to be connected with Nusselt or Stanton numbers. The form of the system (12) suggests that the “length of the duct” must be connected with the quantity:

$$M = \max_{\tilde{x} \in [\tilde{x}_0, \tilde{x}_1]} \left| (4St/Nu)\tilde{\psi}\tilde{\theta} \right| \tag{13}$$

Then, using the decomposition:

$$\sqrt{\frac{Nu}{Nu - 4St\tilde{\psi}\tilde{\theta}}} = \left(1 - \frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^{-1/2} = 1 + \frac{1}{2}\frac{4St}{Nu}\tilde{\psi}\tilde{\theta} + \frac{1\cdot 3}{2\cdot 4}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^2 + \ldots. \tag{14}$$

which is valid for M<1, the last equation of system (12) becomes:

$$\frac{\mathrm{d}\tilde{\psi}}{\tilde{\psi}} = -\frac{Nu}{\tilde{\psi}\tilde{\theta}}\left\{\left[\frac{1}{2}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^3 + ...\right] + \frac{(aRe)^3}{Nu\tilde{\theta}}\right\}\mathrm{d}\bar{x} \tag{15}$$

The duct will be “short” if:

$$\frac{1}{2}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^2 >> \frac{(aRe)^3}{Nu\tilde{\theta}} \tag{16}$$

because the influence of viscous ireversibility can be neglected from the point of view of entropic analisys. By contrast, if:

$$\frac{1}{2}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^2 \approx \frac{(a\,\mathrm{Re})^3}{Nu\tilde{\theta}} >> \frac{1\cdot 3}{2\cdot 4}\left(\frac{4St}{Nu}\tilde{\psi}\tilde{\theta}\right)^3 \tag{17}$$

the duct will be “long”, such that, from the decomposition (14) one will keep only the first two terms. These are two extreme cases for which an analytical solution can be found.

4.1 Optimal solution for “short duct”

As it has been shown previously, in this case, viscous dissipation is negligible in comparison with thermal dissipation. With this assumption, the general solutions for eqs. (12) are:

$$\tilde{\psi} = c_1 / \tilde{\theta} \qquad \tilde{\theta} = c_2 \exp\left[4St\left(1 \Big/ \sqrt{1 - \frac{4St}{Nu}c_1} - 1\right)\bar{x}\right]$$

The constants of integration c_1 and c_2 are determined from boundary conditions (4) so that the solution in this case is:

$$\tilde{\theta}(\bar{x}) = \theta_1^{\bar{x}/\bar{x}_1} \;\; ; \;\; \hat{\psi} = \frac{Nu}{4St}\left[1 - \left(\frac{4St}{\ln\theta_1^{1/\bar{x}_1} + 4St}\right)^2\right]\frac{1}{\hat{\theta}} \;\; ; \;\; \hat{\tau} = \hat{\theta}\left[1 + \frac{\ln\theta_1^{1/\bar{x}_1}}{4St}\right] \tag{18a,b,c}$$

When the wall temperature distribution is $\tilde{\tau}(\bar{x})$, the entropy generation rate can be calculated with:

$$\left[\dot{S}_{gen,Q}\right]_{\min} = \pi\lambda D\bar{x}_1 \frac{Nu\left(\ln\theta_1^{1/\bar{x}_1}\right)^2}{4St\left(4St + \ln\theta_1^{1/\bar{x}_1}\right)} \tag{19}$$

and has a minimum value. For any other wall temperature distribution, $\tau \neq \tilde{\tau}$ that verifies the condition $\theta(1)=\theta_1$, the entropy generation rate is $S_{gen,Q} > [S_{gen,Q}]_{\min}$. One has also to notice that the minimum entropy generation rate decreases when the length of the duct increases due to the fact that the temperature difference between wall and fluid diminishes.

The solution (18) must satisfy the inequality (16), that becomes:

$$\boldsymbol{O}\left[1-\left(\frac{4St}{\ln\theta_1^{1/\bar{x}_1} + 4St}\right)^2\right]^2 >> \boldsymbol{O}\left[\frac{2(aRe)^3}{Nu\hat{\theta}}\right] \tag{20}$$

