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STUDY OF A DISC-SHAPED EARTH ELECTRODE INJECTING CURRENT INTO AN EXPONENTIALLY INCREASING CONDUCTIVITY SOIL

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Lucrarea tratează studiul unei prize de pământ circulară, de grosime neglijabilă, plasată intr-un mediu neomogen cu o conductivitate electrică ce creşte exponenţial în raport cu adâncimea solului. Ecuaţiile diferenţiale de ordinul doi cu derivate parţiale ale potenţialului electric, obţinute din rezolvarea problemei de regim staţionar în domeniul neomogen, sunt determinate în baza metodei separării variabilelor. Soluţia analitică astfel obţinută permite estimarea valorii absolute a intensităţii câmpului electric, a vectorului Poynting precum şi rezistenţei electrice de dispersie a prizei de pământ în mediul considerat.

This article treats the injection of a direct current through an above-ground circular plate earth electrode is studied. A non-homogeneous soil having an exponentially increasing conductivity is considered. The particularities of the problem allow the use of the separation of variables method for solving the homogeneous second order PDE verified by the electric potential. The analytically obtained solution is used to calculate the moduli of the electric intensity and of the Poynting's vector, respectively, and finally the formula of the earth electrode resistance is derived. Evaluating the limit of this relationship, the homogeneous case formula is also obtained.

Keywords: earth electrode, inhomogeneous conductivity soil, earth electrode resistance

1. Introduction

Except earthing, another important use of the earth electrodes concern the measurement of the soil conductivity and the calculation of the earth electrode resistance. These parameters may provide important information regarding the

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geological structure of the tested soil, especially for mineral and fossil deposit detection purposes [1-3]. In many practical cases, due to their complexity, the assumed homogeneous structure of the soil (even if several different layers of the sort are taken into account) is not sufficient for an accurate description of the real problem. For example, the non-uniform absorption of underground water by a dry porous soil to its surface, leads to a continuous variation of the conductivity, even for geologically uniform structures of the terrestrial crust. In spite of the vast literature available nowadays, authors are mainly focusing on the shape of the device, but are still modeling inhomogeneous soil as a stack of homogeneous layers. Several *in-situ* observations lead geologist to the conclusion that under certain physical conditions of the soil a very good approximation of the variation law of the conductivity may be an exponentially increasing one [4].

Therefore, this article aims to study the injection of a direct current of intensity i through an above-ground circular plate of radius a into a soil characterized by the following variation function:

$$\sigma(z) = \sigma_0 \, \mathrm{e}^{+\frac{z}{\lambda}}, \tag{1}$$

where σ_0 is the conductivity at the surface of the ground, z is a spatial coordinate perpendicular to the separation plane, and λ (m) is a real constant, as depicted in Fig. 1.

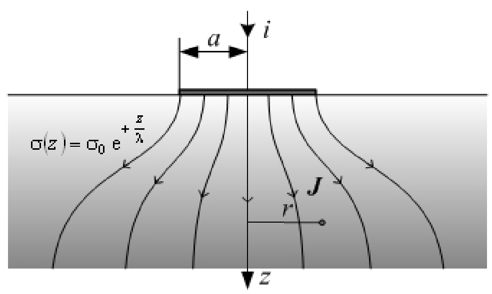

Fig. 1 – Disc-shaped earth electrode injecting current into an inhomogeneous soil.

The study begins from the steady-state local form of the charge conservation law:

$$\mathrm{div}\,\boldsymbol{J} = 0 \tag{2}$$

The left-hand side of (2) is expanded by substituting $\boldsymbol{J} = \sigma\boldsymbol{E}$:

$$\mathrm{div}(\sigma\boldsymbol{E}) = \boldsymbol{E}\cdot\mathrm{grad}\,\sigma + \sigma\,\mathrm{div}\,\boldsymbol{E}, \tag{3}$$

and substituting also $\boldsymbol{E} = -\,\mathrm{grad}\,V$, we have:

$$\Delta V = -\frac{\mathrm{grad}\,\sigma\cdot\mathrm{grad}\,V}{\sigma}, \tag{4}$$

where $\Delta V = \mathrm{div}\,\mathrm{grad}\,V$.