where $\boldsymbol{O}$ denotes the order of magnitude. Usually, the right hand side of inequality (20) is $\boldsymbol{O}(10^{-1})$-$\boldsymbol{O}(10^{-3})$. Then, the validity of the optimal solution (11) is given by $\boldsymbol{O}[(D/L)\ln\theta_1/(4St)]$. Indeed if $L=x_1-x_0$ is small and $\theta_1>>1$, one finds that $\boldsymbol{O}[(D/L)\ln\theta_1/(4St)]=\boldsymbol{O}(10)$, so that the magnitude of the left hand side of inequality (20) is $O(1)$. In these situations the duct can be considered "short" from the point of view of entropy analysis. In the opposite case when L is high and θ_1 is small, it results that solution (11) is given by $\boldsymbol{O}[(D/L)\ln\theta_1/(4St)]$. Indeed, if $L=x_1-x_0$ is high and θ_1 is small, one finds that $\boldsymbol{O}[(D/L)\ln\theta_1/(4St)]=\boldsymbol{O}(10^{-1})$, so that both sides of inequality (20) have the same order of magnitude and the condition (11) cannot be applied. The above analysis shows that the quantity:

$$\tilde{L} = \frac{(D/L)\ln\theta_1}{4St} \tag{21}$$

allows to identify one dimensionless parameter that gives a quantitative understanding to "short duct" or "long duct" notion after how $\boldsymbol{O}(\tilde{L}) = \boldsymbol{O}(10)$ or $\boldsymbol{O}(\tilde{L}) = \boldsymbol{O}(10^{-1})$.

4.2 Optimal solution for "long duct"

In this case the value of viscous dissipation becomes comparable with thermal dissipation value and cannot be neglected. Then, keeping the first two term of decomposition (14), the system (12) becomes:

$$\frac{d\widetilde{\theta}}{\widetilde{\theta}} = \frac{8St^2}{Nu}\widetilde{\psi}\widetilde{\theta}d\overline{x}; \quad \frac{d\widetilde{\psi}}{\widetilde{\psi}} = -\frac{8St^2}{Nu}\widetilde{\psi}\widetilde{\theta}d\overline{x} - (a\,\mathrm{Re})^3 \frac{d\overline{x}}{\widetilde{\theta}^2\widetilde{\psi}} \tag{22}$$

and the expression of optimal control (11) has the form:

$$\widetilde{\tau} = \frac{2St}{Nu}\widetilde{\psi}\widetilde{\theta}^2 + \widetilde{\theta} \tag{23}$$

It is easy to show that the differential system (22) corresponds to Hamilton-Pontryagin's function (8) in which the denominator τθ of the first term was replaced by θ^2. In the beginning of this paper has been shown that this approximation corresponds to the long duct case. This means that approximation, when using the dimensionless quantity $\ln\theta_1/(bSt)$, works also very well. The last approximation is more favorable because it uses the initial values of the problem that define both the flow and the heat transfer processes.

From system (22) one infers the following solution:

$$\widetilde{\theta}(\overline{x}) = \left(\frac{b_1}{4c_1}\right)^2 \left[c_2 \exp(c_1\overline{x}) - \frac{b_2}{c_2}\exp(-c_1\overline{x})\right]^2 \tag{24a}$$

$$\widetilde{\psi}(\overline{x}) = \pm\frac{2|c_1|}{b_1}\left|\frac{c_2 \exp(c_1\overline{x}) + \frac{b_2}{c_2}\exp(-c_1\overline{x})}{c_2 \exp(c_1\overline{x}) - \frac{b_2}{c_2}\exp(-c_1\overline{x})}\right| \frac{1}{\widetilde{\theta}(\overline{x})} \tag{24b}$$

$$\widetilde{\tau}(\overline{x}) = \widetilde{\theta}(\overline{x}) \pm \frac{|c_1|}{2St}\left|\frac{c_2 \exp(c_1\overline{x}) + \frac{b_2}{c_2}\exp(-c_1\overline{x})}{c_2 \exp(c_1\overline{x}) - \frac{b_2}{c_2}\exp(-c_1\overline{x})}\right| \widetilde{\theta}(\overline{x}) \tag{24c}$$

In the above relations, the sign "+" corresponds to the heated fluid and the sign "-" corresponds to the cooled fluid. These signs are selected taking into account that the derivative $d\widetilde{\theta}/d\overline{x}$ must be positive when the fluid is heated and negative in the opposite case. Using the optimal solution (24), the Hamilton-Pontyagin function takes the form:

$$H(\widetilde{\tau}, \widetilde{\theta}, \widetilde{\psi}) = \frac{St^2}{Nu}\left(\frac{4c_1}{b_1}\right)^2 > 0 \tag{25}$$

and it is always greater than zero. The integration constant c_1 and c_2 are established as solution of the algebraic system:

$$\left(\frac{b_1}{4c_1}\right)^2\left(c_2-\frac{b_2}{c_2}\right)^2=1\,;\qquad \left(\frac{b_1}{4c_1}\right)^2\left[c_2\exp(c_1\bar{x}_1)-\frac{b_2}{c_2}\exp(-c_1\bar{x}_1)\right]^2=\theta_1 \qquad (26)$$

which was obtained from boundary conditions (4), where:

$$b_1=\frac{8St^2}{Nu}\;;\qquad \mathrm{b}_2=\frac{2(aRe)^3}{b_1} \qquad (27)$$

The optimal issue has solutions only if the values of the initial conditions ensure the compatibility of system (26), and the auxiliary function $\tilde{\psi}$ exists for $\bar{x}\in[0,\bar{x}_1]$ On the other hand, if there is a set of solutions, the one that satisfies the maximum principle of Pontryagin (10) must be selected. Next, the solutions of system (26) in the case of heating the fluid ($\theta_1>1$) should be discussed. It is obvious that, if (c_1', c_2') represents a solution of system (26), then (c_1', $-c_2'$), ($-c_1'$, b_2/c_2') and ($-c_1'$, $-b_2/c_2'$) are also solutions, but for any of them, the form of optimal solution (24) remains unchanged. For this reason, these solutions are indistinct. It can be shown that, if the initial values of the problem (meaning the fluid nature, l, θ_1 and Re) obey the condition:

$$\sqrt{\theta_1}>1+\frac{b_1\sqrt{b_2}}{2}\bar{x}_1 \qquad (28)$$

then the system (26) has always a solution so that $c_1>0$ and $\mathrm{c}_2>b_2^{1/2}$. This solution always determines the wall optimal temperature distribution $\tilde{\tau}(\bar{x})$ by eq. (24). The minimum value of entropy generation rate can be obtained from:

$$\left(\dot{S}_{gen}\right)_{\min}=\pi\lambda D\left[\frac{St^2}{Nu}\left(c_2-\frac{b_2}{c_2}\right)^2\bar{x}_1+2c_2\left(1-\frac{\exp(c_1\bar{x}_1)}{\sqrt{\theta_1}}\right)\right] \qquad (29)$$

and it can be calculated following the resolution of the algebraic system (26). The thermal component expression of the minimum entropy generation rate has the form:

$$\left(\dot{S}_{gen,Q}\right)_{\min} = \pi\lambda D\left[\frac{St^2}{Nu}\left(c_2 - \frac{b_2}{c_2}\right)^2 \bar{x}_1 + c_2\left(1 - \frac{\exp(c_1\bar{x}_1)}{\sqrt{\theta_1}}\right)\right] \tag{30a}$$

and the viscous component expresion results from:

$$\left(\dot{S}_{gen,V}\right)_{\min} = \pi\lambda D c_2\left(1 - \frac{\exp(c_1\bar{x}_1)}{\sqrt{\theta_1}}\right) \tag{30b}$$

For this case, the thermal component of the entropy generation rate is always greater than the viscous component.

Figure 3 presents the optimal wall and bulk temperature distribution for three values of the Reynolds number. The air, used as thermodynamic agent, is heated from θ_0 = 1.0 until θ_1 = 1.05 in a smooth duct having D = 0.03m and L = 1m. In order to perform calculus, the Colburn analogy has been used. One particularity of the optimal bulk temperature distribution is that it varies approximately linear and the other is that practically it does not depend on Reynolds number. The optimal distribution of wall temperature closely follows the variation of the optimal bulk temperature distribution, but it depends on the Reynolds number because the exchanged heat rate increases with the mass flow rate of the fluid.

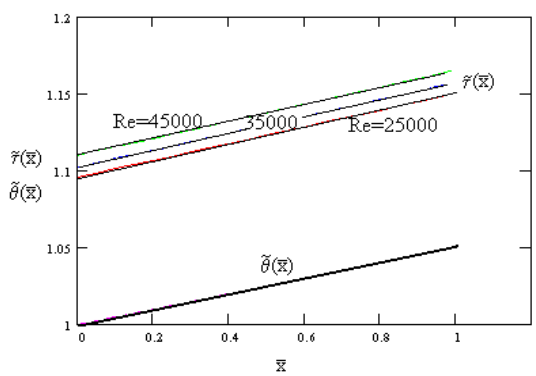

Fig. 3 Optimal temperature distribution in internal forced convection through a smooth duct

5. The problem with free length

In this case, the duct length L=x_1-x_0 becomes one of the variables of the problem. Because the overall heat flux is fixed by the boundary conditions (3), the thermal entropy generation rate component decreases and the viscous entropy generation rate component decreases if the length of the duct grow up. For this reason, the problem with free length will be solved in the "long duct' hypothesis.