The axis-symmetric configuration presented by the geometry of the problem recommends the use of the cylindrical system [5, 6] of coordinates (r, φ, z), where $\dfrac{\partial \sigma}{\partial r} = 0$, $\dfrac{\partial \sigma}{\partial \varphi} = 0$ and $\dfrac{\partial V}{\partial \varphi} = 0$, and therefore:

$$\operatorname{grad}\sigma = \frac{1}{r}\left(r\frac{\partial\sigma}{\partial r}\boldsymbol{u_r} + \frac{\partial\sigma}{\partial\varphi}\boldsymbol{u}_\varphi + r\frac{\partial\sigma}{\partial z}\boldsymbol{u_z} \right) = \frac{\partial\sigma}{\partial z}\boldsymbol{u_z} \tag{5}$$

$$\operatorname{grad} V = \frac{1}{r}\left(r\frac{\partial V}{\partial r}\boldsymbol{u_r} + \frac{\partial V}{\partial\varphi}\boldsymbol{u}_\varphi + r\frac{\partial V}{\partial z}\boldsymbol{u_z} \right) = \frac{\partial V}{\partial r}\boldsymbol{u_r} + \frac{\partial V}{\partial z}\boldsymbol{u}_z \tag{6}$$

and

$$\Delta V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2}, \tag{7}$$

where $\boldsymbol{u}_r$, $\boldsymbol{u}_\varphi$ and $\boldsymbol{u}_z$ are the unit vectors of the cylindrical system of coordinates.

2. Current injection into a exponentially increasing conductivity soil

Taking now into account the assumed variation of conductivity given by (1), and by substituting the gradients given by (5) and (6), the right-hand side of (4) becomes:

$$-\frac{\operatorname{grad}\sigma \cdot \operatorname{grad} V}{\sigma} = -\frac{1}{\lambda\,\sigma(z)}\sigma_0\, e^{+\frac{z}{\lambda}}\, \boldsymbol{u}_z \cdot \left(\frac{\partial V}{\partial r}\boldsymbol{u}_r + \frac{\partial V}{\partial z}\boldsymbol{u}_z \right) = -\frac{1}{\lambda}\frac{\partial V}{\partial z}. \tag{8}$$

The new form of the second order PDE for the electric potential is obtained:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{\lambda}\frac{\partial V}{\partial z} = 0. \tag{9}$$

The homogeneous PDE given by (9) is solved using the separation of variables method, by expressing the potential function as a product of two independent single-variable functions:

$$V(r,z) = R(r)\cdot Z(z). \tag{10}$$

Substitution of (10) in (9) gives:

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} \right) + \frac{1}{Z}\left(\frac{\partial^2 Z}{\partial z^2} + \frac{1}{\lambda}\frac{\partial Z}{\partial z} \right) = 0. \tag{11}$$

By denoting the first term of the left-hand side of (11) by p^2, and the second by $-p^2$, this equation can be divided into two separate ordinary differential equations:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - p^2 R = 0 \quad \text{and} \tag{12}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + \frac{1}{\lambda}\frac{\mathrm{d}Z}{\mathrm{d}z} + p^2 Z = 0, \tag{13}$$

Equations (12) and (13) are solved, and the following general solutions are obtained:

$$R_p(r) = C_{1p}\,\mathrm{I}_0(pr) + C_{2p}\,\mathrm{K}_0(pr) \quad \text{and} \tag{14}$$

$$Z_p(z) = D_{1p}\cdot \mathrm{e}^{\frac{-1+\sqrt{1-4p^2\lambda^2}}{2\lambda}z} + D_{2p}\cdot \mathrm{e}^{\frac{-1-\sqrt{1-4p^2\lambda^2}}{2\lambda}z} \tag{15}$$

The characteristics of the problem determine the following restricting conditions that are to be imposed on the potential two-variable function:

$$V(r,0) = \begin{cases} \text{constant}, & \text{for } r \le a \\ \text{finite}, & \text{for } r > a. \end{cases}$$

At the same time, inside the conducting half-space, the potential decreases to zero for either $r \to \infty$ or $z \to \infty$ (the null reference of the potential is chosen at infinity). One can notice that $\mathrm{I}_0(\infty) \to \infty$ and, for assuring a finite value of the potential, the separation parameter has to be purely imaginary, of the form $p = \mathrm{j}k$, since $I_0(\mathrm{j}k) = \mathrm{j}J_0(k)$ and $\mathrm{J}_0(\infty) \to 0$. Similarly, because $K_0(0) \to \infty$ it results that $C_{2p} = 0$. Note that after substituting the parameter *p*, we have $-1+\sqrt{1+4k^2\lambda^2} > 0$ at the numerator of the first exponential of (15). Therefore, to prevent an infinite value of $Z_p(z)$ for $z \to \infty$, the first term of the right-hand side of (15) must vanish and hence $D_{1p} = 0$. With these adopted constants the general solution for the potential, calculated by integrating over all the positive values of the new considered separation parameter *k*, becomes:

$$V(r,z) = \int_0^\infty A_k\,\mathrm{J}_0(kr)\cdot \mathrm{e}^{\frac{-1-\sqrt{4k^2\lambda^2+1}}{2\lambda}z}\,\mathrm{d}k, \tag{16}$$

where $A_k = C_{1k}D_{2k}$ and $-1-\sqrt{4k^2\lambda^2+1} < 0$ for all possible values of *k*.

The axial component of the electric field intensity results by taking the *z*-component of $\boldsymbol{E} = -\operatorname{grad} V$ (see (6)):

$$E_z(r,z) = -\frac{\partial V(r,z)}{\partial z} = \int_0^\infty A_k\, \mathrm{J}_0(kr)\frac{1+\sqrt{4k^2\lambda^2+1}}{2\lambda}\cdot \mathrm{e}^{\frac{-1-\sqrt{4k^2\lambda^2+1}}{2\lambda}z}\,\mathrm{d}k\,. \quad (17)$$

The z-component of the current density at the surface of the ground ($z = 0$) is:

$$J_z(r,0) = \sigma_0\, E_z(r,0) = \sigma_0 \int_0^\infty A_k\, \mathrm{J}_0(kr)\frac{1+\sqrt{1+4k^2\lambda^2}}{2\lambda}\,\mathrm{d}k\,. \quad (18)$$

In order to determine the constant A_k, restricting conditions for the current density at $z = 0$ have to be established. Let us assume now a uniform current density $\boldsymbol{J}$ inside the circular plate, and that the current enters the soil perpendicularly to the contact area. In other words, in that region $\boldsymbol{J}$ is vertical and downward oriented, so that

$$J_z(r,0) = \begin{cases} \dfrac{i}{\pi a^2}, & \text{for } r \le a \\ 0, & \text{for } r > a \end{cases}. \quad (19)$$

Obviously, (18) and (19) cannot be properly compared to extract the constant A_k. A single compact relationship that replaces (19) is needed. A possible approach is the one given by the Fourier-Bessel integral transform [7], which states that a piecewise continuous function f having a bounded variation in (0, ∞) may admit an expansion of the form:

$$f(r) = \int_0^\infty k\,\mathrm{J}_0(kr)\,\mathrm{d}k \int_0^\infty \rho\,\mathrm{J}_0(k\rho) f(\rho)\,\mathrm{d}\rho\,. \quad (20)$$