The additional condition for the new variable of the problem is [7]:

$$H\left(\tilde{\tau},\tilde{\theta},\tilde{\psi}\right)=\frac{St^2}{Nu}\left(\frac{4c_1}{b_1}\right)^2=\frac{St^2}{Nu}\left(c_2-\frac{b_2}{c_2}\right)^2\equiv 0 \tag{31}$$

the second equality resulting from the first ecuation of algebraic system (26). From the above condition one finds that $c_1=0$ and $c_2=\pm\sqrt{b_2}$, but these values lead to indeterminate operations for the second equation of the algebraic system (26). Using the limit rule and taking into account the corresponding signs of temperature shape for heated or cooled fluid, it results:

$$\sqrt{\theta_1}=1\pm\frac{b_1\sqrt{b_2}}{2}\bar{x}_1 \tag{32}$$

so the optimal length of the duct has the expression:

$$\bar{x}_1=\pm\frac{1}{2}\left(\sqrt{\theta_1}-1\right)\frac{\sqrt{Nu}}{St\sqrt{(aRe)^3}} \tag{33}$$

In this condition, eqs. (24) become:

$$\tilde{\theta}(\bar{x})=\left[1+\left(\sqrt{\theta_1}-1\right)\frac{\bar{x}}{\bar{x}_1}\right]^2 \tag{34a}$$

$$\tilde{\tau}(\bar{x})=\left[1\pm\frac{1}{1+\left(\sqrt{\theta_1}-1\right)\left(\bar{x}/\bar{x}_1\right)}\sqrt{\frac{(aRe)^3}{Nu}}\right]\tilde{\theta}(\bar{x}) \tag{34b}$$

$$\tilde{\psi}(\bar{x})=\pm\frac{\sqrt{Nu(aRe)^3}}{2St\left[1+\left(\sqrt{\theta_1}-1\right)\left(\bar{x}/\bar{x}_1\right)\right]^3} \tag{34c}$$

The function defined by eq.(34c) satisfies the conditions of Pontryagin theorem, so that the functions $\tilde{\theta}(\bar{x})$and $\tilde{\tau}(\bar{x})$ represent the optimal problem solution. This solution allows finding the absolute minimum value of entropy generation rate:

$$\left(\dot{S}_{gen}\right)_{\min,m} = \pi\lambda D\left(1-\frac{1}{\sqrt{\theta_1}}\right)\frac{(aRe)^{3/2}\,Nu^{1/2}}{St} \tag{35}$$

for which:

$$\left(\dot{S}_{gen,Q}\right)_{\min,m} = \left(\dot{S}_{gen,V}\right)_{\min,m} = \tfrac{1}{2}\left(\dot{S}_{gen}\right)_{\min,m} \tag{36}$$

The last relations confirm the equipartition principle of optimal dissipations, stated by Bejan. It is obvious that *Re*, *Nu* and *St* are connected by thermal correlations but their use has been avoided in order to keep the optimal solution

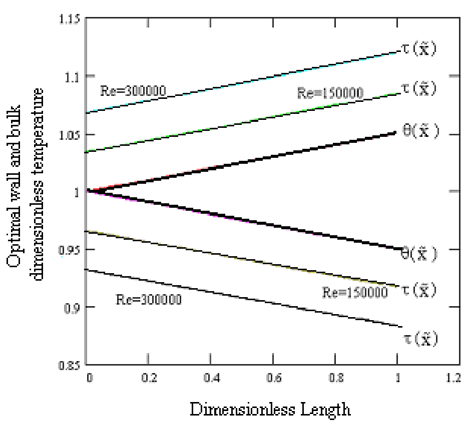

Fig. 4a Optimal wall and bulk temperature distribution for heated and cooled gas

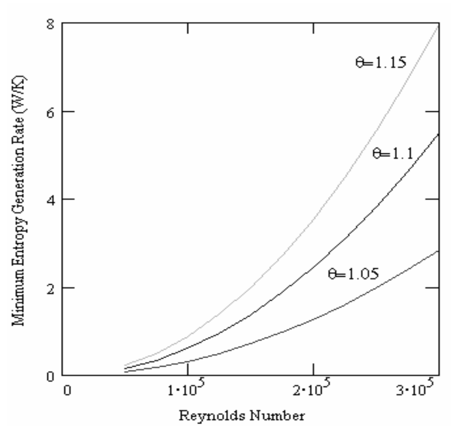

Fig. 4b Minimum entropy generation rate for heated gas

generality.