The stated boundary conditions in (19) impose that

$$f(\rho) = \begin{cases} \dfrac{i}{\pi a^2}, & \text{for } \rho \le a \\ 0, & \text{for } \rho > a \end{cases} \quad \text{Besides, } \int_0^\infty \rho\,\mathrm{J}_0(k\rho)\,\mathrm{d}\rho = \int_0^a \rho\,\mathrm{J}_0(k\rho)\,\mathrm{d}\rho = \frac{a}{k}\mathrm{J}_1(ka).$$

Consequently, (20) becomes:

$$f(r) = J_z(r,0) = \frac{i}{\pi a}\int_0^\infty \mathrm{J}_0(kr)\,\mathrm{J}_1(ka)\,\mathrm{d}k\,. \quad (21)$$

By comparing now (18) and (21) the constant of integration A_k is obtained:

$$A_k = \frac{2i}{\pi\sigma_0}\frac{\lambda}{a}\frac{\mathrm{J}_1(ka)}{1+\sqrt{1+4k^2\lambda^2}}\,. \quad (22)$$

3. Electric potential, electric field intensity and Earth resistance formulas

Substituting now for A_k in (16) the electric potential formula becomes:

$$V(r,z) = \frac{2\,i}{\pi\sigma_0}\frac{\lambda}{a}\int_0^\infty \mathrm{J}_0(k\,r)\mathrm{J}_1(k\,a)\,\mathrm{e}^{-\frac{1+\sqrt{1+4k^2\lambda^2}}{2\lambda}z}\frac{\mathrm{d}k}{1+\sqrt{1+4k^2\lambda^2}}. \quad (23)$$

According to (6) and taking into account that $\mathrm{J}_0'(k\,r) = -\,k\,\mathrm{J}_1(k\,r)$, we get the radial component of the electric field formula:

$$E_r(r,z) = -\frac{\partial V}{\partial r} = \frac{2\,i}{\pi\sigma_0}\frac{\lambda}{a}\int_0^\infty \mathrm{J}_1(k\,r)\mathrm{J}_1(k\,a)\,\mathrm{e}^{-\frac{1+\sqrt{1+4k^2\lambda^2}}{2\lambda}z}\frac{k\,\mathrm{d}k}{1+\sqrt{1+4k^2\lambda^2}}. \quad (24)$$

On the earth's surface ($z = 0$) this last formula becomes:

$$E_r(r,0) = \begin{cases} \dfrac{2\,i}{\pi\sigma_0}\dfrac{\lambda}{a}\displaystyle\int_0^\infty \mathrm{J}_1(k\,r)\mathrm{J}_1(k\,a)\frac{k\,\mathrm{d}k}{1+\sqrt{1+4k^2\lambda^2}}, & \text{for } r > a \\ 0, \quad \text{for } r \le a. \end{cases} \quad (25)$$

Note that the null value of the radial electric field for $r \le a$ is consistent to the assumption made for the current density, namely that it has solely a z-component at the plane interface between the electrode and the soil.

The magnitude of the Poynting's vector on the earth's surface $S(r,0) = E_r(r,0)\cdot H(r,0)$ for $r > a$ may be calculated by considering the simple formula of the magnetic field intensity given by the Ampère's law $H(r,0) = i/(2\pi\,r)$. We get:

$$S(r,0) = \frac{i^2}{\pi^2\sigma_0}\frac{\lambda}{a}\frac{1}{r}\int_0^\infty \mathrm{J}_1(k\,r)\mathrm{J}_1(k\,a)\frac{k\,\mathrm{d}k}{1+\sqrt{1+4k^2\lambda^2}} \quad \text{for } r > a. \quad (26)$$

Integration of (26) over the interval (a, ∞) gives the electromagnetic power transferred to the conducting soil:

$$P = \int_a^\infty S(r,0)\cdot 2\pi\,r\,\mathrm{d}r = \frac{2i^2}{\pi\sigma_0}\frac{\lambda}{a}\int_a^\infty \mathrm{d}r\int_0^\infty \mathrm{J}_1(k\,r)\mathrm{J}_1(k\,a)\frac{k\,\mathrm{d}k}{1+\sqrt{1+4k^2\lambda^2}}. \quad (27)$$