Figure 4 shows the optimal solution for the air flow through a circular duct having D = 0.1 m. The inlet temperature of air is T_0=298 K and for calculus the Colburn analogy has been used. The optimal wall and bulk temperature distributions are quasi-linear with respect to the length of duct. As it is shown in *Figure 4a*, the difference $|\tau-\theta|$ increases with the Reynolds number because the heat flux that must be transferred to fluid (or from fluid) enhances with mass flow rate. As a consequence, the thermal dissipation increases and practically determines the doubling of the entropy generation rate because the irreversibility distribution ratio is equal to the one for the optimal solution (fig. 4b). This still remains the principal reason for which the optimal length decreases when the

Reynolds number increases. The high Reynolds numbers determine either the augmentation of heat transfer rate per unit length or the viscous entropy generation rate per unit length so that the length where it becomes equal to thermal entropy generation rate becomes smaller. The values of the optimal length (33) are very high and rather not utilizable. But it must be emphasized that an absolute minimum value of entropy generation rate in the internal forced convection through a duct exists and the heat transfer design that leads to it is unique.

6. Numerical example

Table 1 **Analysis of the entropy generation rate for some wall temperature distribution**

Wall Temperature Distribution	L [m]	$\dot{S}_{gen,Q}$ [W/K]	$\dot{S}_{gen,V}$ [W/K]	$\dot{S}_{gen}$ [W/K]	$\frac{\dot{S}_{gen}}{\left[\dot{S}_{gen}\right]_{\min,m}}$	$\frac{\dot{S}_{gen}}{\left[\dot{S}_{gen}\right]_{\min}}$
Eq. (33b)	3.92	0.0186	0.0186	0.0372	1	1
Eq. (22c)	1.0	0.0733	0.00474	0.078	2.096	1
$\dot{q}'_w = const$	1.0	0.0772	0.00588	0.0829	2.22	1.062
T_w=const	1.0	0.0786	0.00583	0.0844	2.27	1.083

As an example the connective heat transfer of air which is heated from θ_0=1.0 until θ_1=1.05 in a smooth duct, having D = 0.03m and L = 1m is considered. At x_0 section, the bulk temperature of the flow is T_0=290K. In *Table 1* the entropy generation rate for many wall temperature distributions are compared. The calculus was done using the hypothesis of the "long duct". If the wall temperature distribution corresponds to the problem with fixed length, the overall entropy generation rate (and obviously the lost available work) is twice larger compared to the problem of free length. Additionally, the classical wall temperature distributions $\dot{q}'_w = const.$ or $T_w = const.$ determines an approximately 10% growth for the overall entropy generation rate. Finally one notes that the wall temperature distribution induced by the boundary condition $\dot{q}'_w = const.$ seems to be more advantageous than other technical feasible distributions.

In the case of heat exchangers design, the criterion of minimum entropy generation is very often combined with thermo-economic optimization procedures. In this case, considering in on one hand the costs of investment and on the other the lost exergy costs, the optimal length of ducts (or of heat transfer surface) will be shorter than that resulting from (33b). From this point of view, the entropy generation minimization has the merit of revealing the minimum costs of lost exergy, from which one starts the thermo-economic optimization.

7. Conclusions

The analysis made in this paper shows that, in the case of internal forced convection through a duct, energy dissipation depends on the wall temperature distribution. This dependence allows finding the optimal conditions for which an imposed overall heat flux is transferred with minimum energy dissipation.

For a fixed length of the duct, there is an optimal temperature distribution of the wall that minimizes the entropy generation rate. This temperature distribution is specified by eq. (18) only for thermal dissipation or by eq. (24) when considering both viscous and thermal dissipation and relatively long ducts. For these optimal distributions the entropy generation rate has a minimum value that results from eq. (19) and (29) respectively. On the other hand, *Figure* 3 suggests that the counter-flow heat transfer is more favorable from the point of view of energy dissipation.

In the case of free length, the absolute minimum value of the entropy generation rate is given by eq. (35) which is obtained if the wall temperature distributions correspond to eq. (34b) and the length of the duct has the value obtained from eq. (33). At this optimal regime, the thermal component of entropy generation rate is equal to the viscous component, which confirms the equipartition principle of optimal dissipations. In these equations, the sign "+" corresponds to the heated fluid and the sign"-" correspond to the cooled fluid. Any other wall temperature distribution, that ensure the transfer of the same overall heat flux, leads to a higher value of the entropy generation rate than the value given by eq. (35) and (36).

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