Dividing this last formula by i^2, the earth electrode resistance is also derived:

$$R = \frac{P}{i^2} = \frac{1}{\pi \sigma_0 a} \int_a^{\infty} \mathrm{d}\, r \int_0^{\infty} \mathrm{J}_1(k\, r) \mathrm{J}_1(k\, a) \frac{2\lambda k}{1 + \sqrt{1 + 4k^2\lambda^2}} \mathrm{d}\, k \,. \tag{28}$$

Due to its high degree of complexity, this double integral cannot be evaluated to a simple analytical formula. Nevertheless, a mixed analytical and numerical approach is possible [8]. So, if we firstly perform analytically the double-integral with respect to *r*, a more accessible simple-integral is obtained which for some given values of λ and a may be computed by numerical means. Equation (28) becomes:

$$R = \frac{1}{\pi \sigma_0 a} \int_0^{\infty} \mathrm{J}_0(k\, a) \mathrm{J}_1(k\, a) \frac{2\lambda}{1 + \sqrt{1 + 4k^2\lambda^2}} \mathrm{d}\, k \,. \tag{29}$$

Let us define now the normalized earth electrode resistance R_{norm}, by dividing the right-hand side of (29) by a quantity having the dimension of resistance, namely $(\pi \sigma_0 a)^{-1}$.

Next, in order to give a quantitative illustration, a numerical example will be presented. For the radius $a = 0.5$ m of the circular plate the right-hand side integral of (29) is numerically evaluated for a set of discrete values of parameter λ. The variation graph of the normalized earth electrode resistance is shown in Fig. 2.

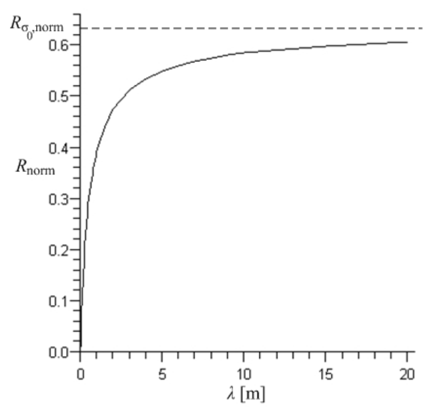

Fig. 2 – Interpolated graph of the normalized earth electrode resistance vs. λ.

Obviously, the parameter λ (having the dimension of length) introduced by (1) defines the “amplification” degree of conductivity along the *z*-axis, into the ground. Computing the limits

$$\lim_{\lambda \to \infty} \sigma_0\, \mathrm{e}^{+\frac{z}{\lambda}} = \sigma_0 \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{2k\lambda}{1 + \sqrt{1 + 4k^2\lambda^2}} = 1 \,,$$

the already reported in the scientific literature formula of the earth electrode resistance in the uniform σ_0 conductivity case is obtained:

$$R_{\sigma_0} = \frac{1}{\pi \sigma_0 a} \int_0^\infty \frac{J_0(ka) J_1(ka)}{k} \mathrm{d}k \,, \tag{30}$$

with its corresponding normalized value $R_{\sigma_{0,}\mathrm{norm}} = 2/\pi \cong 0{,}637$, which is also plotted in Fig. 2.

6. Conclusions

By examining the graph plotted in Fig. 2 two important conclusions may be highlighted. As expected, the smaller the value of parameter λ (corresponding to a rapid rise of conductivity into the ground) the smaller the value of the earth electrode resistance is obtained. In this case due to a significant *z*-direction increase of conductivity the initial half-space may be approximated with a high conductivity plate of finite thickness. With even smaller values of this thickness (λ→0) the resistance of the plate becomes unimportant, becoming practically a superconducting one. Unlike this case, for increasing values of λ the earth electrode resistance grows rapidly by asymptotically converging to the homogeneous conductivity value, obtained from a well known relationship of the earth electrode literature.

R E F E R E N C E S

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